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Abstract. Let k be a totally real quadratic field. Let α be an integer of k and m be a positive rational integer. Denote by $\zeta((\alpha),(m),s)$ be a partial zeta function associated to a ray class containing the principal ideal (α) defined with a conductor (m). We give a formula of the value of $\zeta((\alpha),(m),0)$ by applying the Shintani method to compute special values of partial zeta functions, *c.f.* [4]. It is an analogue to the formula of values of a partial zeta function corresponding to a class contained in $(\boldsymbol{Z}/m\boldsymbol{Z})^{\times}$ of the field of rational numbers, *c.f.* Chap. 4, [5].

1. Introduction. We study the value at zero of a partial zeta function on a real quadratic field in the present paper. The formula of values of a partial zeta function of non-positive integers is given in [4]. We follow it under analogue to the case of cyclotomic field. Namely, we consider the ray class group defined by the conductor (m) and a ray class (α), where m is a positive integer, and where α is an integer of the real quadratic field. Let $\zeta((\alpha),(m),s)$ be the partial zeta function associated to the integral principal ideal (α). Our aim is to give an explicit formula of $\zeta((\alpha),(m),0)$. If this value is described explicitly, we can obtain the Stickelberger element

$$\sum_{(\alpha)}\zeta((\alpha),(m),0)\varphi((\alpha))$$

on the real quadratic field of the class number one in a similar manner how we do in the theory of cyclotomic fields, where φ is the Artin map of the ray class group onto the Galois group of the ray class field defined with a conductor (m). T. Shintani showed a method to calculate special values of a partial zeta function using generalized Bernoulli polynomials. We write out the formula for the ray class (α) on real quadratic field. The point in the computation is to solve a certain Diophantine

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equation. The value of $\zeta((\alpha),(m),0)$ is given as a sum of values for these solutions of generalized Bernoulli polynomials. These polynomials are expressed by Bernoulli polynomials $B_1(x)$ and $B_2(x)$, essentially.

In §2, we determine the set of the complete system of representatives of the principal ray classes as an analogue of $(\boldsymbol{Z}/m\boldsymbol{Z})^{\times}$ in the theory of cyclotomic fields. We review the Shintani method in §3 and solve a Diophantine problem in §4. In §5, by using the results preceding sections, we calculate the value of $\zeta((\alpha),(m),0)$ and show a formula.

2. Ray class groups over a real quadratic field. Let $\mathfrak{m}$ be an integral ideal of a real quadratic field k. Let $\mathcal{O}$ be the ring of integers of k. The set of integers of $\mathcal{O}$ which are prime to $\mathfrak{m}$ forms a multiplicative set. Hence, every fractions constructing from elements of $\mathcal{O}$ whose denominators belong to the multiplicative set form a subring of $k^{\times}$. This ring is a semilocal ring. Denote it by $\mathcal{O}_{\mathfrak{m}}$. We easily verified $\mathcal{O}_{\mathfrak{m}} = \mathcal{O} + \mathfrak{m}\mathcal{O}_{\mathfrak{m}}$ and $\mathfrak{m} = \mathcal{O} \cap \mathfrak{m}\mathcal{O}_{\mathfrak{m}}$. Thus, an isomorphism $\mathcal{O}/\mathfrak{m}\mathcal{O} \cong \mathcal{O}_{\mathfrak{m}}/\mathfrak{m}\mathcal{O}_{\mathfrak{m}}$ is induced from the natural inclusion of $\mathcal{O}$ into $\mathcal{O}_{\mathfrak{m}}$. Since the multiplicative groups of both quotient rings are isomorphic, $(\mathcal{O}/\mathfrak{m}\mathcal{O})^{\times}$ is isomorphic to $\mathcal{O}_{\mathfrak{m}}^{\times}/(1+\mathfrak{m}\mathcal{O}_{\mathfrak{m}})$. Let A be the ideal group generated by every prime ideals which are not dividing $\mathfrak{m}$. Let $(1+\mathfrak{m}\mathcal{O}_{\mathfrak{m}})_{+}$ be a subgroup generated by every totally positive elements contained in $1+\mathfrak{m}\mathcal{O}_{\mathfrak{m}}$. Principal ideals generated by elements contained in this set belong to A, which form a subgroup of A. We denote it by H and define a ray class group in narrow sense with conductor $\mathfrak{m}$ to be the factor group A/H. An abelian extension of k corresponding to A/H called the ray class field is defined in class field theory. We denoted it by $k(\mathfrak{m})$. The relation of the ray class and the Galois group is given by the Artin map. There is an automorphism of $k(\mathfrak{m})$ defined for each prime ideal which is called a Frobenius automorphism. Every Frobenius automotphisms with respect to prime ideals contained in a same ray class are equal. Therefore, we send each ray class to the automorphism and obtain an injective map. The theorem of arithmetic progression is generalized in arbitrary algebraic number fields and every ray classes contain infinitely many prime ideals. Thus, this map is an isomorphism and is called the Artin map. We denote by φ the Artin map:

