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How to sharpen a tridiagonal pair

Tatsuro Ito* and Paul Terwilliger†

Abstract

Let $F$ denote a field and let $V$ denote a vector space over $F$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following conditions: (i) each of $A, A^*$ is diagonalizable; (ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that $A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$, where $V_{-1} = 0$ and $V_{d+1} = 0$; (iii) there exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of $A^*$ such that $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$ for $0 \leq i \leq \delta$, where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$; (iv) there is no subspace $W$ of $V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$. We call such a pair a tridiagonal pair on $V$. It is known that $d = \delta$, and for $0 \leq i \leq d$ the dimensions of $V_i, V_i^*, V_{d-i}, V_{d-i}^*$ coincide. Denote this common dimension by $\rho_i$ and call $A, A^*$ sharp whenever $\rho_0 = 1$. Let $T$ denote the $F$-subalgebra of $\text{End}_F(V)$ generated by $A, A^*$. We show: (i) the center $Z(T)$ is a field whose dimension over $F$ is $\rho_0$; (ii) the field $Z(T)$ is isomorphic to each of $E_0TE_0, E_dTE_d, E_0^*TE_0^*, E_d^*TE_d^*$, where $E_i$ (resp. $E_i^*$) is the primitive idempotent of $A$ (resp. $A^*$) associated with $V_i$ (resp. $V_i^*$); (iii) with respect to the $Z(T)$-vector space $V$ the pair $A, A^*$ is a sharp tridiagonal pair.

Keywords. Tridiagonal pair, Tridiagonal system.


1 Tridiagonal pairs

Throughout this paper $F$ denotes a field and $V$ denotes a vector space over $F$ with finite positive dimension.

We begin by recalling the notion of a tridiagonal pair. We will use the following terms. Let $\text{End}(V) = \text{End}_F(V)$ denote the $F$-algebra of all $F$-linear transformations from $V$ to $V$. Given $A \in \text{End}(V)$ and a subspace $W \subseteq V$, we call $W$ an eigenspace of $A$ whenever $W \neq 0$ and there exists $\theta \in F$ such that $W = \{v \in V | Av = \theta v\}$; in this case $\theta$ is the eigenvalue of $A$ associated with $W$. We say that $A$ is diagonalizable whenever $V$ is spanned by the eigenspaces of $A$.

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**Definition 1.1** [5, Definition 1.1] By a tridiagonal pair (or TD pair) on $V$ we mean an ordered pair $A, A^*$ of elements in $\text{End}(V)$ that satisfy the following four conditions.

(i) Each of $A, A^*$ is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of $A^*$ such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace $W$ of $V$ such that $AW \subseteq W, A^*W \subseteq W, W \neq 0, W \neq V$.

**Note 1.2** According to a common notational convention $A^*$ denotes the conjugate-transpose of $A$. We are not using this convention. In a TD pair $A, A^*$ the linear transformations $A$ and $A^*$ are arbitrary subject to (i)–(iv) above.

We refer the reader to [1, 2, 4–18] and the references therein for background information on TD pairs.

To motivate our results we recall some facts about TD pairs. Let $A, A^*$ denote a TD pair on $V$, as in Definition 1.1. By [5, Lemma 4.5] the integers $d$ and $\delta$ from (ii), (iii) are equal; we call this common value the *diameter* of the pair. An ordering of the eigenspaces of $A$ (resp. $A^*$) is said to be standard whenever it satisfies (1) (resp. (2)). We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of $A$. By [5, Lemma 2.4], the order $\{V_{i-1}^*\}_{i=0}^\delta$ is also standard and no further ordering is standard. A similar result holds for the eigenspaces of $A^*$. Let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) denote a standard ordering of the eigenspaces of $A$ (resp. $A^*$). By [5, Corollary 5.7], for $0 \leq i \leq d$ the spaces $V_i, V_i^*$ have the same dimension; we denote this common dimension by $\rho_i$. By [5, Corollaries 5.7, 6.6] the sequence $\{\rho_i\}_{i=0}^d$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_i \leq \rho_{i+1}$ for $1 \leq i \leq d/2$. We call the sequence $\{\rho_i\}_{i=0}^d$ the *shape* of $A, A^*$. The TD pair $A, A^*$ is called sharp whenever $\rho_0 = 1$. By [15, Theorem 1.3], if $F$ is algebraically closed then $A, A^*$ is sharp. It is an open problem to classify the sharp TD pairs up to isomorphism, but progress is being made [4, 8, 11, 12, 14, 16, 18]. Concerning the nonsharp TD pairs, relatively little research has been done. In this paper we get the ball rolling by proving three results on the subject. Referring to the above TD pair $A, A^*$ let $T$ denote the $F$-subalgebra of $\text{End}(V)$ generated by $A, A^*$. We show: (i) the center $Z(T)$ is a field whose dimension over $F$ is $\rho_0$; (ii) the field $Z(T)$ is isomorphic to each of $E_0 T E_0^*, E_d T E_d, E_0^* T E_0, E_d^* T E_d$, where $E_i$ (resp. $E_i^*$) is the primitive idempotent of $A$ (resp. $A^*$) associated with $V_i$ (resp. $V_i^*$); (iii) with respect to the $Z(T)$-vector space structure on $V$, the pair $A, A^*$ is a sharp TD pair. Our proof is based on the recent discovery that each of $E_0 T E_0, E_d T E_d, E_0^* T E_0, E_d^* T E_d$ is commutative [15, Theorem 2.6]. Essentially our results follow from this and the Wedderburn theory [3, Chapter IV], but for the sake of completeness and accessibility we prove everything from first principles.
2 Tridiagonal systems

