On the values at zero of partial zeta functions for ray classes of a real quadratic field II.

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Abstract. Let k be a real quadratic field. Let α be an integer of k and m be a positive rational integer. Denote by $\zeta(\mathfrak{a},(m),s)$ be a partial zeta function associated to a ray class $\mathfrak{a}$ containing the principal ideal (α) defined with a conductor (m). We give a formula of the value of $\zeta(\mathfrak{a},(m),0)$ by applying the Shintani method to compute special values of partial zeta functions in [2]. In this successive paper, we show a little improvement of the formula and several computational examples by using Pari/GP system.

1. Introduction. We study the value at zero of a partial zeta function on a real quadratic field in [4]. The motivation and aim of this work is to pursue an analogous formula of the relative class number of a cyclotomic field given as a determinant of a rational matrix, which has been studied by many authors. It is interpreted as a product of values at zero of Dirichlet L-functions for odd characters having a same conductor. In other words, the value at zero of the Artin L-function associated with the induced representation of the nontrivial character of the group H generated by complex conjugation. This value is described in terms of representation theory of abelian groups, which means the matrix representation afforded by multiplication of a certain element of the group ring.

We need to study the formula to extend it to the ray class field of a real quadratic field. Since it is well-known that every ray class fields on a real quadratic field are not CM-fields, we can not talk about relative class number formulas in general. Instead of this, we consider a value of the Artin L-function associated with an induced representation. We consider in §2 which subgroup and which character are adequate to this purpose.

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Secondly, we make a little progress to improve the formula in §3. We get several examples by computing with using Pari/GP System and some tables are in §4.

2. Representation of a finite abelian group. Let G be a finite abelian group and H be a subgroup. We fix a complex character ψ of H and denote by $\chi_1, \cdots, \chi_n$ the preimage of ψ by restriction of characters of G onto H. Each character χ_i is a character of a complex representation of a subalgebra of the group ring of G over $\boldsymbol{C}$. Let A_i be the subalgebra. It is of one dimension, where an arbitrary $\sigma \in G$ acts as $\sigma x = \chi_i(\sigma)x$. Let $\widetilde{\psi}$ is the induced character

$$\widetilde{\psi} = \mathrm{Ind}_H^G \psi = \sum_{i=1}^{n} \chi_i.$$

This character is afforded with the G-algebra

$$A_\psi = \oplus_{i=1}^n A_i.$$

Let e be the unit of this algebra. We fix a section of the canonical homomorphism of G onto G/H. The cardinal number of the section equals n. Let $\{\sigma_1, \cdots, \sigma_n\}$ be the section. We see $\{\sigma_1^{-1}e, \cdots, \sigma_n^{-1}e\}$ is a basis of the algebra A_ψ. Therefore, the matrix

$$A = \begin{pmatrix} \chi_1(\sigma_1^{-1}) & \cdots & \chi_n(\sigma_1^{-1}) \\ \vdots & & \vdots \\ \chi_1(\sigma_n^{-1}) & \cdots & \chi_n(\sigma_n^{-1}) \end{pmatrix}$$

is regular. Let $h(i,j) = h(\sigma_i, \sigma_j)$ be the factor system of extension of a group G/H by H associated with the section, that is for σ_i and σ_j, there are σ_l and $h(i,j) \in H$ such that

$$\sigma_i \sigma_j = \sigma_l h(i,j). \tag{1}$$

Here, the index l is equal to one of n numbers $1, \cdots, n$ such that $(\sigma_i\sigma_j)H = \sigma_l H$ holds. We regard this l as a variable which is a function of two variables i and j. When we fix a value of i, the variable l is a partial function of a variable j and this correspondence is bijective.