(1) $\varphi : A/H \longrightarrow G = Gal(k(\mathfrak{m})/k).$

Let P be the subgroup of A consisting of every principal ideals which are prime to $\mathfrak{m}$. We note

$$P/H \cong \mathcal{O}_{\mathfrak{m}}^{\times}/(1+\mathfrak{m}\mathcal{O}_{\mathfrak{m}})_{+}.$$

$A = P$ holds if and only if the class number of k is one.

An equivalence class of valuations of the field k is called a place of k. If a place contains a non-archimedian valuation, it is called a finite place. A place containing an archimedian valuation is called an infinite place. An archimedian place $\mathfrak{p}$ is written as $\mathfrak{p} \mid \infty$, where ∞ means the unique archimedian place of $\boldsymbol{Q}$. On the contrary, a finite place is treated as if it is a prime ideal. We identify it with its valuation ideal. For each place $\mathfrak{p}$, there is a local field $k_\mathfrak{p}$, which is called the completion of the field k at $\mathfrak{p}$ and denoted by $k_\mathfrak{p}$. When $\mathfrak{p}$ is finite, the multiplicative group of the valuation ring of a place $\mathfrak{p}$ is called the group of units of the local field $k_\mathfrak{p}$ and is denoted by $U_\mathfrak{p}$. The factor group $k_\mathfrak{p}^\times/U_\mathfrak{p}$ is isomorphic to the value group of the valuation. It is also isomorphic to a subgroup of A generated by the prime ideal $\mathfrak{p}$. Hence, an isomorphism

$$A \cong \bigoplus_{\mathfrak{p} \nmid \mathfrak{m}\infty} k_\mathfrak{p}^\times/U_\mathfrak{p} \tag{2}$$

is induced. We write the direct sum in the right as $I_\mathfrak{m}$. We choose an embedding of k into $k_\mathfrak{p}$ for each $\mathfrak{p}$ and fix it once for all. Consequently, we obtain

$$k^\times \longrightarrow I_\mathfrak{m} \times \prod_{\mathfrak{p}\mid\mathfrak{m}} k_\mathfrak{p}^\times \times \prod_{\mathfrak{p}\mid\infty} k_\mathfrak{p}^\times$$

through the diagonal embeddings into $k_\mathfrak{p}^\times$'s of k. The Artin map (1) is a surjection of I_m onto the Galois group G. By local class field theory, there is a homomorphism of $k_\mathfrak{p}^\times$ into the decomposition group of the place $\mathfrak{p}$. Denote by $\varphi_\mathfrak{p}$ this homomorphism of $k_\mathfrak{p}^\times$ into G and set

$$\varphi_0 = \prod_{\mathfrak{p}\mid\mathfrak{m}} \varphi_\mathfrak{p}, \qquad \varphi_\infty = \prod_{\mathfrak{p}\mid\infty} \varphi_\mathfrak{p},$$

where the product means that in images of maps taken in the Galois group G. By the reciprocity law in class field theory, a relation

