On Bit Error Probabilities of SSMA Communication Systems Using Spreading Sequences of Markov Chains

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SUMMARY We study asynchronous SSMA communication systems using binary spreading sequences of Markov chains and prove the CLT (central limit theorem) for the empirical distribution of the normalized MAI (multiple-access interference). We also prove that the distribution of the normalized MAI for asynchronous systems can never be Gaussian if chains are irreducible and aperiodic. Based on these results, we propose novel theoretical evaluations of bit error probabilities in such systems based on the CLT and compare these and conventional theoretical estimations based on the SGA (standard Gaussian approximation) with experimental results. Consequently we confirm that the proposed theoretical evaluations based on the CLT agree with the experimental results better than the theoretical evaluations based on the SGA. Accordingly, using the theoretical evaluations based on the CLT, we give the optimum spreading sequences of Markov chains in terms of bit error probabilities.

key words: SSMA (spread spectrum multiple access) communication systems, spreading sequences, Markov chains, MAI (multiple-access interference), bit error probability, CLT (central limit theorem), SGA (standard Gaussian approximation)

1. Introduction

To evaluate bit error probabilities in the system analysis of spread spectrum multiple access (SSMA) communication systems, the standard Gaussian approximation (SGA) is often used. For fixed spreading sequences, Pursley defined the average interference parameter (AIP) and gave the average signal-to-noise ratio (SNR) at the receiver output by using Gaussian distributions whose variance was the AIP for asynchronous SSMA communication systems using binary spreading sequences [1]. Based on this result, bit error probabilities in SSMA communication systems using Gold sequences were examined, and it was pointed out in [2] that evaluations of bit error probabilities based on the SGA with the AIP were not valid for such systems with small number of users, low length of pseudonoise (PN) sequences, and high SNR. This problem has been often discussed (see [3] for instance) and improved Gaussian approximations were given in [4] and [5].

Without evaluating the accuracy of the SGA with the AIP, the problem of finding spreading sequences that minimize the AIP was considered and was successfully solved by introducing chaotic spreading sequences in [6]. As far

as symbolic dynamics of PL (piecewise linear) Markov transformations are concerned, we know that their statistical properties are equivalent to those of Markov chains [7]. Therefore we need not use the word *chaotic* any more. What we have to do is to consider spreading sequences of Markov chains. Under some mild conditions concerning a class of Markov chains, we know that the CLT (central limit theorem) holds for the normalized summand of sequences of Markov chains. By virtue of the CLT, we can explicitly obtain the density of the normalized MAI (multiple-access interference) for SSMA communication systems using binary spreading sequences of Markov chains. Using this density, we can give the optimum binary spreading sequences of Markov chains in terms of bit error probabilities. Practically, this is very useful for applying these sequences to SSMA communication systems because bit error probabilities in such systems caused by the MAI from other channels can be explicitly evaluated in advance.

In this study, we examine evaluations of bit error probabilities in SSMA communication systems using binary spreading sequences of Markov chains based on the CLT and discuss designing binary spreading sequences of Markov chains optimal in terms of bit error probabilities. We consider SSMA communication systems using binary spreading sequences of Markov chains and prove the CLT for the empirical distribution of the normalized MAI. Consequently it is found that the distribution of the normalized MAI for asynchronous systems can never be Gaussian if chains are irreducible and aperiodic, which implies that the SGA with the AIP is not precise. Based on these results, we propose novel theoretical evaluations of bit error probabilities in such systems based on the CLT. We then compare the theoretical evaluations based on the CLT and conventional theoretical estimations based on the SGA with experimental results and confirm that the proposed theoretical evaluations based on the CLT agree with the experimental results better than the theoretical evaluations based on the SGA. Using the theoretical evaluations based on the CLT, we explicitly give the optimum spreading sequences of Markov chains in terms of bit error probabilities. These optimum sequences demonstrate that the spreading sequences of Markov chains characterized by the transition matrix with its second eigenvalue $-2 + \sqrt{3}$, which is known to be optimal based on the SGA [8]–[10], are not always optimal in terms of bit error probabilities.

This report is composed of six sections. In Sect. 2, we consider baseband asynchronous SSMA communication

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systems using binary spreading sequences and recall the representation of MAI for such asynchronous systems in [1]. We also simplify this expression for MAI. In Sect. 3, we consider binary spreading sequences generated by general 2-state Markov chains and give the limit of variance of the real part of the normalized MAI for asynchronous SSMA communication systems using such sequences. We find out that it depends on the fractional part of the relative time delay and prove that the distribution of the normalized MAI for asynchronous systems can never be Gaussian if chains are irreducible and aperiodic. In Sect. 4, we see that the theoretical evaluations of bit error probabilities in such systems based on the CLT agree with the experimental results better than the theoretical evaluations based on the SGA. In Sect. 5, using theoretical evaluations based on the CLT, we give the optimum binary spreading sequences of Markov Chains in terms of bit error probabilities. The report concludes with the summary in Sect. 6. We prove the CLT for the empirical distribution of the normalized MAI in Appendix A.