Let f be a complex valued function of G satisfying a relation

$$f(\sigma h) = f(\sigma)\psi(h) \tag{2}$$

for every $\sigma \in G$ and every $h \in H$. We make this function f correspond with an element of the group ring defined by

$$\theta_f = \sum_{j=1}^{n} f(\sigma_j)\sigma_j^{-1}.$$

We shorten notation in (1) and write as $\sigma_i\sigma_j = \sigma_l h$. With replacing $\sigma_i^{-1}\sigma_l h$ with σ_j, we have

$$\begin{aligned}\theta_f &= \sum_{j=1}^{n} f(\sigma_i^{-1}\sigma_l h)(\sigma_i^{-1}\sigma_l h)^{-1} \\ &= \sum_{j=1}^{n} f(\sigma_i^{-1}\sigma_l)\psi(h)h^{-1}\sigma_l^{-1}\sigma_i.\end{aligned}$$

Thus,

$$\theta_f \sigma_i^{-1} = \sum_{l=1}^{n} f(\sigma_i^{-1}\sigma_l)\psi(h)h^{-1}\sigma_l^{-1}. \tag{3}$$

Let D_f be the matrix of representation obtained by multiplication of θ_f on the algebra A_ψ:

$$D_f = \begin{pmatrix} \chi_1(\theta_f) & & \\ & \ddots & \\ & & \chi_n(\theta_f) \end{pmatrix}.$$

We denote by A_f the matrix $(f(\sigma_p^{-1}\sigma_q))$. We write down the system consisting of the equalities yielded by (3) as

$$A_f A = A D_f$$

We take determinants and divide both sides by $|A|$.

THEOREM 1. *We have* $\widetilde{\psi}(\theta_f) = |A_f|$.

We are interested in this formula in arithmetical context, which is familiar in theory of cyclotomic fields. We recall the relation between the values of a L-function at non-positive integers and the values of Bernoulli polynomials in the view point of Theorem 1. Let m be a positive integer which is greater than two. Set G to the ray class group of $\boldsymbol{Q}$ defined with a conductor m. By class field theory, there is an exact sequence

$$1 \to \{\pm 1\} \to (\boldsymbol{Z}/m\boldsymbol{Z})^\times \times \boldsymbol{R}^\times/(\boldsymbol{R}^\times)^2 \to G \to 1$$

where the factor $\boldsymbol{R}^\times/(\boldsymbol{R}^\times)^2$ in the middle comes from the archimedian place. By virtue of this sequence, we may identify the ray class group with the multiplicative group $(\boldsymbol{Z}/m\boldsymbol{Z})^\times$. Hence, we regard G as a subset

$$\{a : 1 \le a < m,\ (a, m) = 1\}$$

of minimal non-negative residue by m. To avoid confusion, we write σ_a if the integer a means an element of G. We take the subgroup generated by σ_{m-1} as the subgroup H. Let ψ_0 be the non-trivial character of H. We choose the Bernoulli polynomial $B_k(x)$ to define the function f:

$$f(\sigma_a) = B_k\left(\frac{a}{m}\right).$$

Since $B_k(1-x) = (-1)^k B_k(x)$, a reasonable choice of the character ψ is to be $\psi = \psi_0^k$. Namely, this character and the function f satisfy the relation (2). If we consider each character χ_i as a Dirichlet character with a conductor m, the value of the Dirichlet L-function associated with the character χ_j has values

$$L(1-k, \chi_j) = -\frac{m^{k-1}}{k}\chi_j(\theta_f)$$

for positive integers $k = 1, 2, \cdots$, *c.f.* Theorem 2.9 in [3]. Since the Artin L-function $L(s, \widetilde{\psi})$ is a product of the Dirichlet L-functions, its values at non-positive integers are equal to the determinants of the rational matrices in Theorem 1.

We will pursue an analogue of this relation between values of the Artin L-function and certain rational matrices in a real quadratic field. For this purpose, we study the partial zeta function associated with a ray class. Denote by $\mathfrak{a}$ the ray class defined with conducutor $\mathfrak{m}$ which contains a principal ideal (a). When we take real quadratic field to the base field and apply the formula of values of of the partial zeta function, we may be possible to set

$$f(\sigma_a) = -\zeta(\mathfrak{a}, \mathfrak{m}, 1-k).$$

Here, we write the partial zeta function following to notation in [2]. In order to study this relation, we need to obtain valus of the partial zeta function.