When working with a TD pair, it is often convenient to consider a closely related object called a TD system. To define a TD system, we recall a few concepts from linear algebra. Let $A$ denote a diagonalizable element of $\text{End}(V)$. Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of $A$ and let $\{\theta_i\}_{i=0}^d$ denote the corresponding ordering of the eigenvalues of $A$. For $0 \leq i \leq d$ define $E_i \in \text{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_iV_j = 0$ for $j \neq i$ $(0 \leq j \leq d)$. Here $I$ denotes the identity of $\text{End}(V)$. We call $E_i$ the primitive idempotent of $A$ corresponding to $V_i$. Observe that (i) $I = \sum_{i=0}^d E_i$; (ii) $E_iE_j = \delta_{ij}E_i$ $(0 \leq i, j \leq d)$; (iii) $V_i = E_iV$ $(0 \leq i \leq d)$; (iv) $A = \sum_{i=0}^d \theta_iE_i$. Moreover

$$E_i = \prod_{0 \leq j \leq d \atop j \neq i} \frac{A - \theta_jI}{\theta_i - \theta_j}. \quad (3)$$

Note that each of $\{A^i\}_{i=0}^d$, $\{E_i\}_{i=0}^d$ is a basis for the $\mathbb{F}$-subalgebra of $\text{End}(V)$ generated by $A$. Moreover $\prod_{i=0}^d (A - \theta_iI) = 0$. Now let $A, A^*$ denote a TD pair on $V$. An ordering of the primitive idempotents of $A$ (resp. $A^*$) is said to be standard whenever the corresponding ordering of the eigenspaces of $A$ (resp. $A^*$) is standard.

**Definition 2.1** [5, Definition 2.1] By a tridiagonal system (or TD system) on $V$ we mean a sequence $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ that satisfies (i)–(iii) below.

(i) $A, A^*$ is a TD pair on $V$.

(ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of $A$.

(iii) $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of $A^*$.

**Definition 2.2** Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TD system on $V$. For $0 \leq i \leq d$ let $\theta_i$ (resp. $\theta_i^*$) denote the eigenvalue of $A$ (resp. $A^*$) associated with the eigenspace $E_iV$ (resp. $E_i^*V$). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the eigenvalue sequence (resp. dual eigenvalue sequence) of $\Phi$. Observe that $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct and contained in $\mathbb{F}$. By the shape of $\Phi$ we mean the shape of the TD pair $A, A^*$. We say $\Phi$ is sharp whenever $A, A^*$ is sharp.

The following characterization of TD systems will be useful.

**Lemma 2.3** A sequence $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a TD system on $V$ if and only if (i)–(iv) hold below:

(i) Each of $A, A^*$ is a diagonalizable element of $\text{End}(V)$.

(ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of $A$ such that

$$E_iA^*E_j = 0 \quad \text{if} \quad |i - j| > 1, \quad (0 \leq i, j \leq d).$$

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(iii) \( \{E_i^*\}_{i=0}^d \) is an ordering of the primitive idempotents of \( A^* \) such that

\[
E_i^* AE_j^* = 0 \quad \text{if} \quad |i - j| > 1, \quad (0 \leq i, j \leq d).
\]

(iv) There does not exist a subspace \( W \) of \( V \) such that \( AW \subseteq W, A^*W \subseteq W, W \neq 0, W \neq V \).