2. MAI in Asynchronous SSMA Systems

In direct-sequence spread-spectrum multiple-access (DS/SSMA) communication systems, data symbols are directly multiplied by a PN code or a spreading code which is independent of the data. In such systems, spread spectrum signals of J users, $s^{(j)}(t)$ ($j = 0, 1, \dots, J-1$) are transmitted through a common channel simultaneously. For simplicity, we consider baseband communications. We suppose that data symbols are bipolar. To simplify discussions, we suppose that code symbols are also bipolar in this research.

The data signal $d^{(j)}(t)$ of the j -th user and its assigned PN code signal $Y^{(j)}(t)$ ($j = 0, 1, \dots, J-1$) can be respectively written in the form

$$d^{(j)}(t) = \sum_{p=-\infty}^{\infty} d_p^{(j)} u_T(t - pT_d), \quad d_p^{(j)} \in \{1, -1\}, \quad (1)$$

$$Y^{(j)}(t) = \sum_{q=-\infty}^{\infty} Y_q^{(j)} u_{T_c}(t - qT_c), \quad Y_q^{(j)} \in \{1, -1\}, \quad (2)$$

where $u_T(t)$ is defined by

$$u_T(t) = \begin{cases} 1 & \text{for } 0 \leq t < T \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

We assume that the j -th user's PN code sequence $(Y_n^{(j)})_{n=-\infty}^{\infty}$ has period $N = T_d/T_c$, so that there is a PN sequence $Y^{(j)} = (Y_0^{(j)}, Y_1^{(j)}, \dots, Y_{N-1}^{(j)})$ per data symbol. Without loss of generality, we assume that $T_c = 1$. We denote the 0-th user's PN code sequence $Y^{(0)}$ per data symbol by $X = (X_n)_{n=0}^{N-1}$ throughout this study. Thus the baseband spread spectrum signal $s^{(j)}(t)$ is given by

$$s^{(j)}(t) = Y^{(j)}(t)d^{(j)}(t). \quad (4)$$

Generally, the received signal $r(t)$ can be represented as

$$r(t) = \sum_{j=0}^{J-1} Y^{(j)}(t - t_j) d^{(j)}(t - t_j) + n(t), \quad (5)$$

where t_j is the time delay in the j -th channel and $n(t)$ is a common channel noise process.

If the received signal is the input to the correlation receiver matched to $Y^{(0)} = X$, then the output during the p -th time-interval is given by

$$Z_p = \int_{pT}^{(p+1)T} r(t + t_0) X(t) dt = \eta + d_p^{(0)} N + I_{J,p}, \quad (6)$$

where

$$\eta = \int_{pT}^{(p+1)T} n(t + t_0) X(t) dt, \quad (7)$$

and the MAI from other $J-1$ channels during p -th time interval for asynchronous SSMA communication systems is given by

$$I_{J,p} = \sum_{j=1}^{J-1} \int_{pT_d}^{(p+1)T_d} d^{(j)}(t - (t_j - t_0)) \times Y^{(j)}(t - (t_j - t_0)) X(t) dt. \quad (8)$$

Without loss of generality, we assume that $t_0 = 0$. Since the 0-th user's PN code signal $X(t)$ has period $N = T_d$, we take modulo T_d for relative time delays. Then we have $0 \leq t_j - t_0 = t_j < N$. We define $\ell_j = [t_j]$, $\tau_j = \{t_j\}$, where $[x]$ and $\{x\}$ denote the integer part and fractional part of a real number x . Then we have $\ell = 0, 1, 2, \dots, N-1$ and $0 \leq \tau_j < 1$.

To evaluate the MAI, $I_{J,p}$, we use correlation functions for PN sequences introduced in [1].

Definition 1: The *aperiodic cross-correlation function* of time delay ℓ for the sequences $X = (X_n)_{n=0}^{N-1}$ and $Y = (Y_n)_{n=0}^{N-1}$ is defined by

$$R_N^A(\ell; X, Y) = \sum_{n=0}^{N-1-\ell} X_n Y_{n+\ell}, \quad (9)$$

where $\ell = 0, 1, 2, \dots, N-1$. This gives the *even* and *odd cross-correlation functions* of time delay ℓ for the sequences X and Y of period N which are respectively given by

$$R_N^E(\ell; X, Y) = R_N^A(\ell; X, Y) + R_N^A(N - \ell; Y, X), \quad (10)$$

$$R_N^O(\ell; X, Y) = R_N^A(\ell; X, Y) - R_N^A(N - \ell; Y, X). \quad (11)$$

To simplify expression of the MAI, we introduce

Definition 2: Consider a sequence of real numbers $(\alpha(\ell))_{\ell=0}^{N-1}$ and define a point of the line segment joining $\alpha(\ell)$ and $\alpha(\ell+1)$ with the exclusion of the point $\alpha(\ell+1)$ in the real line by

$$S(\tau; \alpha(\ell)) = (1 - \tau)\alpha(\ell) + \tau\alpha(\ell+1), \quad 0 \leq \tau < 1. \quad (12)$$