3. Review on ray class groups on a real quadratic field. Let d be a square free positive integer greater than one. Let $k = \boldsymbol{Q}(\sqrt{d})$ be a real quadratic field. Let $\mathfrak{m}$ be an integral ideal generated by a positive integer m. Denote by $k(\mathfrak{m})$ the ray class field over k defined with a conductor $\mathfrak{m}$. Put $G_{\mathfrak{m}} = Gal(k(\mathfrak{m})/k)$. Let A be the subgroup of the ideal class group of k generated by every prime ideals not dividing $\mathfrak{m}$. The subset of every principal ideals contained in A forms a subgroup. We denote this subgroup by P. Let $H_{\mathfrak{m}}$ be the principal congruent subgroup mod $\mathfrak{m}$, which is a subgroup of P. There is the Artin map of $A/H_{\mathfrak{m}}$ onto $G_{\mathfrak{m}}$ by class field theory. Denote by φ the Artin map of $A/H_{\mathfrak{m}}$. Let $\mathfrak{p}$ be a place of k. When the place is non-archimedian, this symbol means the corresponding valuation ideal of the ring of integers of k, simultaneously. Denote by $k_{\mathfrak{p}}$ the completion of k at a place $\mathfrak{p}$. Let U_0 be the product of groups $U_{\mathfrak{p}}$ of local units of every finite places such that their associated valuation ideals contain m. Let U_∞ be the product of multiplicative groups $k_{\mathfrak{p}}^\times$ of every archimedian places. We have a homomorphisms into $G_{\mathfrak{m}}$

$$\varphi_0 : U_0 \longrightarrow G_{\mathfrak{m}}, \qquad \varphi_\infty : U_\infty \longrightarrow G_{\mathfrak{m}}$$

by virtue of the Hilbert theory concerning ramifications and local class field theory at every $\mathfrak{p}|\mathfrak{m}\infty$. These three maps have a relation

$$\varphi((\alpha))^{-1} = \varphi_0(\alpha)\varphi_\infty(\alpha)$$

for $\alpha \in k^\times$ by reciprocity law of class field theory, where the ideal (α) is prime to $\mathfrak{m}$. Put $U = U_0 \times U_\infty$. The kernel of $\varphi_0\varphi_\infty$ is a subgroup of U generated by a subgroup $U(\mathfrak{m}\infty)$ defined in [4] and the image E of the global unit group of k. Let L be the Hilbert class field of k. We have an isomorphism

$$U/U(\mathfrak{m}\infty)E \cong Gal(k(\mathfrak{m})/L)$$

induced from these homomorphisms φ_0 and φ_∞. We set $G = Gal(k(\mathfrak{m})/L)$. Let $\mathfrak{a}$ be a ray class containing the principal ideal (a). We define the function f from values at zero of the partial zeta function associated with ray class $\mathfrak{a}$:

$$f(\mathfrak{a}) = \zeta(\mathfrak{a}, \mathfrak{m}, 0).$$

We note that the ray class field $k(\mathfrak{m})$ is not a CM-field in general. For example, when $k = \boldsymbol{Q}(\sqrt{2})$ and $m = 8$, we have $\varphi_0(-1) \neq 1$. Hence, by reciprocity law, we see $\varphi_\infty(-1) \neq 1$. This means the complex conjugation maps on the two archimedian places of k define different elements in $G_{\mathfrak{m}}$, which is equivalent to that $k(\mathfrak{m})$ is not a CM-field. We expects the following holds

Let a and b be integers of k which are prime to m and such that $\varphi_0(a) = \varphi_0(b)$. Let (s_1, s_2) be the sign of ab^{-1} in U_∞/U_∞^2. Then, there is a relation $f(\mathfrak{a}) = s_1 s_2 f(\mathfrak{b})$.

