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Abstract

In this paper, we completely determine the structure of the holomorphic automorphism group of a generalized complex ellipsoid. This is a natural generalization of a result due to Landucci. Also this gives an affirmative answer to an open problem posed by Jarnicki and Pflug.

Keywords: Generalized complex ellipsoids; Holomorphic automorphisms

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1 Introduction

In this paper we study the structure of the holomorphic automorphism group of a *generalized complex ellipsoid*

$$E(n_0, \dots, n_K; p_0, \dots, p_K) := \left\{ (z_0, \dots, z_K) \in \mathbf{C}^{n_0} \times \dots \times \mathbf{C}^{n_K} ; \sum_{k=0}^K \|z_k\|^{2p_k} < 1 \right\}$$

in $\mathbf{C}^N = \mathbf{C}^{n_0} \times \dots \times \mathbf{C}^{n_K}$, where $n_0, \dots, n_K$ are positive integers and $p_0, \dots, p_K$ are positive real numbers, and $N = n_0 + \dots + n_K$. In general this domain is not geometrically convex and its boundary is not smooth. In the special case where all the $p_k = 1$, this domain reduces to the unit ball B^N in $\mathbf{C}^N$ and the structure of its holomorphic automorphism group $\text{Aut}(B^N)$ is well-known (cf. [7]). Also, it is known that $E(n_0, \dots, n_K; p_0, \dots, p_K)$ is homogeneous if and only if $p_k = 1$ for all k (cf. [3], [6], [8]).

For convenience and with no loss of generality, in the following we will always assume that $p_0 = 1$, $p_1, \dots, p_K \neq 1$, $n_1, \dots, n_K > 0$. Moreover, after relabeling the indices, if necessary, we may assume that there exist positive integers $k_1, \dots, k_s$ such that

$$\begin{aligned} k_1 + \dots + k_s &= K, \\ n_{k_1 + \dots + k_{j-1} + 1} &= \dots = n_{k_1 + \dots + k_j}, \quad 1 \leq j \leq s, \\ n_{k_1 + \dots + k_j} &< n_{k_1 + \dots + k_j + 1}, \quad 1 \leq j \leq s-1, \end{aligned}$$

where we put $p_0 = 0$.

Now let us choose an arbitrary generalized complex ellipsoid $\mathcal{E}$ in $\mathbf{C}^N$ and write it in the form

$$(*) \quad \mathcal{E} = E(n_0, n_1, \dots, n_K; 1, p_1, \dots, p_K).$$

Here it is understood that 1 does not appear if $n_0 = 0$, and also this domain is the unit ball B^{n_0} in $\mathbf{C}^{n_0} = \mathbf{C}^N$ if $K = 0$.

The purpose of this paper is to establish the following theorem that gives a full description of the holomorphic automorphism group of generalized complex ellipsoids:

THEOREM *Let $\mathcal{E}$ be the generalized complex ellipsoid appearing in (*). Then the holomorphic automorphism group $\text{Aut}(\mathcal{E})$ of $\mathcal{E}$ consists of all transformations*

$$\varphi : (z_0, z_1, \dots, z_K) \mapsto (\tilde{z}_0, \tilde{z}_1, \dots, \tilde{z}_K)$$

of the form

$$\tilde{z}_0 = H(z_0), \quad \tilde{z}_k = \gamma_k(z_0) U_k z_{\sigma(k)}, \quad 1 \leq k \leq K$$

(think of z_k as column vectors), where

$$(1) \quad H \in \text{Aut}(B^{n_0}),$$

$$(2) \quad \gamma_k(z_0) \text{ are nowhere vanishing holomorphic functions on } B^{n_0} \text{ defined by}$$

$$\gamma_k(z_0) = \left(\frac{1 - \|a\|^2}{(1 - \langle z_0, a \rangle)^2} \right)^{1/2p_k}, \quad a = H^{-1}(o) \in B^{n_0},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian inner product on $\mathbf{C}^{n_0}$ and $o \in B^{n_0}$ is the origin of $\mathbf{C}^{n_0}$,

$$(3) \quad U_k \in U(n_k), \text{ the unitary group of degree } n_k, \text{ and}$$

$$(4) \quad \sigma \text{ is a permutation of } \{1, \dots, K\} \text{ satisfying the following:}$$

$$\begin{aligned} \{\sigma(k_1 + \dots + k_{j-1} + 1), \dots, \sigma(k_1 + \dots + k_j)\} = \\ \{k_1 + \dots + k_{j-1} + 1, \dots, k_1 + \dots + k_j\}, \quad 1 \leq j \leq s, \end{aligned}$$

and $\sigma(\mu) = \nu$ can only happen when $p_\mu = p_\nu$.