Thus we obtain for asynchronous SSMA communication systems

$$I_{J,p} = \sum_{j=1}^{J-1} \left\{ \frac{d_p^{(j)} + d_{p+1}^{(j)}}{2} S(\tau_j, R_N^E(\ell_j; X, Y^{(j)})) + \frac{d_p^{(j)} - d_{p+1}^{(j)}}{2} S(\tau_j, R_N^O(\ell_j; X, Y^{(j)})) \right\}. \quad (13)$$

To evaluate bit error probabilities in asynchronous SSMA communication systems based on the central limit theorem, we are concerned with the variance of the normalized MAI $I_{j,p}/\sqrt{N}$. By virtue of the identity $(d_p^{(j)} + d_{p+1}^{(j)})(d_p^{(j)} - d_{p+1}^{(j)}) = 0$, we can evaluate separately the variances of $S(\tau_j, R_N^E(\ell_j; \mathbf{X}, \mathbf{Y}^{(j)}))/\sqrt{N}$ and $S(\tau_j, R_N^O(\ell_j; \mathbf{X}, \mathbf{Y}^{(j)}))/\sqrt{N}$.

3. Binary Spreading Sequence of Markov Chains

Now we consider binary spreading sequences of Markov chains. Let $\mathbf{X} = (X_n)_{n=0}^\infty$ and $\mathbf{Y} = (Y_n)_{n=0}^\infty$ be Markov chains taking their values in $\{-1, 1\}$. We set transition probabilities

$$\begin{aligned} p_{ij} &= P(X_{n+1} = 2j - 3 | X_n = 2i - 3) \\ &= P(Y_{n+1} = 2j - 3 | Y_n = 2i - 3) \end{aligned} \quad (14)$$

for $i, j = 1, 2$. We define the transition matrix by $P = (p_{ij})_{i,j=1}^2$.

Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are stationary Markov chains and mutually independent. Let their stationary distributions be

$$P(X_n = 2i - 3) = P(Y_n = 2i - 3) = 1/2 \quad (15)$$

for $i = 1, 2$. Then we have

$$E[X_n] = E[Y_n] = 0, \quad (16)$$

$$E[X_n Y_{n+\ell}] = E[X_n]E[Y_n] = 0 \quad \text{for } \ell \geq 0, \quad (17)$$

where $E[Z]$ denotes the expected value of random variable Z .

We note here that the transition matrix P is a doubly stochastic matrix by the condition (15).

It follows by (17) that the mean value of the aperiodic cross-correlation function is

$$E[R_N^A(\ell; \mathbf{X}, \mathbf{Y})] = 0 \quad \text{for } \ell \geq 0. \quad (18)$$

Thus the mean values of the even and odd cross-correlation functions are respectively

$$E[R_N^E(\ell; \mathbf{X}, \mathbf{Y})] = E[R_N^O(\ell; \mathbf{X}, \mathbf{Y})] = 0 \quad \text{for } \ell \geq 0, \quad (19)$$

which implies

$$E\left[S\left(\tau, R_N^{E/O}(\ell; \mathbf{X}, \mathbf{Y})/\sqrt{N}\right)\right] = 0 \quad \text{for } \ell \geq 0, \quad (20)$$

where the superscript E/O denotes either the even or odd cross-correlation function.

For simplicity we suppose irreducible, aperiodic chains. Noting a 2×2 doubly stochastic matrix P is a symmetric matrix, which can be diagonalized by orthogonal matrices, we obtain

Lemma 1:

$$E[X_n X_{n+\ell}] = E[Y_n Y_{n+\ell}] = \lambda^\ell \quad \text{for } \ell \geq 0, \quad (21)$$

where λ is the second eigenvalue of the doubly stochastic matrix P .

Thus we obtain

$$\begin{aligned} \lim_{N \rightarrow \infty} V\left[S\left(\tau, R_N^E(\ell; \mathbf{X}, \mathbf{Y})/\sqrt{N}\right)\right] \\ = \lim_{N \rightarrow \infty} V\left[S\left(\tau, R_N^O(\ell; \mathbf{X}, \mathbf{Y})/\sqrt{N}\right)\right] \\ = \{(1 - \tau)^2 + \tau^2\} \left(1 + \frac{2\lambda^2}{1 - \lambda^2}\right) + 2(1 - \tau)\tau \frac{2\lambda}{1 - \lambda^2}, \end{aligned} \quad (22)$$

where $V[Z]$ denotes the variance of random variable Z . We also have the following theorem:

Theorem 1 (Central Limit Theorem (CLT)): In the above-mentioned situation, we have

$$P_{S(\tau, R_N^{E/O}(\ell; \mathbf{X}, \mathbf{Y})/\sqrt{N})} \Rightarrow \mathcal{N}(0, \sigma^{E/O^2}), \quad (23)$$

where P_Z denotes the distribution of random variable Z , $\mathcal{N}(m, \sigma^2)$ denotes the normal distribution with mean m and variance σ^2 , and $\Rightarrow$ denotes weak convergence. The variance σ^{E/O^2} is given by

$$\sigma^{E/O^2} = \lim_{N \rightarrow \infty} V\left[S\left(\tau, R_N^{E/O}(\ell; \mathbf{X}, \mathbf{Y})/\sqrt{N}\right)\right]. \quad (24)$$

To the best of the author's knowledge, the theorem in the above-mentioned situation is not directly proved although it has already been heuristically claimed in [9] and [10]. In general situations, however, the CLT for empirical processes of mixing random variables is well known in probability theory. Actually, this theorem can be regarded as a special case of Theorem 1 in [12]. By virtue of Corollary 1 in [12], we can prove Theorem 1 above as shown in Appendix A. Pedagogically a direct and simple proof of Theorem 1 above is very meaningful, but it is still open to question.

Now we are in a position to obtain the distributions of the normalized MAI for SSMA communication systems using binary spreading sequences of Markov chains characterized by symmetric matrices. To simplify expressions, we denote the limits in (22) by $\sigma(\tau)^2$.

If we regard τ as the outcome of random variable T with uniform distribution on $[0, 1]$, we then have

Lemma 2: Consider SSMA communication systems using binary spreading sequences of Markov chains. The density distribution of the normalized MAI for such systems is given by

$$\int_0^1 \frac{1}{\sqrt{2\pi\sigma(\tau)^2}} \exp\left(-\frac{z^2}{2\sigma(\tau)^2}\right) d\tau, \quad z \in \mathbf{R}. \quad (25)$$

This lemma raises very important questions on conventional optimization procedures. Following conventional system analysis in [1] and [11], we may use

$$\frac{1}{\sqrt{2\pi E[\sigma(T)^2]}} \exp\left(-\frac{z^2}{2E[\sigma(T)^2]}\right), \quad z \in \mathbf{R} \quad (26)$$

as the density distribution of the normalized MAI for SSMA

communication systems. This is called the SGA. On the other hand, Lemma 2 tells us that this is not always true. And hence the results of our previous optimization in [8] and [10] are not precise and are to be regarded as a first approximation since the optimum sequences are given by using the density (26).

If we regard $\tau_1, \tau_2, \dots, \tau_{J-1}$ as the outcomes of mutually independent random variables $T_1, T_2, \dots, T_{J-1}$ with uniform distribution on $[0, 1)$, then we have

Remark 1: Consider SSMA communication systems of J users using binary spreading sequences of Markov chains with a symmetric transition matrix. The density distribution of the normalized MAI for such systems is given by

$$\int_0^1 \int_0^1 \cdots \int_0^1 \frac{1}{\sqrt{2\pi \sum_{j=1}^{J-1} \sigma(\tau_j)^2}} \times \exp\left(-\frac{z^2}{2 \sum_{j=1}^{J-1} \sigma(\tau_j)^2}\right) d\tau_1 d\tau_2 \cdots d\tau_{J-1}, \quad z \in \mathbf{R}. \quad (27)$$

By using Lemma 2, we can obtain

Theorem 2: Consider asynchronous SSMA communication systems using binary spreading sequences of Markov chains. If chains are irreducible and aperiodic, the distributions of the normalized MAI for such systems can never be Gaussian.

Proof: See Appendix B.

Remark 2: If the system is chip-synchronous, i.e. $\tau \equiv 0$, the distributions of the normalized MAI is Gaussian.

For chip-synchronous systems, we obtain

Theorem 3: For chip-synchronous systems, sequences of i.i.d. (independent and identically distributed) random variables are optimal in terms of bit error probabilities.

4. Bit Error Probabilities in SSMA Communication Systems Using Binary Spreading Sequences of Markov Chains

Suppose that J users communicate through a common channel in the SSMA systems independently and asynchronously. To simplify discussions, we consider the ideal noiseless channel. The bit error probability caused by the MAI from other $J-1$ spreading codes may be estimated by

$$P(I_{J,p} > N | d_p^{(0)} = -1) \approx \int_{-\infty}^{\infty} \left(\int_0^1 \int_0^1 \cdots \int_0^1 \frac{1}{\sqrt{2\pi \sum_{j=1}^{J-1} \sigma(\tau_j)^2}} \times \exp\left(-\frac{z^2}{2 \sum_{j=1}^{J-1} \sigma(\tau_j)^2}\right) d\tau_1 d\tau_2 \cdots d\tau_{J-1} \right) dz. \quad (28)$$

On the other hand, it is approximately estimated in [8]–[10] by

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(J-1)E[\sigma(T)^2]}} \times \exp\left(-\frac{z^2}{2(J-1)E[\sigma(T)^2]}\right) dz. \quad (29)$$

The problem of the theoretical evaluations of bit error probabilities (28) are in the forms of multiple integrals. We have recently given approximations for these forms and have shown that (29) is the 0-th order approximation for (28) in [13].