If this holds, we may choose H as a subgroup of $\mathrm{Im}(\varphi_\infty)$.

We recall notation in [4]. We embed the real quadratic field into the real numbers by selecting one of two archimedian places and fixing it once for all. Set

$$w = \begin{cases} \sqrt{d} & \text{if } d \not\equiv 1 \mod 4, \\ \dfrac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \mod 4. \end{cases}$$

The ring of integers of k is a free module of rank two with a basis $\{1, w\}$. Since the multiplicative group of totally positive units is a free abelian group of rank one, we denote by ε_+ a generator such that $\varepsilon_+ = x + yw$ for $x > 0$ and $y > 0$. Let l be the order of $\varepsilon_+ \mod m$ and put $\varepsilon = \varepsilon_+^l$. We see there is positive integers a_1 and b_1 such that

$$\varepsilon = (1 + ma_1) + mb_1 w.$$

Let α be an integer of $k^\times$ such that $((\alpha), (m)) = 1$. α is decomposed into a product

$$\alpha = c'\beta, \quad c' > 0, \quad \beta = f_1 + f_2 w, \ (f_1, f_2) = 1.$$

Let c be the minimal non-negative residue of c' by m. Denote by N_0 the absolute value of norm of β. Let N be the product of b_1 and N_0. Namely,

$$mN = mb_1 N_0.$$

We see $(f_2, N_0) = 1$ from §4 of [4]. Let λ_1 be an arbitrary solution of a congruent equation $f_2 x \equiv f_1 + f_2 Tr(w) \mod N_0$ in the variable x. We define a constant γ to be

$$\gamma = \begin{cases} a_1 & \text{if } f_2 = 0, \\ a_1 + b_1 Tr(w) & \text{if } f_1 = 0, \\ a_1 + b_1 \lambda_1 & \text{if } f_1 f_2 \neq 0 \end{cases}$$

and put $\delta = 1 + m\gamma$. We have $(\gamma, mN) = 1$ by Lemma 8, [4]. A rational number μ_s is defined to be

$$\mu_s' = -\frac{c}{m} + \frac{\delta s}{mN}$$

for each integer s contained in the interval $[0, mN)$. We showed the following formula of the value at zero of the partial zeta function holds in [4]:

THEOREM 2. *We have*

$$\zeta(\mathfrak{a}, (m), 0) = \frac{Tr(\varepsilon)}{12mN} - S + B_1\left(1 - \frac{c}{m}\right),$$

where

$$(4) \qquad S = \sum_{s=0}^{mN-1} B_1(<\mu_s'>) B_1\left(\frac{s}{mN}\right).$$

The value of mN appearing in this formula may be very large. This is an obstacle when we try to calculate the value of the partial zeta function by using this formula. For example, the values of $b = mN/N_0$ are growing as we can observe in the following table when $k = \boldsymbol{Q}(\sqrt{2})$:

l	b
1	2
2	2^2 3
3	2 57
4	2^3 3 17
5	2 29 41
6	2^2 3^2 5 7 11
7	2 13^2 239
8	2^4 3 17 577
9	2 5 7 197 199
10	2^2 3 19 29 41 59
11	2 23 353 5741
12	2^3 3^2 5 7 11 17 1153
13	279 599 33461
14	2^2 3 13^2 113 239 337
15	2 5^2 7 29 31^2 41 269
16	2^5 3 17 577 665857