In particular, considering the special case where $n_k = 1$ and $2 \leq p_k \in \mathbf{N}$ for all k , we obtain a natural generalization of Landucci [4; Corollary to Theorem]. This also gives an affirmative answer to an open problem posed in Jarnicki and Pflug [2; Remark 2.5.11].

In the next Section 2 we prove the Theorem and, in Section 3, we give a concrete example illustrating our result.

2 Proof of the Theorem

As mentioned in the introduction, the structure of the holomorphic automorphism group of the unit ball B^N in $\mathbf{C}^N$ is well-known. So we prove the Theorem in the case where $K \geq 1$.

For the given generalized complex ellipsoid $\mathcal{E}$ in $\mathbf{C}^N = \mathbf{C}^{n_0} \times \cdots \times \mathbf{C}^{n_K}$, let us consider the subset G of $\text{Aut}(\mathcal{E})$ consisting of all elements

$$\varphi : (z_0, z_1, \dots, z_K) \longmapsto (\tilde{z}_0, \tilde{z}_1, \dots, \tilde{z}_K)$$

having the form

$$(2.1) \quad \tilde{z}_0 = H(z_0), \quad \tilde{z}_k = \gamma_k(z_0)U_k z_k, \quad 1 \leq k \leq K,$$

where $H \in \text{Aut}(B^{n_0})$, $U_k \in U(n_k)$ and $\gamma_k(z_0)$ are the same objects appearing in the statement of the Theorem. Then one can see that G is a connected Lie subgroup of the Lie group $\text{Aut}(\mathcal{E})$ of dimension

$$d(\mathcal{E}) := n_0^2 + 2n_0 + \sum_{k=1}^K n_k^2$$

On the other hand, we know from Naruki [6] and Sunada [8] that $\text{Aut}(\mathcal{E})$ is a real Lie group of dimension $d(\mathcal{E})$; hence, G is exactly the identity component of $\text{Aut}(\mathcal{E})$. In particular, G is a normal subgroup of $\text{Aut}(\mathcal{E})$.

By making use of the concrete description in (2.1) of elements of G , it is an easy matter to check that the G -orbit passing through the origin $o \in \mathcal{E} \subset \mathbf{C}^N$ is of lowest dimension in the set of all G -orbits, i.e.,

$$\dim(G \cdot o) < \dim(G \cdot p) \quad \text{for any point } p \in \mathcal{E} \setminus G \cdot o.$$

Hence, recalling the normality of G in $\text{Aut}(\mathcal{E})$, we obtain that

$$(2.2) \quad g \cdot (G \cdot o) = G \cdot o = \{(z_0, 0, \dots, 0) \in \mathbf{C}^{n_0} \times \mathbf{C}^{n_1} \times \cdots \times \mathbf{C}^{n_K} ; \|z_0\| < 1\}$$

for each element $g \in \text{Aut}(\mathcal{E})$. This combined with a well-known theorem of H. Cartan (cf. [5; p. 67] assures us that every element $g \in \text{Aut}(\mathcal{E})$ can be expressed as $g = \psi_g \cdot \ell_g$, where $\psi_g \in G$ and ℓ_g is a linear automorphism of $\mathcal{E}$, that is, a non-singular linear transformation of $\mathbf{C}^N$ leaving $\mathcal{E}$ invariant. Hence, the proof of our Theorem is now reduced to showing the following:

LEMMA *Every linear automorphism $L : (z_0, z_1, \dots, z_K) \mapsto (\tilde{z}_0, \tilde{z}_1, \dots, \tilde{z}_K)$ of $\mathcal{E}$ can be written in the form*

$$(2.3) \quad \tilde{z}_0 = Az_0, \quad \tilde{z}_k = U_k z_{\sigma(k)}, \quad 1 \leq k \leq K,$$

where $A \in U(n_0)$, $U_k \in U(n_k)$ and σ is a permutation of $\{1, \dots, K\}$ satisfying the same condition (4) as in the Theorem.

