The Tilt Formula for Generalized Simplices in Hyperbolic Space

The tilt formula for weighted simplices in hyperbolic space

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Abstract

For a simplex in Lorentzian space whose vertices are in the positive light cone, J. R. Weeks defined the "tilt" relative to each of its faces. It gives an efficient tool for deciding whether or not the dihedral angle between two simplices holding a face in common is convex. He also provided an efficient formula, called the "tilt formula," to obtain tilts from the intrinsic geometry of the simplex when its dimension is two or three. M. Sakuma and J. R. Weeks generalized it to general dimensions. In this paper, we generalize the concept of the tilt and the tilt formula to the case where not all vertices are in the positive light cone. A key to our generalization is to give a correspondence between points and hyperplanes (or half-spaces) in Lorentzian space. In hyperbolic space, we can regard these hyperplanes as geometric objects, points, spheres, geodesic hyperplanes, equidistant hypersurfaces and horospheres.

Key words: tilt formula, canonical decomposition, convex hull construction, simplex, hyperbolic geometry.

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1 Introduction

D. B. A. Epstein and R. C. Penner gave in [EP] a method for decomposing any noncompact complete hyperbolic manifold of finite volume with *weight* at each cusp into ideal polyhedra. This decomposition is called the *Euclidean decomposition*, and defined via a *convex hull construction* in Lorentzian space. Each vertex of the hull is in the positive light cone and corresponds to a lift of a cusp, and each face of the hull corresponds to an ideal polyhedron in the manifold. Especially if all weights are equal, then the decomposition is called the *canonical decomposition*.

For a simplex in Lorentzian space whose vertices are in the positive light cone, J. R. Weeks defined in [We1] the *tilt* relative to each of its faces. It gives an efficient tool for deciding whether or not the dihedral angle between two simplices holding a face in common is convex. So it becomes a useful tool to determine whether or not a given decomposition of the manifold is obtained from a convex hull. He also provided an efficient formula, called the *tilt formula*, to obtain tilts from the intrinsic hyperbolic geometry of the simplex when its dimension is two or three. Using this formula, he made the hyperbolic structures computation program "SnapPea" (cf. [We2]). Then M. Sakuma and J. R. Weeks generalized the tilt formula to general dimensions in [SW] .

S. Kojima gave in [Ko1, Ko2] a method for decomposing any complete hyperbolic manifold of finite volume with non-empty totally geodesic boundary into partially truncated polyhedra. In many cases each polyhedron is a partially truncated simplex. Since such a simplex is lifted to a simplex in Lorentzian space whose vertices may not be in the positive light cone, it is meaningful to generalize the concept of the tilt and to establish the tilt formula for the generalized tilt. The main purpose of the paper is to do it (see Theorem 6.4, Theorem 6.8 and Corollary 6.9).

This paper is organized as follows: in Section 2 we recall some basic facts about Lorentzian space and hyperbolic geometry. One important task of this section is to give a correspondence between points and hyperplanes (or half-spaces) in Lorentzian space. This correspondence is a generalization of the well-known duality between points in the hyperboloid of one sheet and half-spaces in hyperbolic space. In Section 3 we define two values connected with the hyperbolic distance; one is called the *signed distance*, and the other is called the *width*. The former is a extension of the hyperbolic distance between a geodesic hyperplane and a point, and the later is a generalization of the radius of a sphere and the distance between a equidistant hypersurface and its axial hyperplane. We also relate these values with the Lorentzian inner product. In Section 4 we first define a *generalized n-simplex* in the projective model of the n-dimensional hyperbolic space. Roughly speaking, this is a partially truncated n-dimensional simplex of finite volume. Furthermore we extend it to a *weighted n-simplex*. In Section 5 we define the *tilt* of a weighted n-simplex relative to an internal face, and obtain a relationship between tilts and the convexity of two adjoining weighted n-simplices (see Proposition 5.2). In Section 6 we first define a complex number called a *generalized distance*, by unifying the signed distance and the dihedral angle between two geodesic hyperplanes. We next establish an efficient way to obtain tilts of a weighted n-simplex, by imitating the method in [SW] (see Theorem 6.4, Theorem 6.8 and Corollary 6.9).

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2 Lorentzian space and hyperbolic geometry

The $n+1$-*dimensional Lorentzian space* (or simply *Lorentzian $n+1$-space*) $\mathbf{E}^{1,n}$ is the real vector space $\mathbf{R}^{n+1}$ of dimension $n+1$ with the *Lorentzian inner product* $\langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle := -x_0\,y_0 + x_1\,y_1 + \cdots + x_n\,y_n$, where $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_0, y_1, \ldots, y_n)$. Throughout this paper, we assume $n \geq 2$. The *Lorentzian norm* of $\boldsymbol{x}$ in $\mathbf{E}^{1,n}$ is defined to be the complex number $\sqrt{\langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle}$. If the Lorentzian norm of $\boldsymbol{x}$ is zero (resp. positive, imaginary), then $\boldsymbol{x}$ is said to be *light-like* (resp. *space-like*, *time-like*). The coordinate x_0 of $\mathbf{E}^{1,n}$ is called the *height*. Now we define six connected subsets in $\mathbf{E}^{1,n}$ as follows: the set of time-like vectors with positive height is $T^+ := \{\, \boldsymbol{x} \in \mathbf{E}^{1,n} \mid \langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle < 0 \text{ and } x_0 > 0\,\}$, the set of time-like vectors with negative height is $T^- := \{\, \boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle < 0 \text{ and } x_0 < 0 \,\}$, the set of light-like vectors is $L := \{\, \boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle = 0 \,\}$, the set of light-like vectors with positive height is $L^+ := \{\, \boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle = 0 \text{ and } x_0 > 0 \,\}\,(\subset L)$, the set of light-like vectors with negative height is $L^- := \{\, \boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle = 0 \text{ and } x_0 < 0 \,\}\,(\subset L)$, and the set of space-like vectors is $S := \{\, \boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle > 0 \,\}$. Then $\mathbf{E}^{1,n}$ is disjointly divided as follows: $\mathbf{E}^{1,n} = T^+ \sqcup T^- \sqcup L^+ \sqcup \{\boldsymbol{o}\} \sqcup L^- \sqcup S$, where $\boldsymbol{o}$ is the origin $(0,0,\ldots,0)$ of $\mathbf{E}^{1,n}$, and $\cdot \sqcup \cdot$ means the disjoint union of sets. We call T^+ the *future cone*, T^- the *past cone*, L the *light cone*, L^+ the *positive light cone*, L^- the *negative light cone*, and S the *side cone*. For any $\boldsymbol{x} \in \mathbf{E}^{1,n}$ with $\langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle \neq 0$, we denote by $n\,(\boldsymbol{x})$ its normalized vector, that is, $n\,(\boldsymbol{x}) := \frac{\boldsymbol{x}}{\sqrt{|\langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle|}}$.

Let $H_T^+ := \{\, \boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle = -1 \text{ and } x_0 > 0 \,\}$ be the upper sheet of the (standard) hyperboloid of two sheets. The restriction of the quadratic form induced by $\langle\, \cdot, \cdot\,\rangle$ on $\mathbf{E}^{1,n}$ to the tangent space of H_T^+ is positive definite and gives a Riemannian metric on H_T^+. The space obtained from H_T^+ equipped with the metric above is called the *hyperboloid model* of the n-dimensional hyperbolic space, and we denote it by $\mathbf{H}^n$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are points in H_T^+ and d denotes the hyperbolic distance between $\boldsymbol{x}$ and $\boldsymbol{y}$, then the following relation holds (see [Na, p. 45], [Ra, (3.2.2)] or [Th, Proposition 2.4.5(a)]):

$$\langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle = -\cosh d\,. \tag{2.1}$$

A ray in L^+ started from the origin $\boldsymbol{o}$ corresponds to a point in the ideal boundary of $\mathbf{H}^n$. The set of such rays forms the sphere at infinity, and we denote it by S_∞^{n-1}. Then each ray in L^+ becomes a point at infinity of $\mathbf{H}^n$. The (standard) *hyperboloid of one sheet* H_S is defined to be $H_S := \{\, \boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle = 1 \,\}$.

Let us denote by $\mathcal{P}$ the radial projection from $\mathbf{E}^{1,n} - \{\, \boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, x_0 = 0 \,\}$ to an affine hyperplane $\mathbf{P}_1^n := \{\, \boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, x_0 = 1 \,\}$ along the ray from the origin $\boldsymbol{o}$. The projection $\mathcal{P}$ is a homeomorphism on $\mathbf{H}^n$ to the n-dimensional open unit ball $\mathbf{B}^n$ in $\mathbf{P}_1^n$ centered at the origin $\boldsymbol{i} := (1,0,0,\ldots,0)$ of $\mathbf{P}_1^n$, which gives the *projective model* of $\mathbf{H}^n$. The affine hyperplane $\mathbf{P}_1^n$ contains not only $\mathbf{B}^n$ and its set theoretic boundary $\partial\mathbf{B}^n$ in $\mathbf{P}_1^n$, which is canonically identified with S_∞^{n-1}, but also the outside of the compactified projective model $\overline{\mathbf{B}^n} := \mathbf{B}^n \sqcup \partial\mathbf{B}^n \approx \mathbf{H}^n \sqcup S_\infty^{n-1}$. In this identification, the points near the intersection $S \cap \{\, \boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, x_0 = 0 \,\}$ are mapped to an end of $\mathbf{P}_1^n$. So we can naturally extend $\mathcal{P}$ to the mapping from $\mathbf{E}^{1,n} - \{\boldsymbol{o}\}$ to the n-dimensional real projective space $\mathbf{P}^n := \mathbf{P}_1^n \sqcup \mathbf{P}_\infty^n$, where $\mathbf{P}_\infty^n$ is the set of lines in the affine hyperplane $\{\, \boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, x_0 = 0 \,\}$ through $\boldsymbol{o}$. But we use the notation $\mathcal{P}$ for the mapping obtained as above to save letters since there would be no confusion. We denote by $\operatorname{Ext}\overline{\mathbf{B}^n}$ the exterior of $\overline{\mathbf{B}^n}$ in $\mathbf{P}^n$.

We call an affine hyperplane in $\mathbf{E}^{1,n}$ through the origin a *linear* hyperplane. A vector subspace of $\mathbf{E}^{1,n}$ is said to be *time-like* if it has a time-like vector, *space-like* if every nonzero vector in it is space-like, or *light-like* otherwise. Suppose P is a time-like linear hyperplane, and let R be a half-space in $\mathbf{E}^{1,n}$ bounded by P. Then we can associate a unique vector $\boldsymbol{w} \in H_S$ so that $\langle\, \boldsymbol{w}, \boldsymbol{q}\,\rangle \leq 0$ for any $\boldsymbol{q} \in R$. This establishes a well-known duality between points on H_S and half-spaces in $\mathbf{E}^{1,n}$ bounded by time-like linear hyperplanes. Now we give an generalization of this duality. For an arbitrary vector $\boldsymbol{u}$ in $\mathbf{E}^{1,n}$, we define a half-space $R_{\boldsymbol{u}}$ and a hyperplane $P_{\boldsymbol{u}}$ in $\mathbf{E}^{1,n}$ as follows:

$$\begin{aligned} R_{\boldsymbol{u}} &:= \left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \,\middle|\, \langle\, \boldsymbol{x}, \boldsymbol{u}\,\rangle \leq \frac{\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle - 1}{2} \right\}, \\ P_{\boldsymbol{u}} &:= \left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \,\middle|\, \langle\, \boldsymbol{x}, \boldsymbol{u}\,\rangle = \frac{\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle - 1}{2} \right\} = \partial R_{\boldsymbol{u}}\,. \end{aligned}$$

We denote by $\Gamma_{\boldsymbol{u}}$ (resp. $\Pi_{\boldsymbol{u}}$) the intersection of $R_{\boldsymbol{u}}$ (resp. $P_{\boldsymbol{u}}$) and H_T^+. We note that $\Pi_{\boldsymbol{u}} = \{\boldsymbol{u}\}$ when $\boldsymbol{u} \in H_T^+$. By the definition, a hyperplane $P_{\boldsymbol{u}}$ is linear if and only if $\boldsymbol{u} \in H_S$. Then $\Pi_{\boldsymbol{u}}$ is a geodesic hyperplane in $\mathbf{H}^n$. We call $\boldsymbol{u}$ ($\in H_S$) a *normal vector* to $P_{\boldsymbol{u}}$ (or $\Pi_{\boldsymbol{u}}$).