**Proof:** Routine. \( \square \)

### 3 The algebra \( T \) and its center

For the rest of this paper fix a TD system \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)\) on \( V \), with eigenvalue sequence \( \{\theta_i\}_{i=0}^d \), dual eigenvalue sequence \( \{\theta_i^*\}_{i=0}^d \), and shape \( \{\rho_i\}_{i=0}^d \). Let \( T \) denote the \( F \)-subalgebra of \( \text{End}(V) \) generated by \( A, A^* \). By definition \( T \) contains the identity \( I \) of \( \text{End}(V) \).

By (3) the algebra \( T \) contains each of \( E_i, E_i^* \) for \( 0 \leq i \leq d \). By construction the \( T \)-module \( V \) is faithful. The \( T \)-module \( V \) is irreducible by Lemma 2.3(iv).

**Definition 3.1** An element \( z \in T \) is called **central** whenever \( zt = tz \) for all \( t \in T \). The **center** \( Z(T) \) is the \( F \)-subalgebra of \( T \) consisting of the central elements in \( T \).

**Lemma 3.2** \( Z(T) \) is a field.

**Proof:** Since \( Z(T) \) is commutative, it suffices to show that each nonzero element \( z \in Z(T) \) has an inverse in \( Z(T) \). Define \( K = \{v \in V \mid zv = 0\} \) and observe that \( K \) is a \( T \)-submodule of \( V \). Note that \( K \neq V \) since \( z \neq 0 \) and the \( T \)-module \( V \) is faithful. Now \( K = 0 \) since the \( T \)-module \( V \) is irreducible. Therefore \( z \) is invertible on \( V \). Let \( z^{-1} \in \text{End}(V) \) denote the inverse of \( z \) on \( V \). By elementary linear algebra \( z^{-1} \) is a polynomial in \( z \) and is therefore contained in \( Z(T) \). The result follows. \( \square \)

**Definition 3.3** The center \( Z(T) \) is an \( F \)-subalgebra of \( T \) and therefore an \( F \)-vector space. Let \( \rho \) denote the dimension of the \( F \)-vector space \( Z(T) \).

In Lemma 4.11 we will show that the parameter \( \rho \) from Definition 3.3 is equal to \( \rho_0 \).

For the \( T \)-module \( V \) the restriction of the \( T \)-action to \( Z(T) \) gives a \( Z(T) \)-vector space structure on \( V \). Each of \( A, A^* \) commutes with everything in \( Z(T) \) and is therefore a \( Z(T) \)-linear transformation. Let \( \text{End}_{Z(T)}(V) \) denote the \( Z(T) \)-algebra consisting of the \( Z(T) \)-linear transformations from \( V \) to \( V \). By the construction \( A, A^* \in \text{End}_{Z(T)}(V) \).

Of course every \( Z(T) \)-subspace of \( V \) is an \( F \)-subspace of \( V \), but not every \( F \)-subspace of \( V \) is a \( Z(T) \)-subspace of \( V \).

**Lemma 3.4** For each \( Z(T) \)-subspace \( W \) of \( V \),

\[
\rho \dim_{Z(T)}(W) = \dim_F(W).
\]
Proof: $W$ is the direct sum of $\dim_{Z(T)}(W)$ many one-dimensional $Z(T)$-subspaces of $V$. By Definition 3.3, each of these has dimension $\rho$ as an $F$-subspace of $V$. The result follows. \[\square\]

**Lemma 3.5** Let $\psi$ denote any element of $\text{End}_{Z(T)}(V)$. If $\psi$ is diagonalizable when viewed as an $F$-linear transformation on the $F$-vector space $V$, then $\psi$ is diagonalizable when viewed as a $Z(T)$-linear transformation on the $Z(T)$-vector space $V$. In this case the eigenspaces, eigenvalues, and primitive idempotents of $\psi$ are the same for the two points of view.

Proof: For $\theta \in F$ the set $\{v \in V \mid \psi v = \theta v\}$ is the same for the two points of view. \[\square\]

**Lemma 3.6** There does not exist a $Z(T)$-subspace $W$ of $V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

Proof: By Lemma 2.3(iv) there does not exist an $F$-subspace $W$ of $V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$. The result follows since each $Z(T)$-subspace of $V$ is an $F$-subspace of $V$. \[\square\]

**Proposition 3.7** The elements $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ act on the $Z(T)$-vector space $V$ as a TD system. This TD system has eigenvalue sequence $\{\theta_i\}_{i=0}^d$, dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$, and shape $\{\rho_i/\rho\}_{i=0}^d$.