$$\varphi((\alpha))^{-1} = \varphi_0(\alpha)\varphi_\infty(\alpha) \tag{3}$$

holds for every $\alpha \in \mathcal{O}_\mathfrak{m}^\times$. Let

$$\mathfrak{m} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r} \tag{4}$$

be the primary decomposition of $\mathfrak{m}$. The kernel of $\varphi_\mathfrak{p}$ contains the group of e_ith principal units. It is no more than a subgroup $1 + \mathfrak{p}_i^{e_i}\widetilde{\mathcal{O}}_{\mathfrak{p}_i}$ of $U_\mathfrak{p}$ at a finite place, where $\widetilde{\mathcal{O}}_{\mathfrak{p}_i}$ is the valuation ring of $k_{\mathfrak{p}_i}$. This group is denoted by $U_{\mathfrak{p}_i}^{(e_i)}$. For an infinite place $\mathfrak{p}$, we denote by $U_\mathfrak{p}$ the multiplicative group $k_\mathfrak{p}^\times$ and set $U_\mathfrak{p}^{(1)} = U_\mathfrak{p}^2$. Since $\mathcal{O}_\mathfrak{m}^\times$ is mapped into $U_\mathfrak{p}$ for every $\mathfrak{p} \mid \mathfrak{m}\infty$, we restrict the mapping $\varphi_0\varphi_\infty$ onto a subgroup $U = \prod_{\mathfrak{p}\mid\mathfrak{m}\infty} U_\mathfrak{p}$. The kernel of this restricted map contains a subgroup

$$U(\mathfrak{m}\infty) = \prod_{i=1}^{r} U_{\mathfrak{p}_i}^{(e_i)} \times \prod_{\mathfrak{p}\mid\infty} U_\mathfrak{p}^{(1)}. \tag{5}$$

If $\alpha \in \mathcal{O}^\times$, we see $\varphi((\alpha)) = 1$ and the image of $\mathcal{O}^\times$ into U is contained in the kernel of $\varphi_0\varphi_\infty$. The multiplicative group $\mathcal{O}^\times$ is said as the group of units of the field k and we write it as E, usually. It is identified with the image of the embedding into U. We notice the following commutative diagram exists:

$$\begin{array}{ccccc} & & (1+\mathfrak{m}\mathcal{O}_{\mathfrak{m}})_+ & & \\ & & \downarrow & & \\ \mathcal{O}^\times & \longrightarrow & \mathcal{O}_{\mathfrak{m}}^\times & \longrightarrow & P \\ & & \downarrow{\scriptstyle\mu} & & \downarrow \\ & & U/U(\mathfrak{m}\infty) & \xrightarrow[\rho]{} & P/H \end{array}$$

where ρ is composition $\varphi^{-1}\circ(\varphi_0\varphi_\infty)$. We note the kernel of ρ is the image of E into $U/U(\mathfrak{m}\infty)$. Namely,

$$U/U(\mathfrak{m}\infty)E \cong P/H. \tag{6}$$

There is a well-known formula giving the order of P/H, *c.f.* Corollary 4.5.6 in [1]:

THEOREM 1. *P/H is isomorphic to $U/U_0(\mathfrak{m})E$ and the order is equal to*

$$\frac{|(\mathcal{O}/\mathfrak{m}\mathcal{O})^\times|2^2}{|E:E\cap U(\mathfrak{m}\infty)|}.$$

Put $U_0 = \prod_{i=1}^r U_{\mathfrak{p}_i}$. The canonical homomorphisms of $\mathcal{O}/\mathfrak{m}\mathcal{O}$ onto $\mathcal{O}/\mathfrak{p}_i^{e_i}\mathcal{O}$ induce an isomorphism onto $\oplus_{i=1}^r\mathcal{O}/\mathfrak{p}_i^{e_i}\mathcal{O}$ along (4). Each ring $\mathcal{O}/\mathfrak{p}_i^{e_i}\mathcal{O}$ is canonically isomorphic to $\mathcal{O}_{\mathfrak{p}_i}/\mathfrak{p}_i^{e_i}\mathcal{O}_{\mathfrak{p}_i}$. Since $\widetilde{\mathcal{O}}_{\mathfrak{p}_i} = \mathcal{O}_{\mathfrak{p}_i}+\mathfrak{p}_i^{e_i}\widetilde{\mathcal{O}}_{\mathfrak{p}_i}$, a subset $\mathcal{O}_{\mathfrak{p}_i}^\times+\mathfrak{p}_i^{e_i}\widetilde{\mathcal{O}}_{\mathfrak{p}_i}$ coincides with $\widetilde{\mathcal{O}}_{\mathfrak{p}_i}^\times$. Furthermore, since the kernel of the canonical map $\mathcal{O}_{\mathfrak{p}_i}^\times$ into $U_{\mathfrak{p}_i}/U_{\mathfrak{p}_i}^{(e_i)}$ is $1+\mathfrak{p}_i^{e_i}\mathcal{O}_{\mathfrak{p}_i}$, the factor group $\mathcal{O}_{\mathfrak{p}_i}^\times/1+\mathfrak{p}_i^{e_i}\mathcal{O}_{\mathfrak{p}_i}$ is isomorphic to $U_{\mathfrak{p}_i}/U_{\mathfrak{p}_i}^{(e_i)}$. Thus, an isomorphism

$$(\mathcal{O}/\mathfrak{m}\mathcal{O})^\times \cong U_0U(\mathfrak{m}\infty)/U(\mathfrak{m}\infty)$$

is arising by passing throught the multiplicative group of $\mathcal{O}_{\mathfrak{m}}/\mathfrak{m}\mathcal{O}_{\mathfrak{m}}$.