For two different geodesic hyperplanes in $\mathbf{H}^n$, the following theorem is a well-known one:

Theorem 2.1 (see [Ra, Theorem 3.2.6, 3.2.7, 3.2.9]) *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two points in H_S with $\boldsymbol{x} \neq \pm\boldsymbol{y}$, and we denote by N the vector subspace of $\mathbf{E}^{1,n}$ spanned by $\boldsymbol{x}$ and $\boldsymbol{y}$.*

(1) $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| < 1 \iff$ *N is space-like*
$\iff$ *$\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ intersect in H_T^+.*

(2) $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| > 1 \iff$ *N is time-like*
$\iff$ *$\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ are disjoint, and $N \cap H_T^+$ is a unique common orthogonal geodesic line to $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$.*

(3) $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| = 1 \iff$ *N is light-like*
$\iff$ *$P_{\boldsymbol{x}} \cap P_{\boldsymbol{y}}$ is light-like. So $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ meet at infinity.* □

For two geodesic hyperplanes $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ in $\mathbf{H}^n$ (so $\boldsymbol{x}$, $\boldsymbol{y} \in H_S$), we call $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ are *ultraparallel* if the condition of Theorem 2.1(2) holds, and *parallel* if the condition of Theorem 2.1(3) holds. Next we suppose $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ intersect, that is, the condition of Theorem 2.1(1) holds. Then we have the following relation (see [Th, Proposition 2.4.5(c)] and [SW, Lemma 2.7]):

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\cos\theta \,, \tag{2.2}$$

where θ is the dihedral angle between $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ which is measured in $\Gamma_{\boldsymbol{x}} \cap \Gamma_{\boldsymbol{y}}$. We note that this relation holds even if $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ are parallel. In this case we regard θ as 0. Now the following proposition is well-known:

Proposition 2.2 *Let $\boldsymbol{u}$ be a point in H_S, and let N be a 2-dimensional time-like vector subspace of $\mathbf{E}^{1,n}$ containing $\boldsymbol{u}$. Then the geodesic line $N \cap H_T^+$ and the geodesic hyperplane $\Pi_{\boldsymbol{u}}$ intersect orthogonally.*

Proof of Proposition 2.2. Let $\boldsymbol{a}$ be a time-like vector in N. Without loss of generality, we may assume $\boldsymbol{a} \in H_T^+$. Let $\boldsymbol{b}$ be the point in $\mathbf{E}^{1,n}$ defined as follows:

$$\boldsymbol{b} = \frac{1}{\sqrt{\langle \boldsymbol{a}, \boldsymbol{u} \rangle^2 + 1}}\, \boldsymbol{a} - \frac{\langle \boldsymbol{a}, \boldsymbol{u} \rangle}{\sqrt{\langle \boldsymbol{a}, \boldsymbol{u} \rangle^2 + 1}}\, \boldsymbol{u} \,.$$

Then we can easily show that $\boldsymbol{b} \in N \cap P_{\boldsymbol{u}} \cap H_T^+$, and it means that $N \cap H_T^+$ intersects $\Pi_{\boldsymbol{u}}$.

Now we show that $N \cap H_T^+$ and $\Pi_{\boldsymbol{u}}$ intersect orthogonally at $\boldsymbol{b}$. Let M be an arbitrary time-like linear hyperplane in $\mathbf{E}^{1,n}$ containing N, and let $\boldsymbol{m}$ be its normal vector. Since N contains $\boldsymbol{u}$, we have $\langle \boldsymbol{m}, \boldsymbol{u} \rangle = 0$. This equation means that, by (2.2), $M \cap H_T^+$ and $\Pi_{\boldsymbol{u}}$ intersect orthogonally, and so do $N \cap H_T^+$ and $\Pi_{\boldsymbol{u}}$. Thus we have proved Proposition 2.2. □

For a linear hyperplane and the light cone L, the following proposition is also well-known:

Proposition 2.3 *Fix a point $\boldsymbol{x}$ in H_S. Let $\boldsymbol{y}$ be a point in $P_{\boldsymbol{x}} \cap L - \{\boldsymbol{o}\}$. Then the vector subspace of $\mathbf{E}^{1,n}$ spanned by $\boldsymbol{x}$ and $\boldsymbol{y}$ is tangent to L.*

Proof of Proposition 2.3. Let N be the vector subspace of $\mathbf{E}^{1,n}$ spanned by two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\boldsymbol{z}$ a point in $N \cap L$. Then all we have to show is that $\boldsymbol{z} = a\,\boldsymbol{y}$ for some $a \in \mathbf{R}$.

Since $\boldsymbol{z}$ is contained in N, there exist two real numbers a and b such that $\boldsymbol{z} = a\,\boldsymbol{y} + b\,\boldsymbol{x}$. Since $\boldsymbol{z}$ is also contained in L, we obtain the following equation:

$$\langle\, \boldsymbol{z}, \boldsymbol{z}\,\rangle = a^2\ \langle\, \boldsymbol{y}, \boldsymbol{y}\,\rangle + 2\,a\,b\ \langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle + b^2\ \langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle\ = 0.$$

The assumption $\boldsymbol{y} \in P_{\boldsymbol{x}} \cap L$ shows $b = 0$. We have thus proved Proposition 2.3. □

For an arbitrary point $\boldsymbol{u}$ in H_S, $P_{\boldsymbol{u}} \cap \mathbf{P}^n$ becomes a hyperplane in $\mathbf{P}^n$, moreover $P_{\boldsymbol{u}}$ intersects $\mathbf{B}^n$. Since $\mathcal{P}\,(\boldsymbol{u})$ is a point in $\operatorname{Ext} \overline{\mathbf{B}^n}$, Proposition 2.3 shows that the cone consisting of lines through $\mathcal{P}\,(\boldsymbol{u})$ and a point in $P_{\boldsymbol{u}} \cap \partial\mathbf{B}^n$ is tangent to $\partial\mathbf{B}^n$. We call $P_{\boldsymbol{u}} \cap \mathbf{P}^n$ the *polar hyperplane* of $\mathcal{P}\,(\boldsymbol{u})$ in $\mathbf{P}^n$, and $\mathcal{P}\,(\boldsymbol{u})$ the *pole* of $P_{\boldsymbol{u}} \cap \mathbf{P}^n$ (see, for example, [Ke, p. 544]). For an arbitrary point $\boldsymbol{v}$ in $\operatorname{Ext} \overline{\mathbf{B}^n}$, we denote by $\Omega\,(\boldsymbol{v})$ its polar hyperplane and by $\Psi\,(\boldsymbol{v})$ the hyperplane in $\mathbf{B}^n$ with pole $\boldsymbol{v}$, i.e., $\Psi\,(\boldsymbol{v}) := \Omega\,(\boldsymbol{v}) \cap \mathbf{B}^n$. Using Proposition 2.2, we have the following well-known corollary:

Corollary 2.4 *Let $\boldsymbol{v}$ be a point in $\operatorname{Ext} \overline{\mathbf{B}^n}$. If a line in $\mathbf{P}^n$ through $\boldsymbol{v}$ intersects $\mathbf{B}^n$, then the line and the hyperplane $\Psi\,(\boldsymbol{v})$ intersect orthogonally (in the sense of hyperbolic geometry).* □

3 Signed distances and widths

In this section we define two kinds of values connected with the hyperbolic distance; one is called a *signed distance* and the other is called a *width*. Then we give a relationship between each of them and the Lorentzian inner product. We start this section with defining a signed distance.

3.1 Signed distances

Definition 3.1 Fix a point $\boldsymbol{x}$ in H_S. Let $\boldsymbol{y}$ be an arbitrary point in H_T^+. Then the *signed distance between $\Pi_{\boldsymbol{x}}$ and $\boldsymbol{y}$ (or equivalently $\Pi_{\boldsymbol{y}}$)* is defined to be the real number, say d, which satisfies the following two conditions:

(1) Its absolute value $|d|$ is equal to the hyperbolic distance between $\Pi_{\boldsymbol{x}}$ and $\boldsymbol{y}$ in the usual sense. Especially $d = 0$ if and only if $\boldsymbol{y} \in \Pi_{\boldsymbol{x}}$, that is, if and only if $\langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle = 0$.

(2) The sign of d is positive (resp. negative) if $\boldsymbol{y} \in \Gamma_{\boldsymbol{x}} - \Pi_{\boldsymbol{x}}$ (resp. $\boldsymbol{y} \notin \Gamma_{\boldsymbol{x}}$), that is, if $\langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle < 0$ (resp. $\langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle > 0$).

Lemma 3.2 *Let $\boldsymbol{x}$ (resp. $\boldsymbol{y}$) be a point in H_S (resp. H_T^+). Then the signed distance d between $\Pi_{\boldsymbol{x}}$ and $\boldsymbol{y}$ has the following relation to the inner product $\langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle$:*

$$\langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle = -\ \sinh d\,.$$

Proof of Lemma 3.2. Let N be the time-like vector subspace of $\mathbf{E}^{1,n}$ spanned by $\boldsymbol{x}$ and $\boldsymbol{y}$. Then the shortest geodesic between $\Pi_{\boldsymbol{x}}$ and $\boldsymbol{y}$ is the segment in $N \cap H_T^+$ joining $\boldsymbol{y}$ and $\boldsymbol{z}$, where $\boldsymbol{z}$ is the point defined by $N \cap P_{\boldsymbol{x}} \cap H_T^+$. As we saw in the proof of Proposition 2.2, $\boldsymbol{z}$ is expressed as a linear combination of $\boldsymbol{x}$ and $\boldsymbol{y}$ as follows:

$$\boldsymbol{z} = \frac{1}{\sqrt{\langle\, \boldsymbol{x}, \boldsymbol{y} \,\rangle^2 + 1}}\, \boldsymbol{y} - \frac{\langle\, \boldsymbol{x}, \boldsymbol{y} \,\rangle}{\sqrt{\langle\, \boldsymbol{x}, \boldsymbol{y} \,\rangle^2 + 1}}\, \boldsymbol{x}\,.$$

Using the relation (2.1) together with the fact that the hyperbolic cosine is an even function, we obtain $\langle\, \boldsymbol{x}, \boldsymbol{y} \,\rangle$ as follows:

$$\begin{aligned} -\cosh d &= \langle\, \boldsymbol{y}, \boldsymbol{z} \,\rangle \\ \Longleftrightarrow \cosh d &= \sqrt{\langle\, \boldsymbol{x}, \boldsymbol{y} \,\rangle^2 + 1} \\ \Longleftrightarrow \langle\, \boldsymbol{x}, \boldsymbol{y} \,\rangle &= \pm \sinh d\,. \end{aligned}$$

By the definition of the signed distance, the sign must be negative. This completes the proof. $\square$

Definition 3.3 Fix a point $\boldsymbol{x}$ in H_S. Let $\boldsymbol{y}$ be an arbitrary point either in $H_T^+ \sqcup (R_{\boldsymbol{x}} \cap L^+)$ or in $R_{\boldsymbol{x}} \cap H_S$ with $\langle\, \boldsymbol{x}, \boldsymbol{y} \,\rangle \leq -1$. We denote by N the vector subspace of $\mathbf{E}^{1,n}$ spanned by $\boldsymbol{x}$ and $\boldsymbol{y}$. Then, by Theorem 2.1 and Proposition 2.3, N is either time-like or light-like. Now the *signed distance d between $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$* is defined as follows:

Case 1. Suppose N is time-like. Then, by Theorem 2.1(2) and Proposition 2.3, $\boldsymbol{y}$ is either in $H_T^+ \sqcup (R_{\boldsymbol{x}} \cap L^+ - P_{\boldsymbol{x}})$ or in $R_{\boldsymbol{x}} \cap H_S$ with $\langle\, \boldsymbol{x}, \boldsymbol{y} \,\rangle < -1$. In this case, d is defined to be the signed distance between $\Pi_{\boldsymbol{x}}$ and $\boldsymbol{z}$, where $\boldsymbol{z}$ is a unique point defined by $N \cap P_{\boldsymbol{y}} \cap H_T^+$.

Case 2. Suppose N is light-like. Then $\boldsymbol{y}$ is either in $P_{\boldsymbol{x}} \cap L^+$ or $R_{\boldsymbol{x}} \cap H_S$ with $\langle\, \boldsymbol{x}, \boldsymbol{y} \,\rangle = -1$.

Case 2.1. If $\boldsymbol{y} \in P_{\boldsymbol{x}} \cap L^+$, then $d = -\infty$.

Case 2.2. If $\boldsymbol{y} \in R_{\boldsymbol{x}} \cap H_S$ with $\langle\, \boldsymbol{x}, \boldsymbol{y} \,\rangle = -1$, then $d = 0$.

Proposition 3.4 *Let $\boldsymbol{x}$ be a point in H_S. For an arbitrary point $\boldsymbol{y}$ in $H_T^+ \sqcup (R_{\boldsymbol{x}} \cap L^+)$ or in $R_{\boldsymbol{x}} \cap H_S$ with $\langle\, \boldsymbol{x}, \boldsymbol{y} \,\rangle \leq -1$, the following equation holds:*

$$\langle\, \boldsymbol{x}, \boldsymbol{y} \,\rangle = -\frac{e^d + \nu\, e^{-d}}{2}\,,$$

where $\nu := \langle\, \boldsymbol{y}, \boldsymbol{y} \,\rangle$, and d is the signed distance between $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$.