Proof: To verify the first assertion, we check that $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ satisfy the conditions of Lemma 2.3. Concerning condition (i), we mentioned below Definition 3.3 that each of $A, A^*$ is contained in $\text{End}_{Z(T)}(V)$. Using Lemma 3.5 we find that the map $A$ (resp. $A^*$) is diagonalizable as a $Z(T)$-linear transformation on the $Z(T)$-vector space $V$, since $A$ (resp. $A^*$) is diagonalizable as an $F$-linear transformation on the $F$-vector space $V$. Using Lemma 3.5 we find that conditions (ii), (iii) hold when $A, A^*$ are viewed as $Z(T)$-linear transformations on the $Z(T)$-vector space $V$, since they hold when $A, A^*$ are viewed as $F$-linear transformations on the $F$-vector space $V$. Condition (iv) holds by Lemma 3.6. Our first assertion is now verified. The assertions about the eigenvalue sequence and dual eigenvalue sequence follow from Definition 2.2 and Lemma 3.5. The assertion about the shape follows from Lemma 3.4 and since for $0 \leq i \leq d$, each of $E_iV, E_i^*V$ is a $Z(T)$-subspace of $V$ which has dimension $\rho_i$ as an $F$-subspace of $V$. \[\square\]

### 4 The algebras $E_0TE_0$, $E_dTE_d$, $E_0^*TE_0^*$, $E_d^*TE_d^*$

In this section we show how the center $Z(T)$ is related to $E_0TE_0$, $E_dTE_d$, $E_0^*TE_0^*$, $E_d^*TE_d^*$. Invoking symmetry we will restrict our attention to $E_0^*TE_0^*$. We view $E_0^*TE_0^*$ as an $F$-algebra with multiplicative identity $E_0^*$. Our first goal is to show that this algebra is a field. Our point of departure is the following recent discovery.
Lemma 4.1 [15, Theorem 2.6] The algebra $E^*_0TE^*_0$ is commutative and generated by
$$E^*_0AIE^*_0$$ \quad (1 \leq i \leq d).

Lemma 4.2 The $E^*_0TE^*_0$-module $E^*_0V$ is irreducible.

Proof: Let $W$ denote a nonzero $E^*_0TE^*_0$-submodule of $E^*_0V$. We show that $W = E^*_0V$. Pick any nonzero $w \in W$ and observe $E^*_0TE^*_0w \subseteq W$. The element $E^*_0$ acts on $E^*_0V$ as the identity so $E^*_0Tw \subseteq W$. Observe that $Tw$ is a nonzero $T$-submodule of $V$. The $T$-module $V$ is irreducible so $Tw = V$. Therefore $E^*_0Tw = E^*_0V$. By the above comments $W = E^*_0V$ and the result follows. \hfill \Box

Lemma 4.3 For all nonzero $s \in TE^*_0$ the restriction of $s$ to $E^*_0V$ is injective.

Proof: We define $K = \{ v \in E^*_0V | sv = 0 \}$ and show $K = 0$. In view of Lemma 4.2 it suffices to show that $K$ is invariant under $E^*_0TE^*_0$, and that $K \neq E^*_0V$. We now show that $K$ is invariant under $E^*_0TE^*_0$. For all $k \in K$ and $t \in T$ we show $E^*_0te^*_0k \in K$. By construction $E^*_0te^*_0k \in E^*_0V$, so we just have to check that $se^*_0te^*_0k = 0$. Suppose $se^*_0te^*_0k \neq 0$. Then $TsE^*_0te^*_0k = V$ by the irreducibility of the $T$-module $V$, so $E^*_0TsE^*_0te^*_0k = E^*_0V$. Since $E^*_0TE^*_0$ is commutative and $Ts \subseteq T$, each element of $E^*_0TsE^*_0$ commutes with $E^*_0te^*_0$. Using this and $E^*_0^2 = E^*_0$ we obtain
$$E^*_0te^*_0TsE^*_0k = E^*_0V.$$ \hfill (4)