Let δ' be an arbitrary integer satisfying $\delta'\delta \equiv 1 \mod mN$. Since $\delta \equiv 1 \mod m$, we see $\delta' \equiv 1 \mod m$. There is a unique integer t for each s in the interval $[0, mN)$ such that

$$s \equiv cN + \delta' t \mod mN.$$

We rewite the the right hand side in the equality (4) by using this congruence and obtain a formula

$$\begin{aligned} S &= \sum_{t=0}^{mN-1} B_1\left(\frac{t}{mN}\right) B_1\left(\left\langle\frac{c}{m}+\frac{\delta' t}{mN}\right\rangle\right) \\ &= \sum_{t=0}^{mN-1} B_1\left(\left\langle\frac{\delta t}{mN}\right\rangle\right) B_1\left(\left\langle\frac{c}{m}+\frac{t}{mN}\right\rangle\right). \end{aligned}$$

The following lemma is easily proved:

LEMMA 3. *Let x and y be real numbers. Then, we have*

$$< x+y >=< x > + < y > -[< x > + < y >].$$

It follows from this lemma that the value of $B_1\left(\left\langle\frac{c}{m}+\frac{t}{mN}\right\rangle\right)$ equals

$$\frac{c}{m} + B_1\left(\frac{t}{mN}\right) - \left[\frac{c}{m}+\frac{t}{mN}\right].$$

Applying this equality, we obtain

$$S = S_1 - \frac{c}{2m} - T_1,$$

where

$$\begin{aligned} S_1 &= \sum_{t=0}^{mN-1} B_1\left(\left\langle\frac{\delta t}{mN}\right\rangle\right) B_1\left(\frac{t}{mN}\right), \\ T_1 &= \sum_{t=0}^{mN-1} B_1\left(\left\langle\frac{\delta t}{mN}\right\rangle\right) \left[\frac{c}{m}+\frac{t}{mN}\right] \\ &= \sum_{t=(m-c)N}^{mN-1} B_1\left(\left\langle\frac{\delta t}{mN}\right\rangle\right). \end{aligned}$$

Let N_2 be the greatest common divisor of $N = b_1 N_0$ and γ. Their quotients by N_2 are denoted by N_1 and κ, respectively. We have $N = N_1 N_2$ and

$$\frac{\delta}{mN} = \frac{1+m\gamma}{mN} = \frac{1}{mN} + \frac{\kappa}{N_1}$$

By applying Lemma 3 again, we can expand $B_1\left(\left\langle\frac{\delta t}{mN}\right\rangle\right)$ into four terms

$$B_1\left(\frac{t}{mN}\right) + B_1\left(\left\langle\frac{\kappa t}{N_1}\right\rangle\right) + \frac{1}{2} - \left[\frac{t}{mN} + \left\langle\frac{\kappa t}{N_1}\right\rangle\right].$$

We introduce sums

$$\begin{aligned} T_2 &= \sum_{t=0}^{mN-1} \left[\frac{t}{mN} + \left\langle\frac{\kappa t}{N_1}\right\rangle\right] B_1\left(\frac{t}{mN}\right), \\ S_2 &= \sum_{t=0}^{mN-1} B_1\left(\left\langle\frac{\kappa t}{N_1}\right\rangle\right) B_1\left(\frac{t}{mN}\right) \end{aligned}$$

and modify S_1 to be

$$S_1 = \sum_{t=0}^{mN-1} B_1\left(\frac{t}{mN}\right)^2 - \frac{1}{4} + S_2 - T_2.$$

Since $B_1(x)^2 = B_2(x) + \frac{1}{12}$, we see

$$\sum_{t=0}^{mN-1} B_1\left(\frac{t}{mN}\right)^2 = (mN)^{1-2}B_2(0) + \frac{mN}{12}.$$

To reduce the sum S_2, we divide values of the variable t into two parts:

$$t = t_0 + N_1 t_1,$$

where t_0 runs over a set of integers $\{0, \cdots, N_1 - 1\}$ and t_1 dose over a set $\{0, \cdots mN_2 - 1\}$. By this decomposition, the sum is converted to a double sum

$$\sum_{t_0=0}^{N_1-1}\sum_{t_1=0}^{mN_2-1} B_1\left(\left\langle\frac{\kappa t_0}{N_1}\right\rangle\right) B_1\left(\frac{t_0}{mN} + \frac{t_1}{mN_2}\right).$$

By virtue of the distribution relation, this double sum is equal to

$$S_2 = \sum_{t_0=0}^{N_1-1} B_1\left(\left\langle\frac{\kappa t_0}{N_1}\right\rangle\right) B_1\left(\frac{t_0}{N_1}\right).$$

We collect all terms and put together.