Proof. We will show this Lemma by generalizing the argument used in the proofs of [4; Proposition 2.1] and [1; Lemma 8.5.3]. It is clear that the linear

transformation L of $\mathbf{C}^N$ written in the form (2.3) induces a linear automorphism of $\mathcal{E}$. So, taking an arbitrary linear automorphism L of $\mathcal{E}$, we would like to show that L can be described as in (2.3). To this end, we define the coordinate vector subspaces V_k, W_k of $\mathbf{C}^N$ by setting

$$V_k = \{(z_0, z_1, \dots, z_K) \in \mathbf{C}^N; z_j = 0, j \neq k\},$$

$$W_k = \{(z_0, z_1, \dots, z_K) \in \mathbf{C}^N; z_k = 0\}$$

for $0 \leq k \leq K$; accordingly $\bigcap_{j \neq k} W_j = V_k$ for $0 \leq k \leq K$. Here, recalling our assumption that $K \geq 1$ and all the $p_k \neq 1$, we put

$$W = \{(z_0, z_1, \dots, z_K) \in \mathbf{C}^N; \|z_1\| \cdots \|z_K\| = 0\} \quad \text{and} \quad \mathcal{W} = W \cap \partial\mathcal{E},$$

where $\partial\mathcal{E}$ stands for the boundary of $\mathcal{E}$. Then, by routine computations it follows that $\partial\mathcal{E} \setminus \mathcal{W}$ is just the set consisting of all C^ω -smooth strongly pseudoconvex boundary points of $\mathcal{E}$; consequently, $L(\mathcal{W}) = \mathcal{W}$. This, combined with the facts that W is invariant under the dilations $\delta_r : z \mapsto rz$ ($r > 0$) on $\mathbf{C}^N$ and $L(\delta_r(z)) = \delta_r(L(z))$ on $\mathbf{C}^N$, yields at once that $L(W) = W$.

With respect to the coordinate system $(z_0, z_1, \dots, z_K)$ in $\mathbf{C}^N$, the linear automorphism L can be expressed as $L = (L_0, L_1, \dots, L_K)$. Recall here the fact in (2.2). It then follows that

- each L_k ($1 \leq k \leq K$) does not depend on the variable z_0 ,

and

- the restriction $L_0|_{V_0} : V_0 \rightarrow V_0$ of L_0 to V_0 gives rise to a holomorphic automorphism of the unit ball B^{n_0} ; and hence, it has to be a unitary transformation of $V_0 \equiv \mathbf{C}^{n_0}$.

Therefore, one may assume that L has the form:

$$L(z) = (z_0 + A(z_1, \dots, z_K), L_1(z_1, \dots, z_K), \dots, L_K(z_1, \dots, z_K))$$

for $z = (z_0, z_1, \dots, z_K) \in \mathbf{C}^N$, where A, L_k ($1 \leq k \leq K$) are all linear mappings.

Now we will proceed in steps.

1) *There exists a permutation τ of $\{1, \dots, K\}$ such that $L_{\tau(k)}(W_k) = \{0\}$ for every $1 \leq k \leq K$. In particular, we have $L(W_k) \subset W_{\tau(k)}$ for $1 \leq k \leq K$.* Indeed, let $1 \leq k \leq K$ and assume that $L_j(W_k) \neq \{0\}$ for all j , $1 \leq j \leq K$. Then, considering the proper complex analytic subset $\mathcal{A}$ of W_k consisting of all points $z \in W_k$ with $L_j(z) = 0$ for some j , $1 \leq j \leq K$, we have

$$\|L_1(z^o)\| \cdots \|L_K(z^o)\| > 0 \quad \text{for any point } z^o \in W_k \setminus \mathcal{A}.$$