Proof of Proposition 3.4. We first suppose N, the vector subspace of $\mathbf{E}^{1,n}$ spanned by $\boldsymbol{x}$ and $\boldsymbol{y}$, is time-like. Then Lemma 3.2 shows $\langle\, \boldsymbol{x}, \boldsymbol{z} \,\rangle = -\sinh d$, where we recall that $\boldsymbol{z}$ is a unique point defined by $N \cap P_{\boldsymbol{y}} \cap H_T^+$. We here note that $\boldsymbol{x}$ and $\boldsymbol{z}$ also span N. Since $\boldsymbol{y}$ is contained in N, there exist two real numbers a and b such that $\boldsymbol{y} = a\,\boldsymbol{x} + b\,\boldsymbol{z}$.

Since $\boldsymbol{z}$ is contained in $P_{\boldsymbol{y}}$, we have

$$\langle\, \boldsymbol{z}, \boldsymbol{y} \,\rangle = \frac{\langle\, \boldsymbol{y}, \boldsymbol{y} \,\rangle - 1}{2} \Longleftrightarrow b = -\left(a \sinh d + \frac{\nu - 1}{2}\right)\,.$$

Then compute $\langle\, \boldsymbol{y}, \boldsymbol{y}\,\rangle = \nu$ and we obtain $a = \pm\,(\nu+1)\,/\,(2\cosh d)$.

If $\boldsymbol{y} \in H_T^+$, then $a = 0$. From now on we consider the case where $\boldsymbol{y}$ is in $R_{\boldsymbol{x}} \cap L^+ - P_{\boldsymbol{x}}$ or in $R_{\boldsymbol{x}} \cap H_S$ with $\langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle < -1$. Then we obtain $\langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle$ as follows:

$$\begin{aligned} \langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle &= a\,\langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle + b\,\langle\, \boldsymbol{x}, \boldsymbol{z}\,\rangle \\ &= \frac{\nu\, e^{d} + e^{-d}}{2}\,,\; -\frac{e^{d} + \nu\, e^{-d}}{2}\,. \end{aligned}$$

Since $\boldsymbol{y} \in R_{\boldsymbol{x}}$, $\langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle$ must be non-positive, that is, $\langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle$ must be $-(e^{d} + \nu\, e^{-d})/2$. Of course this consequence holds when $\boldsymbol{y} \in H_T^+$.

We next suppose N is light-like. If $\boldsymbol{y}$ is a point in $P_{\boldsymbol{x}} \cap L^+$, then $\langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle = 0 = -e^{-\infty}/2$, and if $\boldsymbol{y}$ is a point in $R_{\boldsymbol{x}} \cap H_S$ with $\langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle = -1$, then $-(e^{0} + 1\, e^{-0})/2 = -1$.

We have thus completed the proof of Proposition 3.4. $\square$

3.2 Widths

We next define the *width* of a point $\boldsymbol{u}$ in $T^+ \sqcup L^+ \sqcup S$, and observe its relationship to the Lorentzian norm.

We first consider the case where $\boldsymbol{u}$ is a time-like vector with positive height, that is, $\boldsymbol{u} \in T^+ = \{\, \boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle < 0 \text{ and } x_0 > 0\,\}$.

Lemma 3.5 *For any $\boldsymbol{u} \in T^+$, $\Pi_{\boldsymbol{u}}$ is a sphere centered at $n\,(\boldsymbol{u}) = \frac{\boldsymbol{u}}{\sqrt{-\langle \boldsymbol{u}, \boldsymbol{u}\rangle}}$.*

Proof of Lemma 3.5. Since $-\,\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle > 0$, we can rewrite the definition of $P_{\boldsymbol{u}}$ as follows:

$$P_{\boldsymbol{u}} = \left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \;\middle|\; \left\langle\, \boldsymbol{x}, \frac{\boldsymbol{u}}{\sqrt{-\,\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle}} \,\right\rangle = \frac{\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle - 1}{2\sqrt{-\,\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle}} \right\}.$$

Thus, by (2.1), all we have to show is that the following inequality holds for every $\boldsymbol{u} \in T^+$:

$$\frac{\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle - 1}{2\sqrt{-\,\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle}} < -1\,.$$

And it holds by the following identity:

$$-1 - \frac{\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle - 1}{2\sqrt{-\,\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle}} = \frac{\left(\sqrt{-\,\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle} - 1\right)^2}{2\sqrt{-\,\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle}}\,. \qquad \square$$

We here note that $\Pi_{\boldsymbol{u}}$ consists of only one point in H_T^+ if and only if $\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle = -1$, and then $\Pi_{\boldsymbol{u}} = \{\boldsymbol{u}\}$.

Definition 3.6 Let $\boldsymbol{u}$ be a point in T^+. Then a real number $\delta_{\boldsymbol{u}}$ is said to be the *width of $\boldsymbol{u}$* if the following two conditions hold:

(1) Its absolute value $|\delta_{\boldsymbol{u}}|$ is equal to the hyperbolic radius of the sphere $\Pi_{\boldsymbol{u}}$ in the usual sense. Especially $\delta_{\boldsymbol{u}} = 0$ if and only if $\boldsymbol{u} \in H_T^+$, that is, if and only if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = -1$.

(2) The sign of $\delta_{\boldsymbol{u}}$ is positive (resp. negative) if $-1 < \langle \boldsymbol{u}, \boldsymbol{u} \rangle < 0$ (resp. $\langle \boldsymbol{u}, \boldsymbol{u} \rangle < -1$).

Lemma 3.7 *For any $\boldsymbol{u} \in T^+$, we have the following relation between its width $\delta_{\boldsymbol{u}}$ and $\langle \boldsymbol{u}, \boldsymbol{u} \rangle$:*

$$- \langle \boldsymbol{u}, \boldsymbol{u} \rangle = e^{-2\delta_{\boldsymbol{u}}}.$$

Proof of Lemma 3.7. Using (2.1), we obtain $e^{\delta_{\boldsymbol{u}}}$ as follows:

$$\frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle - 1}{2\sqrt{- \langle \boldsymbol{u}, \boldsymbol{u} \rangle}} = -\cosh \delta_{\boldsymbol{u}} = -\frac{e^{\delta_{\boldsymbol{u}}} + e^{-\delta_{\boldsymbol{u}}}}{2}$$

$$\Longleftrightarrow e^{\delta_{\boldsymbol{u}}} = (- \langle \boldsymbol{u}, \boldsymbol{u} \rangle)^{1/2} \quad \text{or} \quad e^{\delta_{\boldsymbol{u}}} = (- \langle \boldsymbol{u}, \boldsymbol{u} \rangle)^{-1/2}.$$

The definition of the width $\delta_{\boldsymbol{u}}$ implies $e^{\delta_{\boldsymbol{u}}} = (- \langle \boldsymbol{u}, \boldsymbol{u} \rangle)^{-1/2}$, thereby completing the proof. □

Since the exponential function is injective, the following corollary holds:

Corollary 3.8 *For an arbitrary point $\boldsymbol{v}$ in $\mathbf{B}^n$ and an arbitrary real number t, there exists a unique vector $\boldsymbol{u}$ in T^+ such that $\mathcal{P}(\boldsymbol{u}) = \boldsymbol{v}$ and $\delta_{\boldsymbol{u}} = t$.* □

We secondly consider the case where $\boldsymbol{u}$ is a space-like vector, that is, $\boldsymbol{u} \in S = \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0 \}$. In this case we can rewrite the definition of $P_{\boldsymbol{u}}$ as follows:

$$P_{\boldsymbol{u}} = \left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \;\middle|\; \left\langle \boldsymbol{x}, \frac{\boldsymbol{u}}{\sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle}} \right\rangle = \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle - 1}{2\sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle}} \right\}.$$

Lemma 3.2 shows that $\Pi_{\boldsymbol{u}}$ is a non-empty set of points each of which is at a certain distance away from the geodesic hyperplane $\Pi_{n(\boldsymbol{u})}$. We call such a hypersurface $\Pi_{\boldsymbol{u}}$ an *equidistant hypersurface*, and $\Pi_{n(\boldsymbol{u})}$ the *axial hyperplane* of $\Pi_{\boldsymbol{u}}$ (cf. [Fe, p. 39]).

Definition 3.9 Let $\boldsymbol{u}$ be a point in S. Then a real number $\delta_{\boldsymbol{u}}$ is said to be the *width of* $\boldsymbol{u}$ if the following two conditions hold:

(1) Its absolute value $|\delta_{\boldsymbol{u}}|$ is equal to the hyperbolic distance between $\Pi_{\boldsymbol{u}}$ and its axial hyperplane $\Pi_{n(\boldsymbol{u})}$ in the usual sense. Especially $\delta_{\boldsymbol{u}} = 0$ if and only if $\boldsymbol{u} \in H_S$, that is, if and only if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 1$.

(2) The sign of $\delta_{\boldsymbol{u}}$ is positive (resp. negative) if $0 < \langle \boldsymbol{u}, \boldsymbol{u} \rangle < 1$ (resp. $1 < \langle \boldsymbol{u}, \boldsymbol{u} \rangle$).

Lemma 3.10 *For any $\boldsymbol{u} \in S$, we have the following relation between the width $\delta_{\boldsymbol{u}}$ and $\langle \boldsymbol{u}, \boldsymbol{u} \rangle$:*

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle = e^{-2\delta_{\boldsymbol{u}}}.$$

Proof of Lemma 3.10. By the definition of the width together with Lemma 3.2, we obtain $e^{\delta_{\boldsymbol{u}}}$ as follows:

$$\frac{\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle - 1}{2\sqrt{\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle}} = -\sinh \delta_{\boldsymbol{u}} = -\frac{e^{\delta_{\boldsymbol{u}}} - e^{-\delta_{\boldsymbol{u}}}}{2}$$
$$\Longleftrightarrow e^{\delta_{\boldsymbol{u}}} = -\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle^{1/2} \text{ or } e^{\delta_{\boldsymbol{u}}} = \langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle^{-1/2}.$$

Since the exponential function is always positive, $e^{\delta_{\boldsymbol{u}}}$ must be $\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle^{-1/2}$. This completes the proof. □

Since the exponential function is injective, the following corollary holds:

Corollary 3.11 *For an arbitrary half-space in $\mathbf{B}^n$ bounded by a geodesic hyperplane and an arbitrary real number, there exists a unique vector $\boldsymbol{u}$ in S such that $R_{n(\boldsymbol{u})} \cap \mathbf{B}^n$ coincides with the given half-space, and such that the width $\delta_{\boldsymbol{u}}$ is equal to the given real number.* □

Now we put together Lemma 3.7 and Lemma 3.10, and thus have the following proposition:

Proposition 3.12 *Suppose $\boldsymbol{u} \in T^+ \sqcup S$. Then the sign of its width $\delta_{\boldsymbol{u}}$ is positive if and only if $0 < |\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle| \leq 1$ and negative if and only if $|\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle| > 1$. Furthermore we have the following relation between $\delta_{\boldsymbol{u}}$ and $|\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle|$:*

$$\delta_{\boldsymbol{u}} = -\frac{1}{2}\log|\langle\, \boldsymbol{u}, \boldsymbol{u}\,\rangle|\,.$$ □

We finally consider the case where $\boldsymbol{u}$ is a light-like vector with positive height, that is, $\boldsymbol{u} \in L^+ = \{\, \boldsymbol{x} \in \mathbf{E}^{1,n} \mid \langle\, \boldsymbol{x}, \boldsymbol{x}\,\rangle = 0 \text{ and } x_0 > 0\,\}$. In this case we can rewrite the definition of $P_{\boldsymbol{u}}$ as follows:

$$P_{\boldsymbol{u}} = \left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \,\middle|\, \langle\, \boldsymbol{x}, \boldsymbol{u}\,\rangle = -\frac{1}{2} \right\}.$$

The set $\Pi_{\boldsymbol{u}}$ is called a *horosphere* whose *center* is the ray through $\boldsymbol{u}$. Now the following lemma is a well-known one.

Lemma 3.13 *Let $\boldsymbol{u}$ be a point in L^+.*

(1) Let $\boldsymbol{p}$ be an arbitrary point in T^+. Then the The Lorentzian inner product $\langle\, \boldsymbol{u}, \boldsymbol{p}\,\rangle$ is negative.

(2) The horosphere $\Pi_{\boldsymbol{u}}$ intersects orthogonally any geodesic line one of whose point at infinity is the ray through $\boldsymbol{u}$.

Proof of Lemma 3.13. Since $\boldsymbol{p} = (p_0, p_1, \ldots, p_n) \in T^+$, we have $p_0 > \sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}$. And since $\boldsymbol{u} = (u_0, u_1, \ldots, u_n) \in L^+$, we have $u_0 = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$. So we can compute $\langle\, \boldsymbol{u}, \boldsymbol{p} \,\rangle$ as follows:

$$\begin{aligned} \langle\, \boldsymbol{u}, \boldsymbol{p} \,\rangle &= -\, u_0\, p_0 + u_1\, p_1 + u_2\, p_2 + \cdots + u_n\, p_n \\ &< u_1\, p_1 + u_2\, p_2 + \cdots + u_n\, p_n - \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}\, \sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}\,. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain $\langle\, \boldsymbol{u}, \boldsymbol{p} \,\rangle < 0$, thereby proving part (1).