THEOREM 4. *The main term S in the formula of Theorem 1 equals a sum*

$$\frac{1}{6mN} + \frac{mN}{12} - \frac{3c}{2m} + \frac{1}{4} + S_2 - T_1 - T_2.$$

COROLLARY 5. *We have the value of $\zeta(\mathfrak{a}, (m), 0)$ is equal to a sum*

$$\frac{Tr(\varepsilon) - 2}{12mN} - \frac{mN}{12} + \frac{c}{2m} + \frac{1}{4} - S_2 + T_1 + T_2.$$

4. Some examples.

We obtain values of partial zeta functions at $s = 0$ by making macro programs with Pari/GP that run on Windows 10 and compute values of the formula of Corollary 5 for small m's. The item "order" in the tables is that consisiting of orders of corresponding ray classes. The item $\pm N_0$ in the following tables means the value of $N(\beta)(= \pm N_0)$.

(1) $k = \boldsymbol{Q}(\sqrt{2})$.

$m = 2,\ l = 1$

α	order	N_0	$\zeta(\mathfrak{a}, \mathfrak{m}, 0)$
1	1	1	$1/4$
$3 + w$	2	7	$-1/4$

$m = 3,\ l = 4$

α	order	N_0	$\zeta(\mathfrak{a}, \mathfrak{m}, 0)$
1	1	1	$1/3$
$4 + w$	2	14	$-1/3$

$m = 4,\ l = 2$

α	order	N_0	$\zeta(\mathfrak{a}, \mathfrak{m}, 0)$
1	1	1	$1/8$
$5 + 3w$	2	7	$-1/8$
3	2	1	$1/8$
$5 + w$	2	23	$-1/8$

$m = 5,\ l = 6$

α	order	N_0	$\zeta(\mathfrak{a}, \mathfrak{m}, 0)$
1	1	1	$-1/5$
2	2	1	$1/5$
$5 + 2w$	4	17	$3/5$
$5 + w$	4	23	$-3/5$

$m = 6,\ l = 4$

α	order	N_0	$\zeta(\mathfrak{a}, \mathfrak{m}, 0)$
1	1	1	$1/3$
$3 + w$	2	7	$1/3$
$7 + 2w$	2	41	$-1/3$
$7 + w$	2	47	$-1/3$

$m=7,\ l=3$

α	order	N_0	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	2/7
6	2	1	2/7
$7+2w$	2	41	$-2/7$
$14+5w$	2	146	$-2/7$
2	3	1	1/7
4	3	1	4/7
3	6	1	4/7
5	6	1	1/7
$7+4w$	6	17	$-1/7$
$7+3w$	6	31	$-1/7$
$7+w$	6	47	$-4/7$
$14+6w$	6	124	$-4/7$

$m=8,\ l=4$

α	order	N_0	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	5/16
3	2	1	$-3/16$
5	2	1	$-3/16$
7	2	1	5/16
$3+w$	2	7	3/16
$9+3w$	2	63	$-5/16$
$9+w$	2	79	$-5/16$
$11+3w$	2	103	3/16

$m=9,\ l=12$

α	order	N_0	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	1/9
$10+w$	2	98	
2	3	1	$-5/9$
4	3	1	7/9
$2+w$	6	2	$-7/9$
$4+w$	6	14	5/9

(2) $k=\boldsymbol{Q}(\sqrt{3})$.