However, since $W_k \subset W$ for every $1 \leq k \leq K$ and $L(W) = W$, this is absurd. Therefore we have shown that, for every $1 \leq k \leq K$, there exists at least one integer j , $1 \leq j \leq K$, such that $L_j(W_k) = \{0\}$. Let us fix, once and for all, the correspondence $\tau : k \mapsto j$. Then this τ is injective. Indeed, assume contrarily that $\tau(k) = \tau(\ell) =: j_0$ for some k, ℓ with $1 \leq k \neq \ell \leq K$. Then, since $\mathbf{C}^N =$

$W_k + W_\ell$, the sum of the vector subspaces W_k and W_ℓ , and since $L : \mathbf{C}^N \rightarrow \mathbf{C}^N$ is a linear isomorphism, we obtain a contradiction: $\mathbf{C}^N = L(\mathbf{C}^N) \subset W_{j_0} \subsetneq \mathbf{C}^N$. As a result, τ is a permutation of $\{1, \dots, K\}$ satisfying the condition required in 1).

2) Let τ be the permutation of $\{1, \dots, K\}$ appearing in 1). Then we have

$$\{\tau(k_1 + \dots + k_{j-1} + 1), \dots, \tau(k_1 + \dots + k_j)\} = \{k_1 + \dots + k_{j-1} + 1, \dots, k_1 + \dots + k_j\}, \quad 1 \leq j \leq s,$$

where we put $k_0 = 0$. Indeed, for every $1 \leq k \leq K$, we have

$$L(V_k) = \bigcap_{0 \leq j \leq K, j \neq k} L(W_j) \subset L(W_0) \cap \left(\bigcap_{1 \leq j \leq K, j \neq k} W_{\tau(j)} \right)$$

by 1); consequently,

$$(2.4) \quad L_{\tau(k)}(V_k) \subset V_{\tau(k)} \quad \text{and} \quad L_{\tau(j)}(V_k) = \{0\}, \quad 1 \leq j \leq K, j \neq k.$$

From now on, putting $M = n_1 + \dots + n_K$, we identify in the obvious way $\mathbf{C}^M = \mathbf{C}^{n_1} \times \dots \times \mathbf{C}^{n_K}$ with the coordinate vector subspace W_0 of $\mathbf{C}^N$. Then the linear transformation $\tilde{L} := (L_1, \dots, L_K) : \mathbf{C}^M \rightarrow \mathbf{C}^M$ induced by L is non-singular; and hence, we see that $L_{\tau(k)}(V_k) = V_{\tau(k)}$ in (2.4) and $n_k = n_{\tau(k)}$. This, together with the ordering among the integers $n_1, \dots, n_K$ as in the previous section, guarantees that τ has to satisfy the condition in 2), as desired.

Let $\sigma := \tau^{-1}$ be the inverse of τ in 1). Then, by (2.4) L can be written in the form

$$(2.5) \quad L(z) = (z_0 + A(z_1, \dots, z_K), U_1 z_{\sigma(1)}, \dots, U_K z_{\sigma(K)})$$

for $z = (z_0, z_1, \dots, z_K) \in \mathbf{C}^N$ (think of z_k as column vectors), where U_k are non-singular $n_k \times n_k$ matrices for $1 \leq k \leq K$. Here we wish to verify the following:

3) For every $1 \leq k \leq K$, we have $U_k \in U(n_k)$. To show this, we first assert that $A(z_1, \dots, z_K) \equiv 0$ in (2.5). Indeed, the fact $L(\partial\mathcal{E}) = \partial\mathcal{E}$ yields that

$$\|z_0 + A(z_1, \dots, z_K)\|^2 + \sum_{k=1}^K \|U_k z_{\sigma(k)}\|^{2p_k} = 1, \quad z \in \partial\mathcal{E}.$$