To prove part (2), fix a point, say $\boldsymbol{a}$, in the geodesic line in question. Then the line is obtained by $N \cap H_T^+$, where N is the 2-dimensional vector subspace of $\mathbf{E}^{1,n}$ spanned by $\boldsymbol{a}$ and $\boldsymbol{u}$. Let $\boldsymbol{b}$ be the point in $\mathbf{E}^{1,n}$ defined as follows:

$$\boldsymbol{b} = \left(1 - \frac{1}{4\,\langle\, \boldsymbol{u}, \boldsymbol{a} \,\rangle^2}\right) \boldsymbol{u} - \frac{1}{2\,\langle\, \boldsymbol{u}, \boldsymbol{a} \,\rangle}\, \boldsymbol{a}\,.$$

We note that the definition is well-defined since, by part (1), $\langle\, \boldsymbol{u}, \boldsymbol{a} \,\rangle \neq 0$. Now we can easily show that $\boldsymbol{b} \in N \cap P_{\boldsymbol{u}} \cap H_T^+$, and it means that $\Pi_{\boldsymbol{u}}$ intersects $N \cap H_T^+$.

From now on, we prove that $N \cap H_T^+$ and $\Pi_{\boldsymbol{u}}$ intersects orthogonally at $\boldsymbol{b}$. Let M be a time-like linear hyperplane through $\boldsymbol{b}$ with satisfying that its normal vector $\boldsymbol{m}$ is in $N \cap H_S$. By this definition together with Proposition 2.2, $M \cap H_T^+$ is orthogonal to $N \cap H_T^+$. Therefore, if $M \cap H_T^+$ is tangent to $\Pi_{\boldsymbol{u}}$, then $\Pi_{\boldsymbol{u}}$ must be orthogonal to $N \cap H_T^+$. After this we show that $M \cap H_T^+$ is tangent to $\Pi_{\boldsymbol{u}}$.

Since N is also spanned by $\boldsymbol{u}$ and $\boldsymbol{b}$, we can express $\boldsymbol{m}$ as a linear combination of $\boldsymbol{u}$ and $\boldsymbol{b}$ as follows: $\boldsymbol{m} = -\,2\,\boldsymbol{u} + \boldsymbol{b}$. We denote by $\boldsymbol{b}'$ a point obtained from $M \cap P_{\boldsymbol{u}} \cap H_T^+$. Since $\boldsymbol{b}'$ is contained in $P_{\boldsymbol{u}}$, we have $\langle\, \boldsymbol{b}', \boldsymbol{u} \,\rangle = -1/2$. Since $\boldsymbol{b}'$ is also contained in M, we have $\langle\, \boldsymbol{b}', \boldsymbol{m} \,\rangle = -\,2\,\langle\, \boldsymbol{b}', \boldsymbol{u} \,\rangle + \langle\, \boldsymbol{b}', \boldsymbol{b} \,\rangle = 1 + \langle\, \boldsymbol{b}', \boldsymbol{b} \,\rangle = 0$. Thus we obtain $\langle\, \boldsymbol{b}', \boldsymbol{b} \,\rangle = -\,1$. The relation (2.1) shows that $\boldsymbol{b}' = \boldsymbol{b}$, and it means that $M \cap H_T^+$ is tangent to $\Pi_{\boldsymbol{u}}$ at $\boldsymbol{b}$. Thus we have proved part (2). □

Definition 3.14 Let $\boldsymbol{u}$ be a point in L^+. Then the *width* $\delta_{\boldsymbol{u}}$ *of* $\boldsymbol{u}$ is defined as follows:

$$\delta_{\boldsymbol{u}} := -\,\frac{1}{2}\,\log\,(\,\boldsymbol{u}, \boldsymbol{u}\,)\ ,$$

where $(\,\cdot, \cdot\,)$ means the Euclidean inner product, that is, $(\,\boldsymbol{u}, \boldsymbol{u}\,) := u_0^2 + u_1^2 + \cdots + u_n^2$ if $\boldsymbol{u} = (u_0, u_1, \ldots, u_n)$.

One reason why this definition is fit for us is given by the following proposition:

Proposition 3.15 (cf. [Na, Theorem 2_3 and 3]) *Let $\boldsymbol{u}_1$ be a point in L^+. For an arbitrary $k \geq 1$, we denote by $\boldsymbol{u}_2$ the vector $k\,\boldsymbol{u}_1$. Let N denote an arbitrary 2-dimensional time-like vector subspace containing $\boldsymbol{u}_1$.*

(1) The set $R_{\boldsymbol{u}_1}$ is contained in $R_{\boldsymbol{u}_2}$.

(2) Let $\boldsymbol{b}_1$ (resp. $\boldsymbol{b}_2$) be the point in $N \cap P_{\boldsymbol{u}_1} \cap H_T^+$ (resp. $N \cap P_{\boldsymbol{u}_2} \cap H_T^+$), and let d be the hyperbolic distance between $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$. Then d is independent with the choice of N, and is equal to $\delta_{\boldsymbol{u}_1} - \delta_{\boldsymbol{u}_2}$.

Proof of Proposition 3.15. Let $\boldsymbol{x}$ be a point in $R_{\boldsymbol{u}_1}$. Then it satisfies that $\langle \boldsymbol{x}, \boldsymbol{u}_1 \rangle \leq -1/2$. Now we have $\langle \boldsymbol{x}, \boldsymbol{u}_2 \rangle = k \langle \boldsymbol{x}, \boldsymbol{u}_1 \rangle$ and, since $k \geq 1$, we also have $\langle \boldsymbol{x}, \boldsymbol{u}_2 \rangle \leq -k/2 \leq -1/2$. Thus we obtain $\boldsymbol{x} \in R_{\boldsymbol{u}_2}$, thereby proving part (1).

By the definition of the width, we can easily obtain $\delta_{\boldsymbol{u}_1} - \delta_{\boldsymbol{u}_2} = \log k$. So all we have to show is that $d = \log k$.

Using the computation in the proof of Lemma 3.13, we can express $\boldsymbol{b}_2$ as a linear combination of $\boldsymbol{u}_1$ and $\boldsymbol{b}_1$ as follows:

$$\begin{aligned} \boldsymbol{b}_2 &= \left(1 - \frac{1}{4 \langle \boldsymbol{u}_2, \boldsymbol{b}_1 \rangle^2}\right) \boldsymbol{u}_2 - \frac{1}{2 \langle \boldsymbol{u}_2, \boldsymbol{b}_1 \rangle} \boldsymbol{b}_1 \\ &= \left(k - \frac{1}{k}\right) \boldsymbol{u}_1 + \frac{1}{k} \boldsymbol{b}_1 \,. \end{aligned}$$

Using (2.1), we obtain e^d as follows:

$$\langle \boldsymbol{b}_1, \boldsymbol{b}_2 \rangle = -\frac{e^d + e^{-d}}{2} \iff e^d = k \,,\ \frac{1}{k} \,.$$

Since $d \geq 0$ and $k \geq 1$, e^d must be k, that is, $d = \log k$.

We have thus completed the proof of Proposition 3.15. □

Since the logarithmic function is injective, we have the following corollary, which is the one correspondent with Corollary 3.8 and Corollary 3.11:

Corollary 3.16 *For an arbitrary point $\boldsymbol{v}$ in $\partial \mathbf{B}^n$ and an arbitrary real number t, there exists a unique vector $\boldsymbol{u}$ in L^+ such that $\mathcal{P}(\boldsymbol{u}) = \boldsymbol{v}$ and $\delta_{\boldsymbol{u}} = t$.* □

4 Definition of a weighted n-simplex

The projective model $\mathbf{B}^n$ has the advantage that it enable us to describe polyhedra in $\mathbf{H}^n$ in terms of Euclidean terminology. For example, we can regard an ideal polyhedron in $\mathbf{H}^n$ as an Euclidean polyhedron in $\mathbf{P}^n_1$ whose vertices lie in $\partial \mathbf{B}^n$. We start this section with defining a *generalized n-simplex* in $\mathbf{B}^n$.

Let $V = \{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a set of independent points in $\mathbf{P}^n$, and let $V_{\text{in}} := \left\{\boldsymbol{v} \in V \,\middle|\, \boldsymbol{v} \in \overline{\mathbf{B}^n}\right\}$ and $V_{\text{ex}} := \left\{\boldsymbol{v} \in V \,\middle|\, \boldsymbol{v} \in \operatorname{Ext} \overline{\mathbf{B}^n}\right\} = V - V_{\text{in}}$. Without loss of generality, we may assume $V_{\text{ex}} = \{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ and $V_{\text{in}} = \{\boldsymbol{v}_{k+1}, \boldsymbol{v}_{k+2}, \ldots, \boldsymbol{v}_n\}$ for some $k \in \{-1, 0, 1, \ldots, n\}$, by changing indices if necessary. This notation means that $V_{\text{ex}} = \emptyset$ and $V_{\text{in}} = V$ when $k = -1$, and that $V_{\text{ex}} = V$ and $V_{\text{in}} = \emptyset$ when $k = n$. Now we suppose V satisfies the following two conditions:

Condition 1. If V_{ex} has more than one point, then for arbitrary different points $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ in V_{ex} hyperplanes $\Psi(\boldsymbol{v}_i)$ and $\Psi(\boldsymbol{v}_j)$ with poles $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ respectively are parallel or ultraparallel, that is, they do not intersect in $\mathbf{B}^n$.

Condition 2. The set V_{in} is wholly contained in one connected component of $\overline{\mathbf{B}^n} - \bigcup_{i=0}^k \Omega(\boldsymbol{v}_i)$.

We note that, when $k = -1$, Condition 2 means that $V \subset \overline{\mathbf{B}^n}$. We also note that Condition 1 is equivalent to the following one:

Condition 1′. If V_{ex} has more than one point, then for arbitrary different points $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ in V_{ex} the intersection of $\overline{\mathbf{B}^n}$ and the line through $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ is not an empty set.

For each point $\boldsymbol{v}_i$ in V_{ex}, there is a unique point $\boldsymbol{v}'_i$ in H_S such that $\mathcal{P}(\boldsymbol{v}'_i) = \boldsymbol{v}_i$ and $V_{\mathrm{in}} \subset R_{\boldsymbol{v}'_i}$. Let $|\boldsymbol{v}'_0\boldsymbol{v}'_1\cdots\boldsymbol{v}'_k\boldsymbol{v}_{k+1}\boldsymbol{v}_{k+2}\cdots\boldsymbol{v}_n|$ be the affine simplex in $\mathbf{E}^{1,n}$ with vertex set $\{\boldsymbol{v}'_0, \boldsymbol{v}'_1, \ldots, \boldsymbol{v}'_k, \boldsymbol{v}_{k+1}, \boldsymbol{v}_{k+2}, \ldots, \boldsymbol{v}_n\}$. Since the points in V are independent in $\mathbf{P}^n$, vectors $\{\boldsymbol{v}'_0, \boldsymbol{v}'_1, \ldots, \boldsymbol{v}'_k, \boldsymbol{v}_{k+1}, \boldsymbol{v}_{k+2}, \ldots, \boldsymbol{v}_n\}$ are linearly independent in $\mathbf{E}^{1,n}$, namely the hyperplane through $n+1$-points $\boldsymbol{v}'_0, \boldsymbol{v}'_1, \ldots, \boldsymbol{v}'_k, \boldsymbol{v}_{k+1}, \boldsymbol{v}_{k+2}, \ldots, \boldsymbol{v}_n$ does not contain the origin $\boldsymbol{o}$. Thus we can define $\mathcal{P}(|\boldsymbol{v}'_0\boldsymbol{v}'_1\cdots\boldsymbol{v}'_k\boldsymbol{v}_{k+1}\boldsymbol{v}_{k+2}\cdots\boldsymbol{v}_n|)$, an n-simplex in $\mathbf{P}^n$ with vertex set V, and denote it by $|\boldsymbol{v}_0\boldsymbol{v}_1\cdots\boldsymbol{v}_n|$. We note that, if $V_{\mathrm{ex}} = \emptyset$, $|\boldsymbol{v}_0\boldsymbol{v}_1\cdots\boldsymbol{v}_n|$ is just the n-dimensional affine simplex in $\mathbf{P}^n_1 \approx \mathbf{R}^n$ with vertex set V.