Let us examine the accuracy of theoretical estimations of bit error probabilities based on (28) and (29) by comparing them with experimental results.

Figure 1 shows the theoretical estimations based on (28) and (29) and experimental results of bit error probabilities in asynchronous SSMA communication systems using binary spreading sequences of i.i.d. random variables as a function of the number of users J for $N = 32$. Figure 2 shows the theoretical estimations based on (28) and (29) and experimental results of bit error probabilities in asynchronous SSMA communication systems using binary spreading sequences of Markov chain with the second eigenvalue $\lambda = -2 + \sqrt{3}$ of its transition matrix as a function of the number of users J for $N = 32$. In both figures, the theoretical estimations based on (28) agree properly with experimental results better than the ones based on (29) except for $J = 3$.

Figure 3 and Fig. 4 show the theoretical estimations based on (28) and (29) and experimental results of bit error probabilities in asynchronous SSMA communication systems using binary spreading sequences of i.i.d. random variables as a function of the period of PN sequences N for $J = 5$ and $J = 4$, respectively. In both cases, the theoretical estimations based on (28) agree properly with the experimental results better than the ones based on (29).

Figure 5 and Fig. 6 show the theoretical estimations

Fig. 1 Bit error probabilities in SSMA communication systems using binary spreading sequences of i.i.d. random variables for $N = 32$.

Fig. 2 Bit error probabilities in SSMA communication systems using binary spreading sequences of Markov chains uniquely characterized by transition matrix P with its second eigenvalue $\lambda = -2 + \sqrt{3}$ for $N = 32$.

Fig. 3 Bit error probabilities in SSMA communication systems using binary spreading sequences of i.i.d. random variables for $J = 5$.

Fig. 4 Bit error probabilities in SSMA communication systems using binary spreading sequences of i.i.d. random variables for $J = 4$.

Fig. 5 Bit error probabilities in SSMA communication systems using binary spreading sequences of Markov chains uniquely characterized by transition matrix P with its second eigenvalue $\lambda = -2 + \sqrt{3}$ for $J = 5$.

Fig. 6 Bit error probabilities in SSMA communication systems using binary spreading sequences of Markov chains uniquely characterized by transition matrix P with its second eigenvalue $\lambda = -2 + \sqrt{3}$ for $J = 4$.

based on (28) and (29) and experimental results of bit error probabilities in asynchronous SSMA communication systems using binary spreading sequences of Markov chains uniquely characterized by transition matrix P with its second eigenvalue $\lambda = -2 + \sqrt{3}$ as a function of the period of PN sequences N for $J = 5$ and $J = 4$, respectively. In both cases, the theoretical estimations based on (28) agree properly with the experimental results better than the ones based on (29).

Figure 7 shows the theoretical estimations based on (28) and (29) and experimental results of bit error probabilities in asynchronous SSMA communication systems using binary spreading sequences of i.i.d. random variables as a function of the period of PN sequences N for $J = 3$. Figure 8 shows the theoretical estimations based on (28) and (29) and experimental results of bit error probabilities in asynchronous SSMA communication systems using binary spreading sequences of Markov chains uniquely char-

Fig. 7 Bit error probabilities in SSMA communication systems using binary spreading sequences of i.i.d. random variables for $J = 3$.

Fig. 8 Bit error probabilities in SSMA communication systems using binary spreading sequences of Markov chains uniquely characterized by transition matrix P with its second eigenvalue $\lambda = -2 + \sqrt{3}$ for $J = 3$.

acterized by transition matrix P with its second eigenvalue $\lambda = -2 + \sqrt{3}$ as a function of the period of PN sequences N for $J = 3$. In these two cases, comparing with experimental results, the theoretical estimations based on (28) are slightly overestimated while the ones based on (29) are too underestimated. In other words, the former estimations are conservative, but the latter estimations are optimistic.

5. On Optimum Binary Spreading Sequences of Markov Chains

Let us recall our previous optimization in [8] and [10] regarding it as a first approximation. The expected value of the variance $\sigma(T)^2$ is given by

$$E[\sigma(T)^2] = \frac{2}{3} \left(1 + 2 \frac{\lambda^2}{1 - \lambda^2} + \frac{\lambda}{1 - \lambda^2} \right), \quad (30)$$

which takes the minimum value, $2/3 - 1/6(\sqrt{3} - 1)^2$ for $\lambda = -2 + \sqrt{3}$. We stress here that, as far as the random variable

Fig. 9 Bit error probabilities in SSMA communication systems using binary spreading sequences of i.i.d. random variables and binary spreading sequences of Markov chains uniquely characterized by transition matrix P with its second eigenvalue $\lambda = -2 + \sqrt{3}$ for $N = 32$.