For any point $z = (z_0, z_1, \dots, z_K) \in \partial\mathcal{E}$, write $z_0 = (z_0^1, \dots, z_0^{n_0})$. Then, by taking a suitable point $\hat{z}_0$ of the form

$$\hat{z}_0 = (\xi_1 z_0^1, \dots, \xi_{n_0} z_0^{n_0}), \quad \xi_j \in \mathbf{C}, |\xi_j| = 1, \quad 1 \leq j \leq n_0,$$

we see that $\text{Re}\langle z_0, A(z_1, \dots, z_K) \rangle = 0$; and hence,

$$(2.6) \quad -\sum_{k=1}^K \|z_k\|^{2p_k} + \|A(z_1, \dots, z_K)\|^2 + \sum_{k=1}^K \|U_k z_{\sigma(k)}\|^{2p_k} = 0, \quad z \in \partial\mathcal{E}.$$

Notice that this equality holds also for any point

$$(z_1, \dots, z_K) \in \mathbf{C}^M \quad \text{with} \quad \sum_{k=1}^K \|z_k\|^{2p_k} \leq 1,$$

because one can always find a point $z_0 \in \mathbf{C}^{n_0}$ such that $(z_0, z_1, \dots, z_K) \in \partial\mathcal{E}$. Now, in order to prove that $A(z_1, \dots, z_K) \equiv 0$, take an arbitrary point $z_1 \in \mathbf{C}^{n_1}$ with $\|z_1\| = 1$ and set $j = \sigma^{-1}(1)$, for simplicity. Then

$$-x^{2p_1} + x^2 \|A(z_1, 0, \dots, 0)\|^2 + x^{2p_j} \|U_j z_1\|^{2p_j} = 0, \quad 0 \leq x \leq 1.$$

Since all the $p_k \neq 1$, this says that $A(z_1, 0, \dots, 0) = 0$. Analogously, for every $2 \leq k \leq K$ one can show that $A(0, \dots, 0, z_k, 0, \dots, 0) = 0$ for $z_k \in \mathbf{C}^{n_k}$ with $\|z_k\| = 1$. Obviously this means that $A(z_1, \dots, z_K) \equiv 0$ on $\mathbf{C}^M$, as asserted.

Next, put $j = \sigma(k)$ for a given k , $1 \leq k \leq K$. It then follows from (2.6) that

$$\|U_k z_j\| = 1 \quad \text{for all } z_j \in V_j, \|z_j\| = 1;$$

which implies that $U_k \in U(n_k)$ for every $1 \leq k \leq K$; verifying the assertion 3).

Summarizing the above, we have shown that L has the form

$$L(z) = (z_0, U_1 z_{\sigma(1)}, \dots, U_K z_{\sigma(K)}), \quad z = (z_0, z_1, \dots, z_K) \in \mathbf{C}^N,$$

where $U_k \in U(n_k)$, $1 \leq k \leq K$, and σ is a permutation of $\{1, \dots, K\}$ satisfying the condition:

$$\begin{aligned} \{\sigma(k_1 + \dots + k_{j-1} + 1), \dots, \sigma(k_1 + \dots + k_j)\} = \\ \{k_1 + \dots + k_{j-1} + 1, \dots, k_1 + \dots + k_j\}, \quad 1 \leq j \leq s. \end{aligned}$$

Therefore, in order to complete the proof of the Lemma, we have only to show the following assertion:

4) *Let $k_1 + \dots + k_{j-1} + 1 \leq \mu$, $\nu \leq k_1 + \dots + k_j$, $1 \leq j \leq s$. Then $\sigma(\mu) = \nu$ can only happen when $p_\mu = p_\nu$.* We verify this only in the case where $j = 1$, since the verification in the general case is almost identical. Moreover, once the proof of 4) for $k_1 \geq 4$ is accomplished, then that for $1 \leq k_1 \leq 3$ follows by a simple modification of it. Taking these into account, we will carry out the proof of 4) in the case where $j = 1$ and $k_1 \geq 4$. Clearly $\sigma(\mu) = \nu$ is possible when $p_\mu = p_\nu$. So, assuming that $\sigma(\mu) = \nu$ for $1 \leq \mu, \nu \leq k_1$, $\mu \neq \nu$, we wish to prove that $p_\mu = p_\nu$. For this purpose, we first remark the following: Since $L(\partial\mathcal{E}) = \partial\mathcal{E}$, with exactly the same argument as in the proof of 3), we can see that

$$(2.7) \quad \sum_{1 \leq k \leq k_1, k \neq \mu} \|z_{\sigma(k)}\|^{2p_k} + \left(1 - \sum_{1 \leq j \leq k_1, j \neq \nu} \|z_j\|^{2p_j}\right)^{p_\mu/p_\nu} = 1$$

for any point

$$(z_1, \dots, z_{\nu-1}, z_{\nu+1}, \dots, z_{k_1}) \quad \text{with} \quad \sum_{1 \leq j \leq k_1, j \neq \nu} \|z_j\|^{2p_j} \leq 1.$$