Definition 4.1 Under the assumptions stated above, the *generalized n-simplex Δ_V in $\mathbf{B}^n$ with vertex set V* is defined as follows:

$$\Delta_V := \begin{cases} \mathbf{B}^n \cap |\boldsymbol{v}_0\boldsymbol{v}_1\cdots\boldsymbol{v}_n| & \text{if } \ V \subset \overline{\mathbf{B}^n}\,, \\ \mathbf{B}^n \cap |\boldsymbol{v}_0\boldsymbol{v}_1\cdots\boldsymbol{v}_n| \cap \displaystyle\bigcap_{i=0}^{k} R_{\boldsymbol{v}'_i} & \text{if } \ V \cap \operatorname{Ext}\overline{\mathbf{B}^n} \neq \emptyset \quad \text{(see Figure 1).} \end{cases}$$

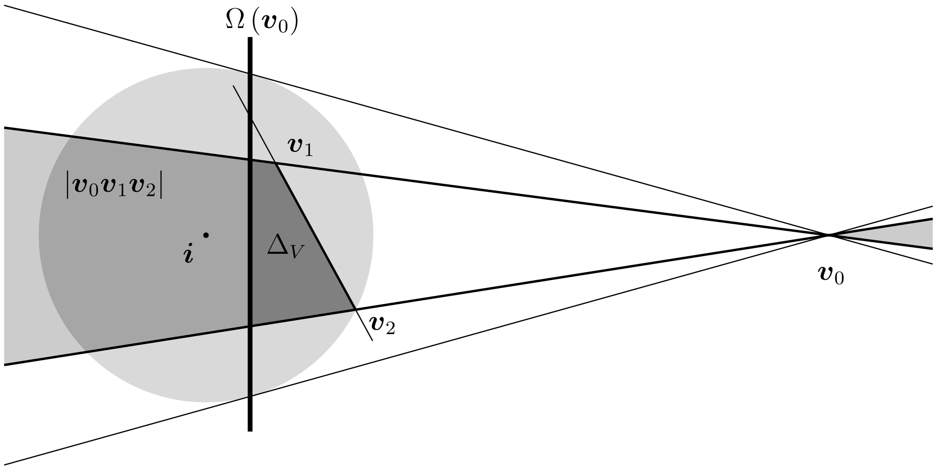

Figure 1: An example of a generalized 2-simplex in $\mathbf{B}^2$

Each face of Δ_V is either contained in a face of $|\boldsymbol{v}_0\boldsymbol{v}_1\cdots\boldsymbol{v}_n|$ or in $\Psi(\boldsymbol{v}_i)$ for some $\boldsymbol{v}_i \in V_{\mathrm{ex}}$. We call the former an *internal face* of Δ_V, and the later an *external face* of Δ_V (cf. [Ko1, Ko2]). For each vertex $\boldsymbol{v}_i$ of Δ_V, we denote by $\mathcal{F}_i$ the hyperplane in $\mathbf{P}^n$ through n points $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_n\}$. If an internal face of Δ_V coincides with $\mathcal{F}_i \cap \Delta_V$ for some $\boldsymbol{v}_i \in V$, then we call the face the *opposite face* of $\boldsymbol{v}_i$, and denote it by Φ_i. By the definitions of the notation, we have an injective correspondence from the internal faces of Δ_V to the vertex set. We here note that this correspondence may not be surjective (see Figure 2). We may use the symbol of opposite faces to denote internal faces without referring to vertices. Let Φ_i and Φ_j be internal faces, and $\mathcal{F}_i$ and $\mathcal{F}_j$ their corresponding geodesic hyperplanes in the previous sense. Then we say that Φ_i and Φ_j (with $i \neq j$) are *parallel* (resp. *ultraparallel*, *intersecting*) if $\mathcal{P}^{-1}(\mathcal{F}_i) \cap H^+_T$ and $\mathcal{P}^{-1}(\mathcal{F}_j) \cap H^+_T$ are parallel (resp. ultraparallel, intersecting) (cf. Theorem 2.1). The *dihedral angle between* Φ_i *and* Φ_j is defined to be the dihedral angle between $\mathcal{P}^{-1}(\mathcal{F}_i) \cap H_T^+$ and $\mathcal{P}^{-1}(\mathcal{F}_j) \cap H_T^+$ measured in $\mathcal{P}^{-1}(\Delta_V) \cap H_T^+$. By Condition 1, we can see that each connected component of external faces is totally geodesic. We also note that a vertex of Δ_V as a polyhedron in hyperbolic space is not a "vertex" of the generalized n-simplex Δ_V if it is made from the intersection of an external face and an edge of $|\boldsymbol{v}_0 \boldsymbol{v}_1 \cdots \boldsymbol{v}_n|$ (see Definition 4.1).

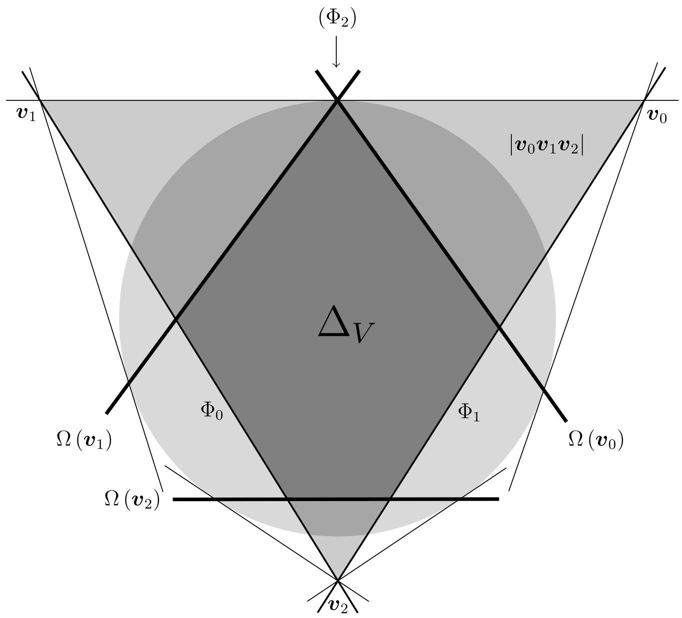

Figure 2: A generalized 2-simplex with one degenerate internal face

We next define a *weighted n-simplex*. We recall that Δ_V is a generalized n-simplex with vertex set V. At each vertex, we give a real number called *weight*. Let W be the set of weights of all vertices.

Definition 4.2 Under the assumptions stated above, we call a triplet (Δ_V, V, W) a *weighted n-simplex in* $\mathbf{B}^n$.

Corollary 3.8, 3.11 and 3.16 imply the following proposition:

Proposition 4.3 (lift proposition) *For a weighted n-simplex (Δ_V, V, W) in the projective model $\mathbf{B}^n$, there exists a unique affine n-simplex $\widehat{\Delta_V}$ in $\mathbf{E}^{1,n} - \{\boldsymbol{o}\}$ with vertex set $\widehat{V}$ satisfying the following four conditions:*

(1) $\widehat{V} \subset T^+ \sqcup L^+ \sqcup S$;

(2) $\mathcal{P}\left(\widehat{V}\right) = V$;

(3) For any $\boldsymbol{u} \in \widehat{V} \cap S$, *we have* $\Delta_V \subset R_{n(\boldsymbol{u})} \cap \mathbf{B}^n$;

(4) For any $\boldsymbol{u} \in \widehat{V}$, *the width* $\delta_{\boldsymbol{u}}$ *is equal to the weight of* $\mathcal{P}(\boldsymbol{u})$. □

We call $\widehat{\Delta_V}$ the *lift* of the weighted n-simplex (Δ_V, V, W) in $\mathbf{B}^n$, $\widehat{V}$ the *lift* of the vertex set V, and $\boldsymbol{u}$ the lift of the vertex $\mathcal{P}(\boldsymbol{u}) \in V$. We here note that condition (2) means $\widehat{V}$ is a set of linearly independent vectors in $\mathbf{E}^{1,n}$. We also note that $\mathcal{P}\left(\widehat{\Delta_V}\right)$ does not always coincide with Δ_V, though $\mathcal{P}\left(\widehat{V}\right) = V$.

5 Definition of a tilt

D. B. A. Epstein and R. C. Penner gave in [EP] a method for decomposing any noncompact complete hyperbolic n-manifold of finite volume with weight at each cusp into ideal polyhedra. This decomposition is called the *Euclidean decomposition* and defined via a *convex hull construction* in Lorentzian space. Especially if all weights are equal, then the decomposition is invariant under the action of the fundamental group of the manifold. In this case it is called the *canonical decomposition*. S. Kojima gave in [Ko1, Ko2] a method for decomposing any complete hyperbolic manifold of finite volume with non-empty totally geodesic boundary into partially truncated polyhedra. This decomposition is also called the *canonical decomposition*, and defined via a *convex hull construction* in Lorentzian space. We give a brief review of the canonical decomposition of a compact hyperbolic n-manifold, say M, with non-empty totally geodesic boundary.

We regard the universal cover $\widetilde{M}$ of M as a subset of the hyperboloid model $\mathbf{H}^n$. To each component of $\partial\widetilde{M}$, assign a label. To each component of $\partial\widetilde{M}$ labeled by α, there exists a unique space-like vector $\boldsymbol{v}_\alpha \in H_S$ such that $P_{\boldsymbol{v}_\alpha}$ contains the boundary component, and such that $R_{\boldsymbol{v}_\alpha}$ contains $\widetilde{M}$. Let $\mathcal{A}$ be the set of dual vectors $\{\boldsymbol{v}_\alpha\}$ on H_S. Then $\mathcal{A}$ is invariant under the action of the covering transformation group. Let $\mathcal{H}_\mathcal{A}$ be the closed convex hull of $\mathcal{A}$ in $\mathbf{E}^{1,n}$. The projection $\mathcal{P}(\partial\mathcal{H}_\mathcal{A})$ contains $\mathbf{B}^n$ (see [Ko1, Lemma 4.3]), and the intersection of $\mathcal{P}(\partial\mathcal{H}_\mathcal{A})$ with $\mathcal{P}(\widetilde{M})$ in $\mathbf{B}^n$ defines a $\pi_1(M)$-equivalent polyhedral decomposition on $\widetilde{M}$. It induces a truncated polyhedral decomposition of M (see [Ko1, Theorem 4.8]), which is the canonical decomposition of M.

In many cases the canonical decomposition for a compact hyperbolic n-manifold with non-empty totally geodesic boundary consists of truncated n-simplices (see, for example, [Us1, Theorem 2.2]). Furthermore if the manifold has cusps, then the simplices have several ideal vertices instead of external faces. Namely they are "partially truncated ideal n-simplices." In the previous section we defined the weighted n-simplex, a unity of such simplices and ideal n-simplices with weights. So when two weighted n-simplices are adjacent to each other along a face, it is meaningful to provide an efficient tool to decide whether or not their lifts form a convex dihedral angle.

R. C. Penner gave in [Pe, Proposition 2.6(b)] a criterion of convexity when simplices are (2-dimensional) ideal triangles. J. R. Weeks independently gave in [We1, Proposition 3.1] a criterion of convexity when simplices are 2 and 3-dimensional ideal simplices. This criterion is expressed by using "tilts," and allow him to make the hyperbolic structures computation program "SnapPea" (cf. [We2]). He also provided an efficient formula, called the *tilt formula*, to obtain tilts from the intrinsic hyperbolic geometry of the simplex when its dimension is two (see [We1, Theorem 3.2]) and three (see [We1, Theorem 5.1]). M. Sakuma and J. R. Weeks generalized the tilt formula to general dimensions in [SW]. The idea of R. C. Penner is translated by M. Näätänen in [Nä, Lemma 3.3] into the case where simplices are triangles, and by the author in [Us2, Proposition 3.5(2)] into the case where simplices are truncated triangles (i.e., orthogonal hexagons). In this section, using

Weeks' method, we will obtain a criterion of convexity when two weighted n-simplices in $\mathbf{B}^n$ are "adjacent along faces." So we start this section with the definition of the tilt of a weighted n-simplex in $\mathbf{B}^n$ relative to an internal face.

Fix a weighted n-simplex (Δ_V, V, W) in $\mathbf{B}^n$, and take an internal face Φ_i of Δ_V. Then there is a unique point $\boldsymbol{m}_i$ in H_S such that $\Phi_i \subset P_{\boldsymbol{m}_i} \cap \mathbf{B}^n$ and $\Delta_V \subset R_{\boldsymbol{m}_i} \cap \mathbf{B}^n$. We define the *normal* vector $\boldsymbol{p}$ to the lift $\widehat{\Delta_V}$ of (Δ_V, V, W) by the condition that $\langle\, \boldsymbol{p}, \boldsymbol{x} \,\rangle = -1$ for all $\boldsymbol{x} \in \widehat{\Delta_V}$.

Definition 5.1 Under the assumptions stated above, the *tilt* t_i *of* (Δ_V, V, W) *relative to* Φ_i is defined as follows:

$$t_i := \langle\, \boldsymbol{m}_i, \boldsymbol{p} \,\rangle \ .$$

Let (Δ_{V_0}, V_0, W_0) and (Δ_{V_1}, V_1, W_1) be two weighted n-simplices in $\mathbf{B}^n$, and let Φ_0 (resp. Φ_1) be an internal face of (Δ_{V_0}, V_0, W_0) (resp. (Δ_{V_1}, V_1, W_1)). Then we say that (Δ_{V_0}, V_0, W_0) and (Δ_{V_1}, V_1, W_1) are *adjacent along* Φ_0 and Φ_1 if $\widehat{\Delta_{V_0}} \cap \widehat{\Delta_{V_1}} = \widehat{\Phi_0} = \widehat{\Phi_1}$, where $\widehat{\Phi_0}$ (resp. $\widehat{\Phi_1}$) is the lift of Φ_0 (resp. Φ_1) in $\widehat{\Delta_{V_0}}$ (resp. $\widehat{\Delta_{V_1}}$). Now we call Φ_0 and Φ_1 *joint faces*. For convenience we additionally assume that $V_0 = \{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$, $V_1 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n, \boldsymbol{v}_{n+1}\}$, and that the joint faces are opposite faces of $\boldsymbol{v}_0$ and $\boldsymbol{v}_{n+1}$. We denote by t_0 (resp. t_1) the tilt of (Δ_{V_0}, V_0, W_0) (resp. (Δ_{V_1}, V_1, W_1)) relative to Φ_0 (resp. Φ_1). Then the following proposition correspondent with [We1, Proposition 3.1] holds.