T is concerned, the AIP is derived in the same manner as this expectation. Interestingly, this very number $-2 + \sqrt{3}$ is the minimizer in both [6] and [8]–[10].

This result says as a first approximation that the binary sequences generated by the unique Markov chain whose transition matrix P with its second eigenvalue $\lambda = -2 + \sqrt{3}$ are optimal and hence are better than binary sequences of i.i.d. random variables in terms of bit error probabilities. We examine this fact by using (28).

Figure 9 shows that the theoretical estimation of bit error probabilities in asynchronous SSMA communication systems using binary spreading sequences of Markov chains including binary spreading sequences of i.i.d. random variables as a function of the number of users J for $N = 32$ based on (28). Points $\times$ indicate the bit error probabilities in the system using binary spreading sequences of i.i.d. random variables while points $+$ indicate the ones in the system using the binary spreading sequences of Markov chains with the second eigenvalue $\lambda = -2 + \sqrt{3}$ of its transition matrix. Figure 9 tells us that the binary spreading sequences of the Markov chain with the second eigenvalue $\lambda = -2 + \sqrt{3}$ of its transition matrix are not always better than binary spreading sequences of i.i.d. random variables in terms of bit error probabilities. In fact, the binary spreading sequences of i.i.d. random variables are better for $J = 2$ and 3.

Now let us discuss optimum binary spreading sequences of Markov chains in terms of bit error probabilities based on (28) for $J = 3$.

Table 1 shows theoretical estimations based on (28) of the second eigenvalue λ of transition matrix P which minimizes bit error probabilities in SSMA communication systems using binary spreading sequences of Markov chains uniquely characterized by λ of P for $J = 3$. This implies the existence of the optimum Markov chains characterized by P with $\lambda \neq 0$. Note that binary sequences of Markov chains becomes binary sequences of i.i.d. random variables if $\lambda = 0$.

Table 1 The second eigenvalue λ of transition matrix P which minimizes bit error probabilities in SSMA communication systems using binary spreading sequences of Markov chains characterized by P with λ for $J = 3$.

N	λ	N	λ
24	-0.123764	31	-0.0998438
25	-0.125686	32	-0.102411
26	-0.127679	33	-0.104954
27	-0.129721	34	-0.0845418
28	-0.0920871	35	-0.0944466
29	-0.0946709	36	-0.0930879
30	-0.0972605

Fig. 10 Theoretical estimations of bit error probabilities in SSMA communication systems using binary spreading sequences of Markov chains for $J = 3$.

Fig. 11 Experimental results of bit error probabilities in SSMA communication systems using binary spreading sequences of Markov chains for $J = 3$.

Figure 10 shows the theoretical estimations based on (28) of bit error probabilities in asynchronous SSMA communication systems using binary spreading sequences of Markov chains as a function of the period of PN sequences N for $J = 3$. Figure 11 shows experimental results of bit error probabilities in asynchronous SSMA communication systems using binary spreading sequences of Markov

chains. In Fig. 11, points $\times$ indicate the bit error probabilities in the system using binary spreading sequences of i.i.d. random variables, while points $+$ indicate the ones in the system using binary spreading sequences of Markov chains characterized by P with $\lambda = -2 + \sqrt{3}$. Points $*$ indicate the bit error probabilities in the system using the optimum binary spreading sequences of Markov chains based on (28). Figure 11 shows that for all N , the optimum Markov chain based on (28) is better than the other two chains including the one with i.i.d. random variables.

Lastly, we note that the optimum sequences of Markov chains can be easily obtained by Kalman's method which gives one-dimensional piecewise linear transformations generating sequences of Markov chains with arbitrarily prescribed transition matrix [7]. A 2×2 doubly stochastic matrix P is uniquely written by using its second eigenvalue λ in the form:

$$P = \frac{1}{2} \begin{pmatrix} 1 + \lambda & 1 - \lambda \\ 1 - \lambda & 1 + \lambda \end{pmatrix}. \quad (31)$$

6. Conclusions

In this research, we considered SSMA communication systems using binary spreading sequences of Markov chains and proved the CLT for the empirical distribution of the normalized MAI. We also proved that the distribution of the normalized MAI for asynchronous systems can never be Gaussian if chains are irreducible and aperiodic. Based on these results, we proposed novel theoretical evaluations of bit error probabilities in such systems based on the CLT. We then compared theoretical evaluations based on the CLT and those based on the SGA with experimental results. Consequently we confirmed that the proposed theoretical evaluations of bit error probabilities in such systems based on the CLT agree with the experimental results better than the theoretical evaluations based on the SGA. Finally, we derived the optimum Markov chains based on the CLT and showed experimentally that they are better than the chains optimally based on the SGA and i.i.d. random variables.