We have $se^*_0 = s$ since $s \in TE^*_0$, and $sk = 0$ by construction, so the left-hand side of (4) is zero. Of course the right-hand side of (4) is nonzero and we have a contradiction. Therefore $se^*_0te^*_0k = 0$, so $E^*_0te^*_0k \in K$. We have now shown that $K$ is invariant under $E^*_0TE^*_0$. We now show that $K \neq E^*_0V$. Suppose $K = E^*_0V$. Then $s$ is zero on $E^*_0V$ so $se^*_0 = 0$. But $se^*_0 = s$, so $s = 0$ for a contradiction. Therefore $K \neq E^*_0V$. We have shown that $K$ is invariant under $E^*_0TE^*_0$, and that $K \neq E^*_0V$. Therefore $K = 0$ in view of Lemma 4.2. The result follows. \hfill \Box

Observe that $TE^*_0$ has a $T$-module structure such that $t.s = ts$ for all $t \in T$ and $s \in TE^*_0$.

Lemma 4.4 For all nonzero $v \in E^*_0V$ the map
$$TE^*_0 \rightarrow V$$
$$s \mapsto sv$$

is an isomorphism of $T$-modules.

Proof: By construction the map is a homomorphism of $T$-modules. The map is injective by Lemma 4.3. To see that the map is surjective, note that its image is a $T$-submodule of $V$, and nonzero since it contains $v$. This image is equal to $V$ by the irreducibility of the $T$-module $V$. \hfill \Box

We emphasize one point from Lemma 4.4.
Corollary 4.5 The $T$-module $TE_0^*$ is irreducible and faithful.

Proof: Immediate from Lemma 4.4 and since the $T$-module $V$ is irreducible and faithful. \(\square\)

Proposition 4.6 $E_0^*TE_0^*$ is a field.

Proof: Since $E_0^*TE_0^*$ is commutative, it suffices to show that each nonzero $a \in E_0^*TE_0^*$ has an inverse in $E_0^*TE_0^*$. Since $a$ is nonzero and contained in $TE_0^*$, the space $Ta$ is a nonzero $T$-submodule of $TE_0^*$. The $T$-module $TE_0^*$ is irreducible so $Ta = TE_0^*$. Therefore there exists $t \in T$ such that $ta = E_0^*$. Define $b = E_0^*tE_0^*$ and observe $b \in E_0^*TE_0^*$. Since $a \in E_0^*TE_0^*$ we find $E_0^*a = a$; using this and $ta = E_0^*$ we find $ba = E_0^*$. Therefore $b$ is the desired inverse of $a$ and the result follows. \(\square\)

Our next goal is to show that the fields $Z(T)$, $E_0^*TE_0^*$ are isomorphic. To this end it is helpful to define a certain bilinear form.

Definition 4.7 Observe that $TE_0^*$ (resp. $E_0^*T$) has a right (resp. left) $E_0^*TE_0^*$-module structure. Using these structures we view each of $TE_0^*$, $E_0^*T$ as a vector space over the field $E_0^*$. Over this field we define a bilinear form $(,)$: $E_0^*T \times TE_0^* \rightarrow E_0^*TE_0^*$ such that $(r,s) = rs$ for all $r \in E_0^*T$ and $s \in TE_0^*$.

Lemma 4.8 The bilinear form $(,)$ from Definition 4.7 is nondegenerate.

Proof: Given $r \in E_0^*T$ such that $(r,s) = 0$ for all $s \in TE_0^*$, we show $r = 0$. By the construction $rTE_0^* = 0$. Now $r = 0$ since the action of $T$ on $TE_0^*$ is faithful. Given $s \in TE_0^*$ such that $(r,s) = 0$ for all $r \in E_0^*T$, we show $s = 0$. By construction $E_0^*Ts = 0$. Suppose $s \neq 0$. Then $Ts$ is a nonzero $T$-submodule of $TE_0^*$, so $Ts = TE_0^*$ by the irreducibility of the $T$-module $TE_0^*$. Therefore $E_0^*Ts = E_0^*TE_0^*$. In this equation the left hand side is zero and the right hand side is nonzero, for a contradiction. Hence $s = 0$ and the result follows. \(\square\)

Definition 4.9 By Lemma 4.8 the $E_0^*TE_0^*$-vector spaces $TE_0^*$ and $E_0^*T$ have the same dimension which we denote by $n$. Let $\{x_i\}_{i=1}^n$ denote a basis for the $E_0^*TE_0^*$-vector space $TE_0^*$ and let $\{x_i\}_{i=1}^n$ denote the basis for the $E_0^*TE_0^*$-vector space $E_0^*T$ which is dual to $\{x_i\}_{i=1}^n$ with respect to $(,)$. By construction

\[ x_i^*x_j = \delta_{i,j}E_0^* \quad (1 \leq i, j \leq n). \quad (5) \]