Proposition 5.2 (tilt proposition) *Under the assumptions stated above, the dihedral angle formed by* $\widehat{\Delta_{V_0}}$ *and* $\widehat{\Delta_{V_1}}$ *is convex (flat, concave respectively) in* $\mathbf{E}^{1,n}$ *if and only if* $t_0 + t_1 < 0$ *(*$= 0$*,* > 0 *respectively).*

Proof of Proposition 5.2. Let $\boldsymbol{u}_i$ be the lift of $\boldsymbol{v}_i$, where $i \in \{0, 1, \ldots, n+1\}$. So the lift $\widehat{V_0}$ of V_0 is $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$, and the lift $\widehat{V_1}$ of V_1 is $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{n+1}\}$. We denote by $\boldsymbol{m}$ the orthogonal vector to the hyperplane containing joint faces of Δ_{V_0} and Δ_{V_1} with satisfying that $\widehat{\Delta_{V_0}} \subset R_{\boldsymbol{m}}$. Then, for an arbitrary $i \in \{1, 2, \ldots, n\}$, we have $\langle\, \boldsymbol{u}_i, \boldsymbol{m} \,\rangle = 0$. Furthermore we have $\langle\, \boldsymbol{u}_0, \boldsymbol{m} \,\rangle < 0$ and $\langle\, \boldsymbol{u}_{n+1}, \boldsymbol{m} \,\rangle > 0$.

Since vectors $\boldsymbol{m}, \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ form a basis of $\mathbf{E}^{1,n}$, there exist unique real numbers $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that

$$\boldsymbol{u}_0 = \alpha_0\, \boldsymbol{m} + \alpha_1\, \boldsymbol{u}_1 + \alpha_2\, \boldsymbol{u}_2 + \cdots + \alpha_n\, \boldsymbol{u}_n \,.$$

Similarly we have

$$\boldsymbol{u}_{n+1} = \alpha'_0\, \boldsymbol{m} + \alpha'_1\, \boldsymbol{u}_1 + \alpha'_2\, \boldsymbol{u}_2 + \cdots + \alpha'_n\, \boldsymbol{u}_n$$

for some $\alpha'_0, \alpha'_1, \ldots, \alpha'_n \in \mathbf{R}$. We note that $\alpha'_0 > 0$, since $\langle\, \boldsymbol{u}_{n+1}, \boldsymbol{m} \,\rangle > 0$.

Let $\boldsymbol{p}_0$ (resp. $\boldsymbol{p}_1$) be the normal vector to $\widehat{\Delta_{V_0}}$ (resp. $\widehat{\Delta_{V_1}}$). Now $\boldsymbol{p}_0$ is also expressed as a linear combination of $\boldsymbol{m}, \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ as follows:

$$\boldsymbol{p}_0 = \beta_0\, \boldsymbol{m} + \beta_1\, \boldsymbol{u}_1 + \beta_2\, \boldsymbol{u}_2 + \cdots + \beta_n\, \boldsymbol{u}_n \,,$$

where $\beta_0, \beta_1, \ldots, \beta_n \in \mathbf{R}$. Then we have the following:

$$\begin{pmatrix} \langle\, \boldsymbol{p}_0, \boldsymbol{u}_1 \,\rangle \\ \langle\, \boldsymbol{p}_0, \boldsymbol{u}_2 \,\rangle \\ \vdots \\ \langle\, \boldsymbol{p}_0, \boldsymbol{u}_n \,\rangle \end{pmatrix} = \begin{pmatrix} \langle\, \boldsymbol{u}_1, \boldsymbol{u}_1 \,\rangle & \langle\, \boldsymbol{u}_1, \boldsymbol{u}_2 \,\rangle & \cdots & \langle\, \boldsymbol{u}_1, \boldsymbol{u}_n \,\rangle \\ \langle\, \boldsymbol{u}_2, \boldsymbol{u}_1 \,\rangle & \langle\, \boldsymbol{u}_2, \boldsymbol{u}_2 \,\rangle & \cdots & \langle\, \boldsymbol{u}_2, \boldsymbol{u}_n \,\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\, \boldsymbol{u}_n, \boldsymbol{u}_1 \,\rangle & \langle\, \boldsymbol{u}_n, \boldsymbol{u}_2 \,\rangle & \cdots & \langle\, \boldsymbol{u}_n, \boldsymbol{u}_n \,\rangle \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix},$$

that is,

$$\sum_{i=1}^{n} \beta_i \, \langle \boldsymbol{u}_j, \boldsymbol{u}_i \rangle = -1 \quad \text{for} \quad j \in \{1, 2, \ldots, n\} \,. \tag{5.1}$$

Similarly we have

$$\boldsymbol{p}_1 = \beta'_0 \, \boldsymbol{m} + \beta'_1 \, \boldsymbol{u}_1 + \beta'_2 \, \boldsymbol{u}_2 + \cdots + \beta'_n \, \boldsymbol{u}_n$$

for some $\beta'_0, \beta'_1, \ldots, \beta'_n \in \mathbf{R}$. Then, in the same fashion as (5.1), we have

$$\sum_{i=1}^{n} \beta'_i \, \langle \boldsymbol{u}_j, \boldsymbol{u}_i \rangle = -1 \quad \text{for} \quad j \in \{1, 2, \ldots, n\} \,.$$

Using the relations above, we can compute $\langle \boldsymbol{p}_1, \boldsymbol{u}_{n+1} \rangle$ as follows:

$$\begin{aligned} -1 = \langle \boldsymbol{p}_1, \boldsymbol{u}_{n+1} \rangle &= \alpha'_0 \, \beta'_0 + \alpha'_1 \, (\beta'_1 \, \langle \boldsymbol{u}_1, \boldsymbol{u}_1 \rangle + \beta'_2 \, \langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle + \cdots + \beta'_n \, \langle \boldsymbol{u}_1, \boldsymbol{u}_n \rangle) \\ &\quad + \alpha'_2 \, (\beta'_1 \, \langle \boldsymbol{u}_2, \boldsymbol{u}_1 \rangle + \beta'_2 \, \langle \boldsymbol{u}_2, \boldsymbol{u}_2 \rangle + \cdots + \beta'_n \, \langle \boldsymbol{u}_2, \boldsymbol{u}_n \rangle) \\ &\quad + \cdots \\ &\quad + \alpha'_n \, (\beta'_1 \, \langle \boldsymbol{u}_n, \boldsymbol{u}_1 \rangle + \beta'_2 \, \langle \boldsymbol{u}_n, \boldsymbol{u}_2 \rangle + \cdots + \beta'_n \, \langle \boldsymbol{u}_n, \boldsymbol{u}_n \rangle) \\ &= \alpha'_0 \, \beta'_0 - \sum_{i=1}^{n} \alpha'_i \,. \end{aligned}$$

Thus we have the following relation:

$$\sum_{i=1}^{n} \alpha'_i = \alpha'_0 \, \beta'_0 + 1 \,. \tag{5.2}$$

The hyperplane P_i, where $i = 1$ or 2, is defined to be $P_i := \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle \boldsymbol{x}, \boldsymbol{p}_i \rangle = -1 \}$. Then P_i contains $\widehat{\Delta_{V_i}}$. Now the dihedral angle formed by $\widehat{\Delta_{V_0}}$ and $\widehat{\Delta_{V_1}}$ is convex if and only if P_0 separates $\boldsymbol{u}_{n+1}$ from the origin $\boldsymbol{o}$, or equivalently, if and only if P_1 separates $\boldsymbol{u}_0$ from $\boldsymbol{o}$. And this condition is equivalent to the one that $\langle \boldsymbol{u}_i, \boldsymbol{p}_j \rangle < -1$, where $(i, j) = (n+1, 0)$ or $(0, 1)$. So, using (5.1) and (5.2), we can compute $\langle \boldsymbol{u}_{n+1}, \boldsymbol{p}_0 \rangle$ as follows:

$$\begin{aligned} -1 > \langle \boldsymbol{u}_{n+1}, \boldsymbol{p}_0 \rangle &= \alpha'_0 \, \beta_0 + \alpha'_1 \, (\beta_1 \, \langle \boldsymbol{u}_1, \boldsymbol{u}_1 \rangle + \beta_2 \, \langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle + \cdots + \beta_n \, \langle \boldsymbol{u}_1, \boldsymbol{u}_n \rangle) \\ &\quad + \alpha'_2 \, (\beta_1 \, \langle \boldsymbol{u}_2, \boldsymbol{u}_1 \rangle + \beta_2 \, \langle \boldsymbol{u}_2, \boldsymbol{u}_2 \rangle + \cdots + \beta_n \, \langle \boldsymbol{u}_2, \boldsymbol{u}_n \rangle) \\ &\quad + \cdots \\ &\quad + \alpha'_n \, (\beta_1 \, \langle \boldsymbol{u}_n, \boldsymbol{u}_1 \rangle + \beta_2 \, \langle \boldsymbol{u}_n, \boldsymbol{u}_2 \rangle + \cdots + \beta_n \, \langle \boldsymbol{u}_n, \boldsymbol{u}_n \rangle) \\ &= \alpha'_0 \, \beta_0 - \sum_{i=1}^{n} \alpha'_i \\ &= \alpha'_0 \, \beta_0 - (\alpha'_0 \, \beta'_0 + 1) \,. \end{aligned}$$

Thus we obtain $\beta_0 - \beta'_0 < 0$, since $\alpha'_0 > 0$. Furthermore $t_0 = \langle \boldsymbol{m}, \boldsymbol{p}_0 \rangle = \beta_0$ and $t_1 = \langle -\boldsymbol{m}, \boldsymbol{p}_1 \rangle = -\beta'_0$, we have $t_0 + t_1 < 0$ if and only if the dihedral angle formed by $\widehat{\Delta_{V_0}}$ and $\widehat{\Delta_{V_1}}$ is convex. The proofs of other cases are analogous. We have thus proved Proposition 5.2. □

6 Tilt formulas

6.1 Generalized distances

Definition 6.1 Fix a point $\boldsymbol{x}$ in H_S. Let $\boldsymbol{y}$ be an arbitrary point in $T^+ \sqcup (R_{\boldsymbol{x}} \cap L^+) \sqcup (R_{\boldsymbol{x}} \cap S)$. Then the *generalized distance d between $\boldsymbol{x}$ and $\boldsymbol{y}$* is defined as follows:

Case 1. If $\boldsymbol{y} \in R_{\boldsymbol{x}} \cap L^{+}$, then d is defined to be the signed distance between $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$.

Case 2. If $\boldsymbol{y} \in T^{+}$ or $\boldsymbol{y} \in S$ with $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq -\sqrt{\langle \boldsymbol{y}, \boldsymbol{y} \rangle}$ (that is, $\Pi_{\boldsymbol{x}}$ and $\Pi_{n(\boldsymbol{y})}$ are parallel or ultraparallel), then $d = d_n - \delta_{\boldsymbol{y}}$, where d_n is the signed distance between $\Pi_{\boldsymbol{x}}$ and $\Pi_{n(\boldsymbol{y})}$, and $\delta_{\boldsymbol{y}}$ is the width of $\boldsymbol{y}$.

Case 3. If $\boldsymbol{y} \in S$ with $(0 \geq) \langle \boldsymbol{x}, \boldsymbol{y} \rangle > -\sqrt{\langle \boldsymbol{y}, \boldsymbol{y} \rangle}$, that is, if $\Pi_{\boldsymbol{x}}$ and $\Pi_{n(\boldsymbol{y})}$ intersect, then $d = \sqrt{-1}\,\theta - \delta_{\boldsymbol{y}}$, where θ is the dihedral angle between $\Pi_{\boldsymbol{x}}$ and $\Pi_{n(\boldsymbol{y})}$ measured in $\Gamma_{\boldsymbol{x}} \cap \Gamma_{n(\boldsymbol{y})}$.

Now we have the following proposition, which is a generalization of Proposition 3.4:

Proposition 6.2 *Let $\boldsymbol{x}$ be a point in H_S. For an arbitrary point $\boldsymbol{y} \in T^{+} \sqcup (R_{\boldsymbol{x}} \cap L^{+}) \sqcup (R_{\boldsymbol{x}} \cap S)$, the following equality holds:*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\frac{e^{d} + \nu\, e^{-d}}{2},$$

where $\nu := \langle \boldsymbol{y}, \boldsymbol{y} \rangle$, and d is the generalized distance between $\boldsymbol{x}$ and $\boldsymbol{y}$.

Proof of Proposition 6.2. We first consider the case where $\boldsymbol{y} \in R_{\boldsymbol{x}} \cap L^{+}$. In this case, the proposition is a direct consequence of Proposition 3.4.