$m=2,\ l=2$

α	order	$\pm N_0$	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	1/6
$1+2w$	2	-11	$-1/6$

$m=3,\ l=6$

α	order	$\pm N_0$	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	1/6
$1+3w$	2	-26	$-1/6$

$m=4,\ l=4$

α	order	$\pm N_0$	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	7/12
w	2	-3	5/12
$4+w$	2	13	$-5/12$
$1+4w$	2	-47	$-7/12$

$m=5,\ l=3$

α	order	$\pm N_0$	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	9/20
4	2	1	9/20
$1+5w$	2	-74	$-9/20$
$4+5w$	2	-59	$-9/20$
2	4	1	1/20
3	4	1	1/20
$2+5w$	4	-71	$-1/20$
$3+5w$	4	-66	$-1/20$
w	8	-3	7/20
$2w$	8	-3	3/20
$3w$	8	-3	3/20
$4w$	8	-3	7/20
$5+w$	8	22	$-7/20$
$5+2w$	8	13	$-3/20$
$10+3w$	8	73	$-3/20$
$10+4w$	8	13	$-7/20$

$m=6,\ l=6$

α	order	$\pm N_0$	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	1/6
5	2	1	1/6
$1+6w$	2	-107	$-1/6$
$5+6w$	2	-83	$-1/6$

$m=7,\ l=8$

α	order	$\pm N_0$	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	17/21
$3+w$	2	6	11/21
$1+7w$	2	−146	−17/21
$3+8w$	2	−183	−11/21
$1+w$	6	−2	13/21
$1+2w$	6	−11	19/21
$8+w$	6	61	−13/21
$2+7w$	6	−143	−5/21
$3+7w$	6	−138	1/21
$8+2w$	6	13	−19/21

$m=8,\ l=4$

α	order	$\pm N_0$	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	19/48
3	2	1	−5/48
5	2	1	−5/48
7	2	49	19/48
$1+8w$	2	−191	−19/48
$3+8w$	2	−183	5/48
$5+8w$	2	−169	5/48
$7+8w$	2	−143	−19/48
w	4	−3	−7/48
$3w$	4	−3	17/48
$5w$	4	−3	17/48
$7w$	4	−3	−7/48
$8+w$	4	61	7/48
$8+3w$	4	37	−17/48
$16+5w$	4	181	−17/48
$16+7w$	4	109	7/48

$m=9,\ l=18$

α	order	$\pm N_0$	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	
$1+9w$	2	−242	
2	3	1	
4	3	1	
$2+9w$	6	−239	
$4+9w$	6	−227	

(3) $k=\boldsymbol{Q}(\sqrt{5})$.

$m=2,\ l=3$

α	order	N_0	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	0

$m=3,\ l=4$

α	order	N_0	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	1/3
$3+w$	2	11	−1/3

$m=4,\ l=3$

α	order	N_0	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	1/4
3	2	1	1/4
$4+w$	2	19	−1/4
$4+3w$	2	19	−1/4

$m=5,\ l=10$

α	order	N_0	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	1/5
2	2	1	−1/5

$m=6,\ l=12$

α	order	N_0	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	0
$6+w$	2	41	0

$m=7$, $l=8$

α	order	N_0	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	3/7
$7+w$	2	55	$-3/7$
2	3	1	5/7
3	3	1	$-1/7$
$7+2w$	6	59	$-5/7$
$7+3w$	6	61	1/7

$m=8$, $l=6$

α	order	N_0	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	3/8
3	2	1	3/8
$2+w$	2	5	$-1/8$
$3+w$	2	11	1/8
$3+2w$	2	11	1/8
$5+w$	2	29	$-1/8$
$8+w$	2	71	$-3/8$
$8+3w$	2	79	$-3/8$

$m=9$, $l=12$

α	order	N_0	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	7/9
$9+w$	2	89	$-7/9$
2	3	1	1/9
4	3	1	$-5/9$
$9+2w$	6	95	$-1/9$
$19+4w$	6	101	5/9

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