Now, since $k_1 \geq 4$, we can always choose an integer m , $1 \leq m \leq k_1$, in such a way that

$$m \neq \mu, \nu \quad \text{and} \quad j := \sigma(m) \neq \mu, \nu.$$

Then, putting $z_\ell = 0$ for $\ell \neq j$ in (2.7), we obtain that

$$\|z_j\|^{2p_m} + (1 - \|z_j\|^{2p_j})^{p_\mu/p_\nu} = 1, \quad \|z_j\| \leq 1.$$

Accordingly, by taking the points xz_j^o with $0 \leq x \leq 1$, $\|z_j^o\| = 1$, we have

$$x^{2p_m} + (1 - x^{2p_j})^{p_\mu/p_\nu} = 1, \quad 0 \leq x \leq 1.$$

A simple computation shows that this can only happen when $p_m = p_j$ and $p_\mu = p_\nu$; completing the proof of the Lemma. $\square$

Hence we have completed the proof of our Theorem.

3 An example

As a concrete example illustrating our result, we here give the following generalized complex ellipsoid $\mathcal{E}$ in $\mathbf{C}^{11}$ defined by

$$\mathcal{E} = \left\{ (z, w_1, w_2, w_3, w_4, w_5, w_6) \in \mathbf{C} \times \mathbf{C} \times \mathbf{C} \times \mathbf{C} \times \mathbf{C}^2 \times \mathbf{C}^2 \times \mathbf{C}^3; \right. \\ \left. |z|^2 + |w_1|^{2/3} + |w_2|^3 + |w_3|^{2/3} + \|w_4\|^3 + \|w_5\|^3 + \|w_6\|^3 < 1 \right\}.$$

So, with the notation of the introduction, we have:

$$K = 6, \quad n_1 = n_2 = n_3 = 1 < n_4 = n_5 = 2 < n_6 = 3, \quad k_1 = 3, \quad k_2 = 2, \quad k_3 = 1 \\ \text{and} \quad \mathcal{E} = E(1, 1, 1, 1, 2, 2, 3; 1, 1/3, 3/2, 1/3, 3/2, 3/2, 3/2).$$

And our Theorem tells us that every element φ of $\text{Aut}(\mathcal{E})$ can be described as

$$\varphi(u) = \left(\xi \frac{z-a}{1-\bar{a}z}, \rho(z)^{3/2} \xi_1 w_{\sigma(1)}, \rho(z)^{1/3} \xi_2 w_{\sigma(2)}, \rho(z)^{3/2} \xi_3 w_{\sigma(3)}, \right. \\ \left. \rho(z)^{1/3} U_4 w_{\sigma(4)}, \rho(z)^{1/3} U_5 w_{\sigma(5)}, \rho(z)^{1/3} U_6 w_{\sigma(6)} \right)$$

for $u = (z, w_1, w_2, w_3, w_4, w_5, w_6) \in \mathcal{E}$, where

$$a, \xi, \xi_1, \xi_2, \xi_3 \in \mathbf{C} \quad \text{with} \quad |a| < 1, \quad |\xi| = |\xi_1| = |\xi_2| = |\xi_3| = 1, \\ U_4, U_5 \in U(2), \quad U_6 \in U(3), \quad \rho(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}, \quad |z| < 1,$$

and σ is a permutation of $\{1, \dots, 6\}$ such that

$$\{\sigma(1), \sigma(3)\} = \{1, 3\}, \quad \{\sigma(4), \sigma(5)\} = \{4, 5\}, \quad \sigma(2) = 2, \quad \sigma(6) = 6.$$

Therefore we conclude that $\text{Aut}(\mathcal{E})$ is a 23-dimensional Lie group with four connected components.

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