We secondly consider the case where $\boldsymbol{y} \in S$ with $\langle \boldsymbol{x}, \boldsymbol{y} \rangle > -\sqrt{\langle \boldsymbol{y}, \boldsymbol{y} \rangle}$. Then, using the relation (2.2) together with Lemma 3.10, we obtain

$$\begin{aligned}
\langle \boldsymbol{x}, \boldsymbol{y} \rangle &= -\sqrt{\langle \boldsymbol{y}, \boldsymbol{y} \rangle}\cos\theta \\
&= -e^{-\delta_{\boldsymbol{y}}}\,\frac{e^{\sqrt{-1}\,\theta} + e^{-\sqrt{-1}\,\theta}}{2} \\
&= -\frac{e^{\sqrt{-1}\,\theta - \delta_{\boldsymbol{y}}} + e^{-2\delta_{\boldsymbol{y}}}\, e^{-\left(\sqrt{-1}\,\theta - \delta_{\boldsymbol{y}}\right)}}{2} \\
&= -\frac{e^{d} + \nu\, e^{-d}}{2}.
\end{aligned}$$

We thirdly consider the case where $\boldsymbol{y} \in S$ with $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq -\sqrt{\langle \boldsymbol{y}, \boldsymbol{y} \rangle}$. Then, using Proposition 3.4 and Lemma 3.10, we obtain

$$\begin{aligned}
\langle \boldsymbol{x}, \boldsymbol{y} \rangle &= -\sqrt{\langle \boldsymbol{y}, \boldsymbol{y} \rangle}\,\frac{e^{d_n} + \left\langle \frac{\boldsymbol{y}}{\sqrt{\langle \boldsymbol{y}, \boldsymbol{y} \rangle}}, \frac{\boldsymbol{y}}{\sqrt{\langle \boldsymbol{y}, \boldsymbol{y} \rangle}} \right\rangle e^{-d_n}}{2} \\
&= -e^{-\delta_{\boldsymbol{y}}}\,\frac{e^{d_n} + \langle \boldsymbol{y}, \boldsymbol{y} \rangle\, e^{2\delta_{\boldsymbol{y}}}\, e^{-d_n}}{2} \\
&= -\frac{e^{\left(d_n - \delta_{\boldsymbol{y}}\right)} + \nu\, e^{-\left(d_n - \delta_{\boldsymbol{y}}\right)}}{2} \\
&= -\frac{e^{d} + \nu\, e^{-d}}{2}.
\end{aligned}$$

We finally consider the case where $\boldsymbol{y} \in T^{+}$. Then, using Proposition 3.4 and Lemma 3.7, we obtain

$$
\begin{aligned}
\langle \boldsymbol{x}, \boldsymbol{y} \rangle &= -\sqrt{-\langle \boldsymbol{y}, \boldsymbol{y} \rangle}\, \frac{e^{d_n} + \left\langle \frac{\boldsymbol{y}}{\sqrt{-\langle \boldsymbol{y}, \boldsymbol{y} \rangle}}, \frac{\boldsymbol{y}}{\sqrt{-\langle \boldsymbol{y}, \boldsymbol{y} \rangle}} \right\rangle e^{-d_n}}{2} \\
&= -\frac{e^{d} + \nu\, e^{-d}}{2}\,.
\end{aligned}
$$

We have thus proved Proposition 6.2. $\square$

As Figure 2 in Section 4, if the dimension n is equal to two, internal faces may be degenerate, that is, some of opposite faces may not exist in $\mathbf{B}^n$. But n is greater than two, all opposite faces exist.

Proposition 6.3 *Suppose n is greater than or equal to three. Then, for any weighted n-simplex (Δ_V, V, W) in $\mathbf{B}^n$, the opposite face Φ_i of an arbitrary vertex $\boldsymbol{v}_i \in V$ exists in $\mathbf{B}^n$.*

Proof of Proposition 6.3. All we have to show is that the opposite face Φ_n intersects $\mathbf{B}^n$ when $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1} \in \operatorname{Ext} \overline{\mathbf{B}^n}$ and each line $l\,(\boldsymbol{v}_i \boldsymbol{v}_j)$ in $\mathbf{P}^n$ through $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$, where $0 \leq i < j \leq n-1$, touches $\partial \mathbf{B}^n$. Let $\boldsymbol{w}_1$ (resp. $\boldsymbol{w}_2$) be the tangent point of $\partial \mathbf{B}^n$ and $l\,(\boldsymbol{v}_0 \boldsymbol{v}_1)$ (resp. $l\,(\boldsymbol{v}_0 \boldsymbol{v}_2)$). Then $\boldsymbol{w}_1$ does not coincide with $\boldsymbol{w}_2$ when $n \geq 3$. Since n-dimensional ball $\overline{\mathbf{B}^n}$ is convex, the line $l\,(\boldsymbol{w}_1 \boldsymbol{w}_2)$ intersects $\mathbf{B}^n$. Thus $l\,(\boldsymbol{w}_1 \boldsymbol{w}_2) \cap \mathbf{B}^n$ is a (non-empty) segment contained in the opposite face Φ_n. This completes the proof. $\square$

6.2 The case where the dimension is greater than two

In this subsection we suppose the dimension n is greater than or equal to three. Fix a weighted n-simplex (Δ_V, V, W) in $\mathbf{B}^n$. Then Proposition 6.3 guarantees that all internal faces of Δ_V exist in $\mathbf{B}^n$, namely we can always define the tilt t_i for each internal face Φ_i. We denote by $\widehat{V} = \{\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ the lift of V, and we define $\nu_i := \langle \boldsymbol{u}_i, \boldsymbol{u}_i \rangle$. Let d_i be the generalized distance between $\boldsymbol{m}_i$ and $\boldsymbol{u}_i$, where we recall that $\boldsymbol{m}_i$ is the point in H_S such that $\Phi_i \subset P_{\boldsymbol{m}_i} \cap \mathbf{B}^n$ and $\Delta_V \subset R_{\boldsymbol{m}_i} \cap \mathbf{B}^n$. Now we define Q_i as follows:

$$
Q_i := \frac{2}{e^{d_i} + \nu_i\, e^{-d_i}}\,.
$$

We denote by $\theta_{i\,j}$ the dihedral angle between Φ_i and Φ_j, that is, the dihedral angle between $\Pi_{\boldsymbol{m}_i}$ and $\Pi_{\boldsymbol{m}_j}$ measured in $\Gamma_{\boldsymbol{m}_i} \cap \Gamma_{\boldsymbol{m}_j}$. We note that $\theta_{i\,j} = 0$ if Φ_i and Φ_j are parallel. Then we have the following Theorem 6.4, the main theorem of this paper and a generalization of Theorem 2.1 in [SW]:

Theorem 6.4 (tilt formula for $n \geq 3$) *Under the notation defined above, the tilt of a weighted n-simplex relative to each of its (codimension one) internal faces may be computed as follows:*

$$
\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} 1 & -\cos\theta_{0\,1} & -\cos\theta_{0\,2} & \cdots & -\cos\theta_{0\,n} \\ -\cos\theta_{1\,0} & 1 & -\cos\theta_{1\,2} & \cdots & -\cos\theta_{1\,n} \\ -\cos\theta_{2\,0} & -\cos\theta_{2\,1} & 1 & \cdots & -\cos\theta_{2\,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\cos\theta_{n\,0} & -\cos\theta_{n\,1} & -\cos\theta_{n\,2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{pmatrix}.
$$

We may say the $(n+1)\times(n+1)$ matrix on the right side of the formula denoted above the *Gram matrix of the generalized n-simplex* Δ_V (cf. [Vi, p. 39]).

We prove this theorem by imitating the method of Section 2 in [SW]. So we also organize the proof of this theorem and its supporting lemmas in a top-down fashion. The actual logical dependent among the lemmas is as follows:

$$
\begin{array}{ccccccc}
 & & & & \boxed{\text{Theorem 6.4}} & & \\
 & & & \nearrow & \uparrow & \nwarrow & \\
(\text{Proposition 6.2} & \rightarrow) & \text{Lemma 6.5} & \rightarrow & \text{Lemma 6.6} & & \text{Lemma 6.7.}
\end{array}
$$

Proof of Theorem 6.4. Lemma 6.5 shows that vectors $\boldsymbol{m}_0, \boldsymbol{m}_1, \ldots, \boldsymbol{m}_n$ form a basis of $\mathbf{E}^{1,n}$. Relative to this basis, $\boldsymbol{m}_k = (0,\ldots,0,1,0,\ldots,0)$ and $\boldsymbol{p} = (Q_0, Q_1, \ldots, Q_n)$ by Lemma 6.6, and the metric is given by the matrix computed in Lemma 6.7. Therefore

$$
\begin{aligned}
t_k &= \langle\, \boldsymbol{m}_k, \boldsymbol{p}\,\rangle \\
&= (0,\ldots,0,1,0,\ldots,0) \\
&\quad \times \begin{pmatrix} 1 & -\cos\theta_{0\,1} & -\cos\theta_{0\,2} & \cdots & -\cos\theta_{0\,n} \\ -\cos\theta_{1\,0} & 1 & -\cos\theta_{1\,2} & \cdots & -\cos\theta_{1\,n} \\ -\cos\theta_{2\,0} & -\cos\theta_{2\,1} & 1 & \cdots & -\cos\theta_{2\,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\cos\theta_{n\,0} & -\cos\theta_{n\,1} & -\cos\theta_{n\,2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{pmatrix},
\end{aligned}
$$

thereby completing the proof of Theorem 6.4. □

Lemma 6.5 *The set* $\{\boldsymbol{m}_0, \boldsymbol{m}_1, \ldots, \boldsymbol{m}_n\}$ *forms a basis of* $\mathbf{E}^{1,n}$*, and is dual to the basis* $\{-Q_0\,\boldsymbol{u}_0, -Q_1\,\boldsymbol{u}_1, \ldots, -Q_n\,\boldsymbol{u}_n\}$ *in respect of the Lorentzian inner product.*

Proof of Lemma 6.5. As we saw in Proposition 4.3, $\widehat{V} = \{\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ is a set of linearly independent vectors in $\mathbf{E}^{1,n}$. Furthermore, since $\boldsymbol{u}_j$ lies in $P_{\boldsymbol{m}_i}$ for $i \neq j$, we have $\langle\, \boldsymbol{m}_i, \boldsymbol{u}_j\,\rangle = 0$. Now $\boldsymbol{u}_i$ does not lie in $P_{\boldsymbol{m}_i}$, that is, $\langle\, \boldsymbol{m}_i, \boldsymbol{u}_i\,\rangle \neq 0$. Moreover, using Proposition 6.2, we obtain $\langle\, \boldsymbol{m}_i, \boldsymbol{u}_i\,\rangle = -Q_i^{-1}$. It follows that sets of vectors $\{\boldsymbol{m}_0, \boldsymbol{m}_1, \ldots, \boldsymbol{m}_n\}$ and $\{-Q_0\,\boldsymbol{u}_0, -Q_1\,\boldsymbol{u}_1, \ldots, -Q_n\,\boldsymbol{u}_n\}$ are dual, that is, $\langle\, \boldsymbol{m}_i, -Q_j\,\boldsymbol{u}_j\,\rangle = 1$ if $i = j$ and $\langle\, \boldsymbol{m}_i, -Q_j\,\boldsymbol{u}_j\,\rangle = 0$ if $i \neq j$. This duality implies that each set is linearly independent, and therefore forms a basis of $\mathbf{E}^{1,n}$. □

Lemma 6.6 *Relative to the basis* $\{\boldsymbol{m}_0, \boldsymbol{m}_1, \ldots, \boldsymbol{m}_n\}$*, the vector* $\boldsymbol{p}$ *is* $(Q_0, Q_1, \ldots, Q_n)$*.*

Proof of Lemma 6.6. Since bases $\{\boldsymbol{m}_0, \boldsymbol{m}_1, \ldots, \boldsymbol{m}_n\}$ and $\{-\,Q_0\,\boldsymbol{u}_0, -\,Q_1\,\boldsymbol{u}_1, \ldots, -\,Q_n\,\boldsymbol{u}_n\}$ are dual by Lemma 6.5, the following equation holds:

$$\boldsymbol{p} = \sum_{i=0}^{n} \langle\, \boldsymbol{p}, -\,Q_i\,\boldsymbol{u}_i \,\rangle\; \boldsymbol{m}_i\,.$$

By the definition of $\boldsymbol{p}$, we have $\langle\, \boldsymbol{p}, \boldsymbol{u}_i \,\rangle = -\,1$. So we obtain

$$\sum_{i=0}^{n} \langle\, \boldsymbol{p}, -\,Q_i\,\boldsymbol{u}_i \,\rangle\; \boldsymbol{m}_i = \sum_{i=0}^{n} Q_i\,\boldsymbol{m}_i\,. \qquad \Box$$

Lemma 6.7 *Relative to the basis* $\{\boldsymbol{m}_0, \boldsymbol{m}_1, \ldots, \boldsymbol{m}_n\}$*, the Lorentzian space metric is*

$$\begin{pmatrix} 1 & -\cos\theta_{0\,1} & -\cos\theta_{0\,2} & \cdots & -\cos\theta_{0\,n} \\ -\cos\theta_{1\,0} & 1 & -\cos\theta_{1\,2} & \cdots & -\cos\theta_{1\,n} \\ -\cos\theta_{2\,0} & -\cos\theta_{2\,1} & 1 & \cdots & -\cos\theta_{2\,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\cos\theta_{n\,0} & -\cos\theta_{n\,1} & -\cos\theta_{n\,2} & \cdots & 1 \end{pmatrix}.$$

Proof of Lemma 6.7. Since $\boldsymbol{m}_i \in H_S$, we have $\langle\, \boldsymbol{m}_i, \boldsymbol{m}_i \,\rangle = 1$. Now we suppose $i \neq j$. Then, by the definition of the weighted n-simplex together with the assumption $n \geq 3$, $\Pi_{\boldsymbol{m}_i}$ and $\Pi_{\boldsymbol{m}_j}$ are not ultraparallel. So, using (2.2), we obtain $\langle\, \boldsymbol{m}_i, \boldsymbol{m}_j \,\rangle = -\,\cos\theta_{i\,j}$. $\Box$

6.3 The case where the dimension is two

As we saw in Figure 2, some internal faces of a weighted 2-simplex (Δ_V, V, W) in $\mathbf{B}^2$ may be degenerate. So Theorem 6.4 does not always hold when the dimension n is two. But under the assumption that all internal faces exist, it holds that the following Theorem 6.8, an analogue of Theorem 6.4. We here note that $\Pi_{\boldsymbol{m}_i}$ and $\Pi_{\boldsymbol{m}_j}$ may be ultraparallel for some $\boldsymbol{m}_i, \boldsymbol{m}_j \in H_S$ with $i \neq j$ (see Figure 1 again). We denote by $\delta_{i\,j}$ the generalized distance between $\boldsymbol{m}_i$ and $\boldsymbol{m}_j$.