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Appendix A: Proof of Theorem 1

We only give a proof of the CLT for the normalized even cross-correlation terms in the MAI. We can prove the CLT for the normalized odd cross-correlation parts in exactly the same manner as follows for the even parts.

For fixed ℓ ($0 \leq \ell \leq N-1$), we set

$$Z_n = X_{n+N-\ell \pmod N} \{(1-\tau)Y_n + \tau Y_{n+1 \pmod N}\}, \\ n = 0, 1, \dots, N-1. \quad (\text{A} \cdot 1)$$

Thus we obtain a sequence of random variables $(Z_n)_{n=0}^{N-1}$ taking values in $\{\pm 1, \pm(1-2\tau)\}$.

Consider a nonnegative function α_N of nonnegative integers. The condition for $\alpha_N(i)$ -mixing is

$$\sup\{|P(A \cap B) - P(A)P(B)| \mid A \in \sigma(Z_0, \dots, Z_{k-1}), \\ B \in \sigma(Z_{k+i-1}, \dots, Z_{N-1}), 1 \leq k \leq N-i\} \leq \alpha_N(i). \quad (\text{A} \cdot 2)$$

To prove $(Z_n)_{n=0}^{N-1}$ is $\alpha_N(i)$ -mixing, we first look at the mixing properties of $(Y_n)_{n=0}^{N-1}$. For the special elements

$A = \{Y_0 = y_0, \dots, Y_{k-1} = y_{k-1}\} \in \sigma(Y_0, \dots, Y_{k-1})$ and $B = \{Y_{k+i-1} = y_{k+i-1}, \dots, Y_{N-1} = y_{N-1}\} \in \sigma(Y_{k+i-1}, \dots, Y_{N-1})$, there exists a positive constant C such that

$$|P(A \cap B) - P(A)P(B)| \leq C|\lambda|^i P(A)P(B), \quad (\text{A} \cdot 3)$$

where $\sigma(Y_0, \dots, Y_{k-1})$ and $\sigma(Y_{k+i-1}, \dots, Y_{N-1})$ denote the σ -algebra generated by $Y_0, \dots, Y_{k-1}$ and $Y_{k+i-1}, \dots, Y_{N-1}$ respectively. Thus for any events $A \in \sigma(Y_0, \dots, Y_{k-1})$ and $B \in \sigma(Y_{k+i-1}, \dots, Y_{N-1})$, we obtain (A·3) as well.

We continue to use C for suitable positive constants where the actual value of C might change each time it is used.

Next let $L_N = \lfloor N^{1/4} \rfloor$ and consider the sequence $(X_{N-\ell+n \pmod N})_{n=0}^{N-L_N-1}$. For the special elements $A = \{X_{N-\ell} = x_{N-\ell}, \dots, X_{N-\ell+k-1 \pmod N} = x_{N-\ell+k-1 \pmod N}\} \in \sigma(X_{N-\ell}, \dots, X_{N-\ell+k-1 \pmod N})$ and $B = \{X_{N-\ell+k+i-1 \pmod N} = x_{N-\ell+k+i-1 \pmod N}, \dots, X_{N-L_N-1} = x_{N-L_N-1}\} \in \sigma(X_{N-\ell+k+i-1 \pmod N}, \dots, X_{N-L_N-1 \pmod N})$, we obtain

$$|P(A \cap B) - P(A)P(B)| \leq C|\lambda|^{\min\{i, L_N\}} P(A)P(B). \quad (\text{A} \cdot 4)$$

Thus for any events $A \in \sigma(X_{N-\ell}, \dots, X_{N-\ell+k-1 \pmod N})$ and $B \in \sigma(X_{N-\ell+k+i-1 \pmod N}, \dots, X_{N-L_N-1 \pmod N})$, we obtain (A·4) as well.

Noting that $(X_n)_{n=0}^{N-1}$ and $(Y_n)_{n=0}^{N-1}$ are mutually independent, for the product sequence $(Y_0, Y_1, X_{N-\ell}, (Y_1, Y_2, X_{N-\ell+1}), \dots, (Y_{\ell-1}, Y_{\ell}, X_{N-1}), (Y_{\ell}, Y_{\ell+1}, X_0), \dots, (Y_{N-L_N-1}, Y_{N-L_N}, X_{N-L_N-1 \pmod N}))$, we obtain

$$|P(A \cap B) - P(A)P(B)| \leq C|\lambda|^{\min\{i, L_N\}} P(A)P(B), \quad (\text{A} \cdot 5)$$

for any $A \in \sigma((Y_0, Y_1, X_{N-\ell}), \dots, (Y_{k-1}, Y_k, X_{N-\ell+k-1 \pmod N}))$ and $B \in \sigma((Y_{k+i-1}, Y_{k+i}, X_{N-\ell+k+i-1 \pmod N}), \dots, (Y_{N-L_N-1}, Y_{N-L_N}, X_{N-L_N-1 \pmod N}))$.