Lemma 4.10 The map

\[ Z(T) \rightarrow E_0^*TE_0^* \]
\[ z \mapsto zE_0^* \quad (6) \]

is an isomorphism of fields.
Proof: For \( z \in Z(T) \) we have \( zE_0^* = E_0^*zE_0^* \), since \( E_0^*E_0^* = E_0^* \) and \( zE_0^* = E_0^*z \). Therefore \( zE_0^* \in E_0^*T E_0^* \). By construction the map defined in line (6) is an \( F \)-algebra homomorphism. We now show that this map is a bijection. The map is injective since a field has no nonzero proper ideals. To see that the map is surjective, for a given \( a \in E_0^*T E_0^* \) we display \( z \in Z(T) \) such that \( zE_0^* = a \). Define \( z = \sum_{i=1}^{n} x_i a x_i^t \) where \( \{ x_i \}_{i=1}^{n} \) and \( \{ x_i^t \}_{i=1}^{n} \) are from Definition 4.9. Using (5) and \( a = aE_0^* \) we obtain \( zz_j = x_j a \) for \( 1 \leq j \leq n \). Recall \( \{ x_j \}_{j=1}^{n} \) is a basis for the \( E_0^*T E_0^* \)-vector space \( T E_0^* \), so \( z \) acts on \( T E_0^* \) as \( s \mapsto sa \). Using this we find that for all \( t \in T \) the expression \( zt - tz \) vanishes on \( T E_0^* \), yielding \( zt = tz \) since the \( T \)-module \( T E_0^* \) is faithful. Therefore \( z \in Z(T) \). We mentioned above that \( zs = sa \) for all \( s \in T E_0^* \); taking \( s = E_0^* \) we find \( zE_0^* = E_0^*a = a \). We have shown \( z \in Z(T) \) and \( zE_0^* = a \), so the the map is surjective. The result follows. \( \square \)

We need one fact about dimensions and then we will be ready for the main result.

**Lemma 4.11** With reference to Definition 3.3, we have \( \rho = \rho_0 \).

*Proof:* By Definition 3.3 the parameter \( \rho \) is the dimension of the \( F \)-vector space \( Z(T) \). The map \( Z(T) \to E_0^*T E_0^* \) from Lemma 4.10 is an isomorphism of \( F \)-vector spaces, so the \( F \)-vector spaces \( Z(T) \), \( E_0^*T E_0^* \) have the same dimension. The map \( T E_0^* \to V \) from Lemma 4.4 is an isomorphism of \( T \)-modules and hence an isomorphism of \( F \)-vector spaces. Under this map the image of \( E_0^*T E_0^* \) is \( E_0^*V \), so the \( F \)-vector spaces \( E_0^*T E_0^* \) and \( E_0^*V \) have the same dimension. By construction the \( F \)-vector space \( E_0^*V \) has dimension \( \rho_0 \). The result follows. \( \square \)

Combining our above results we immediately obtain the following.

**Theorem 4.12** Let \( (A; \{ E_i \}_{i=0}^{d}; A^*; \{ E_i^* \}_{i=0}^{d}) \) denote a TD system on the \( F \)-vector space \( V \), with eigenvalue sequence \( \{ \theta_i \}_{i=0}^{d} \), dual eigenvalue sequence \( \{ \theta_i^* \}_{i=0}^{d} \), and shape \( \{ \rho_i \}_{i=0}^{d} \). Let \( T \) denote the \( F \)-subalgebra of \( \text{End}_F(V) \) generated by \( A, A^* \). Then the following (i)–(iii) hold.

(i) The center \( Z(T) \) is a field whose dimension over \( F \) is \( \rho_0 \).

(ii) The field \( Z(T) \) is isomorphic to each of \( E_0^*T E_0^* \), \( E_0^*E_0^* \), \( E_0^*T E_0^* \), \( E_0^*E_0^*T E_0^* \).

(iii) The elements \( (A; \{ E_i \}_{i=0}^{d}; A^*; \{ E_i^* \}_{i=0}^{d}) \) act on the \( Z(T) \)-vector space \( V \) as a TD system that has eigenvalue sequence \( \{ \theta_i \}_{i=0}^{d} \), dual eigenvalue sequence \( \{ \theta_i^* \}_{i=0}^{d} \), and shape \( \{ \rho_i/\rho_0 \}_{i=0}^{d} \). In particular this TD system is sharp.

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