Theorem 6.8 (tilt formula for $n = 2$ when all internal faces exist) *Under the assumptions stated above, the following relation holds:*

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 1 & -\cosh\delta_{0\,1} & -\cosh\delta_{0\,2} \\ -\cosh\delta_{1\,0} & 1 & -\cosh\delta_{1\,2} \\ -\cosh\delta_{2\,0} & -\cosh\delta_{2\,1} & 1 \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \end{pmatrix}. \qquad \Box$$

From now on, we consider the case where some internal faces are degenerate. For example we assume that only the opposite face of the vertex $\boldsymbol{v}_2 \in V$ is degenerate (see Figure 2 again). In this case, we put $\boldsymbol{m}_2 := \sqrt{\nu_1}\,\boldsymbol{u}_0 + \sqrt{\nu_0}\,\boldsymbol{u}_1$. Then $\boldsymbol{m}_2$ is a non-zero vector in L. Now, by a similar argument to the proof of Lemma 6.5, we can show that two sets $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $\{-\,Q_0\,\boldsymbol{m}_0, -\,Q_1\,\boldsymbol{m}_1, -\,Q_2\,\boldsymbol{m}_2\}$ form two bases of $\mathbf{E}^{1,2}$ and are dual to each other, where $Q_2 := -\,\langle\, \boldsymbol{m}_2, \boldsymbol{u}_2 \,\rangle^{-1} = -\,\left(\langle\, \boldsymbol{u}_0, \boldsymbol{u}_2 \,\rangle\,\sqrt{\nu_1} + \langle\, \boldsymbol{u}_1, \boldsymbol{u}_2 \,\rangle\,\sqrt{\nu_0}\right)^{-1} (\neq 0)$. Now using equations $\langle\, \boldsymbol{m}_0, \boldsymbol{m}_2 \,\rangle = -\,{Q_0}^{-1}\sqrt{\nu_1}$ and $\langle\, \boldsymbol{m}_1, \boldsymbol{m}_2 \,\rangle = -\,{Q_1}^{-1}\sqrt{\nu_0}$, we can easily obtain the following corollary:

Corollary 6.9 (tilt formula for $n = 2$ with one degenerate internal face) *Under the assumptions stated above, the following relation holds:*

$$\begin{pmatrix} t_0 \\ t_1 \end{pmatrix} = \begin{pmatrix} 1 & -\cosh\delta_{0\,1} & -{Q_0}^{-1}\sqrt{\nu_1} \\ -\cosh\delta_{1\,0} & 1 & -{Q_1}^{-1}\sqrt{\nu_0} \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \end{pmatrix}. \qquad \square$$

A List of notation

$\mathbf{E}^{1,n}$: $n+1$-dimensional Lorentzian space,

$\mathbf{R}^{n+1}$: $n+1$-dimensional Euclidean space,

$\langle\,\boldsymbol{x},\boldsymbol{y}\,\rangle$ $:= -\,x_0\,y_0 + x_1\,y_1 + \cdots + x_n\,y_n$; Lorentzian inner product of $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_0, y_1, \ldots, y_n)$,

T^+ $:= \{\,\boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\,\boldsymbol{x},\boldsymbol{x}\,\rangle < 0$ and $x_0 > 0\,\}$; future cone,

T^- $:= \{\,\boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\,\boldsymbol{x},\boldsymbol{x}\,\rangle < 0$ and $x_0 < 0\,\}$; past cone,

L $:= \{\,\boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\,\boldsymbol{x},\boldsymbol{x}\,\rangle = 0\,\}$; light cone,

L^+ $:= \{\,\boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\,\boldsymbol{x},\boldsymbol{x}\,\rangle = 0$ and $x_0 > 0\,\}\,(\subset L)$; positive light cone,

L^- $:= \{\,\boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\,\boldsymbol{x},\boldsymbol{x}\,\rangle = 0$ and $x_0 < 0\,\}\,(\subset L)$; negative light cone,

S $:= \{\,\boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\,\boldsymbol{x},\boldsymbol{x}\,\rangle > 0\,\}$; side cone,

$\boldsymbol{o}$ $:= (0, 0, \ldots, 0)$; origin of $\mathbf{E}^{1,n}$,

$A \sqcup B$: disjoint union of two sets A and B,

$n\,(\boldsymbol{x})$ $:= \dfrac{\boldsymbol{x}}{\sqrt{|\langle\boldsymbol{x},\boldsymbol{x}\rangle|}}$; normalized vector of $\boldsymbol{x}$ in $\mathbf{E}^{1,n}$ with $\langle\,\boldsymbol{x},\boldsymbol{x}\,\rangle \neq 0$,

H_T^+ $:= \{\,\boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\,\boldsymbol{x},\boldsymbol{x}\,\rangle = -\,1$ and $x_0 > 0\,\}\,(\subset T^+)$; upper sheet of the (standard) hyperboloid of two sheets,

$\mathbf{H}^n$: hyperboloid model of the n-dimensional hyperbolic space,

∂A : boundary of a set A,

S_∞^{n-1} $:= \{$ ray in L^+ started from $\boldsymbol{o}$ $\}$; sphere at infinity of $\mathbf{H}^n$,

H_S $:= \{\,\boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, \langle\,\boldsymbol{x},\boldsymbol{x}\,\rangle = 1\,\}\,(\subset S)$; (standard) hyperboloid of one sheet,

$\mathbf{P}_1^n$ $:= \{\,\boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, x_0 = 1\,\}$,

$\boldsymbol{i}$ $:= (1, 0, 0, \ldots, 0)$; origin of $\mathbf{P}_1^n$,

$\mathbf{B}^n$: n-dimensional open unit ball in $\mathbf{P}_1^n$ centered at $\boldsymbol{i}$,

$\overline{\mathbf{B}^n}$ $:= \mathbf{B}^n \sqcup \partial\mathbf{B}^n\,(\approx \mathbf{H}^n \sqcup S_\infty^{n-1})$,

$\mathbf{P}_\infty^n$ $:= \{$ line in the affine hyperplane $\{\,\boldsymbol{x} \in \mathbf{E}^{1,n} \,|\, x_0 = 0\,\}$ through $\boldsymbol{o}$ $\}$,

$\mathbf{P}^n$: n-dimensional real projective space obtained by $\mathbf{P}^n_1 \sqcup \mathbf{P}^n_\infty$,

$\mathcal{P}$: $\mathbf{E}^{1,n} - \{\boldsymbol{o}\} \longrightarrow \mathbf{P}^n$; radial projection along the ray from $\boldsymbol{o}$,

$\operatorname{Ext} \overline{\mathbf{B}^n}$:= $\mathbf{P}^n - \overline{\mathbf{B}^n}$,

$R_{\boldsymbol{u}}$:= $\left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{u} \rangle \le \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle - 1}{2} \right\}$, where $\boldsymbol{u} \in \mathbf{E}^{1,n}$,

$P_{\boldsymbol{u}}$:= $\left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{u} \rangle = \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle - 1}{2} \right\}$ $(= \partial R_{\boldsymbol{u}})$, where $\boldsymbol{u} \in \mathbf{E}^{1,n}$,

$\Gamma_{\boldsymbol{u}}$:= $R_{\boldsymbol{u}} \cap H^+_T$,

$\Pi_{\boldsymbol{u}}$:= $P_{\boldsymbol{u}} \cap H^+_T$,

$\Omega(\boldsymbol{v})$: polar hyperplane of $\boldsymbol{v} \in \operatorname{Ext} \overline{\mathbf{B}^n}$,

$\Psi(\boldsymbol{v})$:= $\Omega(\boldsymbol{v}) \cap \mathbf{B}^n$; hyperplane in $\mathbf{B}^n$ with pole $\boldsymbol{v}$,

ν := $\langle \boldsymbol{y}, \boldsymbol{y} \rangle$ for $\boldsymbol{y} \in T^+ \sqcup S$,

$(\boldsymbol{x}, \boldsymbol{y})$:= $x_0 y_0 + x_1 y_1 + \cdots + x_n y_n$; Euclidean inner product of $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_0, y_1, \ldots, y_n)$,

$\delta_{\boldsymbol{u}}$: width of $\boldsymbol{u} \in T^+ \sqcup L^+ \sqcup S$, that is,
$\delta_{\boldsymbol{u}} := -\frac{1}{2} \log |\langle \boldsymbol{u}, \boldsymbol{u} \rangle|$ if $\boldsymbol{u} \in T^+ \sqcup S$
$\delta_{\boldsymbol{u}} := -\frac{1}{2} \log (\boldsymbol{u}, \boldsymbol{u}) \left(= -\log(u_0 \sqrt{2})\right)$ if $\boldsymbol{u} = (u_0, u_1, \ldots, u_n) \in L^+$,

V := $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$; set of independent points in $\mathbf{P}^n$,

V_{in} := $\left\{ \boldsymbol{v} \in V \,\middle|\, \boldsymbol{v} \in \overline{\mathbf{B}^n} \right\}$,

V_{ex} := $\left\{ \boldsymbol{v} \in V \,\middle|\, \boldsymbol{v} \in \operatorname{Ext} \overline{\mathbf{B}^n} \right\}$ $(= V - V_{\mathrm{in}})$,

$|\boldsymbol{v}_0 \boldsymbol{v}_1 \cdots \boldsymbol{v}_n|$: n-simplex in $\mathbf{P}^n$ with a vertex set V,

Δ_V : generalized n-simplex in $\mathbf{B}^n$ with a vertex set V,

$\mathcal{F}_i$: hyperplane in $\mathbf{P}^n$ through $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_n\} \subset V$,

Φ_i := $\mathcal{F}_i \cap \Delta_V$; opposite face of $\boldsymbol{v}_i \in V$ (i.e., the internal face opposite to $\boldsymbol{v}_i$),

W : set of weights,

(Δ_V, V, W) : weighted n-simplex in $\mathbf{B}^n$,

$\widehat{\Delta_V}$: lift of (Δ_V, V, W),

$\widehat{V}$: lift of V,

$\boldsymbol{m}_i$: point in H_S with $\Phi_i \subset P_{\boldsymbol{m}_i}$ and $\Delta_V \subset R_{\boldsymbol{m}_i}$,

$\boldsymbol{p}$: normal vector to $\widehat{\Delta_V}$ with $\langle \boldsymbol{p}, \boldsymbol{x} \rangle = -1$ for $\boldsymbol{x} \in \widehat{\Delta_V}$,

t_i := $\langle \boldsymbol{m}_i, \boldsymbol{p} \rangle$; tilt of (Δ_V, V, W) relative to Φ_i,

$\nu_i \;:= \langle\, \boldsymbol{u}_i, \boldsymbol{u}_i \,\rangle$, where $\boldsymbol{u}_i \in \widehat{V}$,

$d_i\;$: generalized distance between $\boldsymbol{m}_i \in H_S$ and $\boldsymbol{u}_i \in \widehat{V}$,

$Q_i \;:= \frac{2}{e^{d_i}+\nu_i\, e^{-\,d_i}}$,

$\theta_{i\,j}\;$: dihedral angle between Φ_i and Φ_j,

$\delta_{i\,j}\;$: generalized distance between $\boldsymbol{m}_i \in H_S$ and $\boldsymbol{m}_j \in H_S$.

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