By definition (A·1) of $(Z_n)_{n=0}^{N-1}$, we have $\sigma(Z_0, \dots, Z_{k-1}) \subset \sigma((Y_0, Y_1, X_{N-\ell}), \dots, (Y_{k-1}, Y_k, X_{N-\ell+k-1 \pmod N}))$ and $\sigma(Z_{k+i-1}, \dots, Z_{N-L_N-1}) \subset \sigma((Y_{k+i-1}, Y_{k+i}, X_{N-\ell+k+i-1 \pmod N}), \dots, (Y_{N-L_N-1}, Y_{N-L_N}, X_{N-L_N-1 \pmod N}))$. Thus we obtain

$$|P(A \cap B) - P(A)P(B)| \leq C|\lambda|^{\min\{i, L_N\}} \quad (\text{A} \cdot 6)$$

for all $A \in \sigma(Z_0, \dots, Z_{k-1})$ and $B \in \sigma(Z_{k+i-1}, \dots, Z_{N-L_N-1})$, where C is a constant. This implies the sequence $(Z_n)_{n=0}^{N-L_N-1} = (X_{n+N-\ell \pmod N} \{(1-\tau)Y_n + \tau Y_{n+1 \pmod N}\})_{n=0}^{N-L_N-1}$ is $\alpha_N(i)$ -mixing.

We remark here that we can similarly show the process $(X_{n+N-\ell \pmod N} Y_n)_{n=0}^{N-1}$ is $\alpha_N(i)$ -mixing if we put $\tau \equiv 0$ in (A·1).

We can easily check that $\sum_{i=1}^{N-1} \alpha_N(i) \leq K < \infty$ holds for a constant K and that $\lim_{N \rightarrow \infty} N^{1-a} \alpha_N([N^b]) = 0$ also holds where $0 < 2b < a < 1 - b$. We also have $E \left[\frac{1}{\sqrt{N-L_N}} \sum_{n=0}^{N-L_N-1} Z_n \right] = 0$. Noting that $\lim_{N \rightarrow \infty} L_N / \sqrt{N} = 0$, we obtain $\lim_{N \rightarrow \infty} V \left[\frac{1}{\sqrt{N-L_N}} \sum_{n=0}^{N-L_N-1} Z_n \right] = \lim_{N \rightarrow \infty}$

$V\left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} Z_n\right] < \infty$ in view of (22). Therefore, by Corollary 1 in [12], we obtain

$$P_{\frac{1}{\sqrt{N-L_N}} \sum_{n=0}^{N-L_N-1} Z_n} \Rightarrow \mathcal{N}(0, \sigma^2) \quad (\text{A} \cdot 7)$$

where $\sigma^2 = \lim_{N \rightarrow \infty} V\left[\frac{1}{\sqrt{N-L_N}} \sum_{n=0}^{N-L_N-1} Z_n\right]$. Noting again that $\lim_{N \rightarrow \infty} L_N / \sqrt{N} = 0$, we obtain

$$P_{\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} Z_n} \Rightarrow \mathcal{N}(0, \sigma^2) \quad (\text{A} \cdot 8)$$

where $\sigma^2 = \lim_{N \rightarrow \infty} V\left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} Z_n\right]$. This completes the proof.

Appendix B: Proof of Theorem 2

Let Z be a real-valued random variable representing the real part of the normalized MAI with random variable T . The density (25) is normal if and only if its Laplace transform $E[e^{sZ}]$ is given by

$$\exp\left(\frac{1}{2}s^2 \int_0^1 \sigma(\tau)^2 d\tau\right) = \exp\left(\frac{1}{2}s^2 E[\sigma(T)^2]\right),$$

which is in fact the Laplace transform of (26). On the other hand, the Laplace transform of (25) is actually computed as

$$\int_0^1 \exp\left(\frac{1}{2}s^2 \sigma(\tau)^2\right) d\tau = E\left[\exp\left(\frac{1}{2}s^2 \sigma(T)^2\right)\right].$$

Since the exponential function is convex, the use of Jensen's inequality gives

$$\exp\left(\frac{1}{2}s^2 \int_0^1 \sigma(\tau)^2 d\tau\right) \leq \int_0^1 \exp\left(\frac{1}{2}s^2 \sigma(\tau)^2\right) d\tau \quad (\text{A} \cdot 9)$$

with equality if and only if $\sigma(\tau)^2$ does not depend on τ . In view of (22), we obtain $\lambda = 1$ which is impossible since we suppose chains are irreducible and aperiodic. This completes the proof.

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