We study this isomorphism in connection with the homomorphism μ in the above diagram. Take an arbitrary positive integer m from the ideal $\mathfrak{m}$. Let $[1,w]$ be an integral basis of $\mathcal{O}$. We take w so that its signature is $(1,-1)$, which means that we give an order on the set consisting of two archimedian places and fix it once for all. w is embedded into the positive real numbers concerning the first place and the negative real numbers relative to the second. Note $1+mw$ is congruent to 1 with modulo $\mathfrak{m}$ and has the same signature as w. Let w' be the conjugate of w. If an element α of $\mathcal{O}\cap\mathcal{O}_{\mathfrak{m}}^\times$ has a signature (s_1,s_2), we modify it and convert it to a totally positive element as

$$\alpha_+ = \alpha(1+mw)^{\frac{1-s_2}{2}}(1+mw')^{\frac{1-s_1}{2}},$$

which belongs to the same congruent class of α with modulo $\mathfrak{m}$. Let $\mathcal{O}_+$ be the subset of every totally positive elements contained in $\mathcal{O}\cap\mathcal{O}_{\mathfrak{m}}^\times$. In the isomorphism

$$\mathcal{O}_{\mathfrak{m}}^\times/1+\mathfrak{m}\mathcal{O}_{\mathfrak{m}} \cong (\mathcal{O}/\mathfrak{m}\mathcal{O})^\times,$$

each element $\alpha(1+\mathfrak{m}\mathcal{O}_{\mathfrak{m}})$ contains an element α_+ of $\mathcal{O}_+$, which means to $\mathcal{O}_+(1+\mathfrak{m}\mathcal{O}) = \mathcal{O}_{\mathfrak{m}}^\times$. Hence, $\mathcal{O}_+U(\mathfrak{m}\infty) = U_0U(\mathfrak{m}\infty)$. We have

$$\mathcal{O}_+(1+\mathfrak{m}\mathcal{O}_{\mathfrak{m}})_+/(1+\mathfrak{m}\mathcal{O}_{\mathfrak{m}})_+ \cong (\mathcal{O}/\mathfrak{m}\mathcal{O})^\times.$$

It is a preimage in $U/U(\mathfrak{m}\infty)$ of $U_0U(\mathfrak{m}\infty)/U(\mathfrak{m}\infty)$ by μ in the above diagram.

We will determine an integral principal ideal containing in each class of

P/H. We do not study P/H but $UU(\mathfrak{m}\infty)/U(\mathfrak{m}\infty)E$ on account of (6). The set of every totally positive units of k forms a subgroup of E, which is denoted by E^+. This subgroup acts on $\mathcal{O}/\mathfrak{m}\mathcal{O}$ by multiplication. Hence, its multiplicative group also becomes an E^+-set. We decompose this E^+-set into disjoint union of orbits:

$$(7) \qquad (\mathcal{O}/\mathfrak{m}\mathcal{O})^{\times} = \bigcup_{i=1}^{n} O_i$$

These orbits corresponds to elements of the subgroup $EU_0U(\mathfrak{m}\infty)/U(\mathfrak{m}\infty)E$ of $UU(\mathfrak{m}\infty)/U(\mathfrak{m}\infty)E$. We choose a representative of each orbit and denote it by α_i. When norm of the fundamental unit of k is positive, we add $1+mw$ to obtain the complete set. On the contrary, when the norm is negative, we see $EU_0U(\mathfrak{m}\infty) = U$ and α_i's form the complete set.

THEOREM 2. Let $\{\alpha_1, \cdots, \alpha_n\}$ be a complete set of representatives of orbits in (7) which consists of totally positive elements. If norm of the fundamental unit of k is negative, a complete system of representatives of P/H is just the set of n principal ideals generated by them. If the norm is positive, we need to add a principal ideal $(1+mw)$ to obtain a complete system.

4. The Shintani method. T. Shintani developed a method to compute values of a partial zeta function over a totally real number field at non-positive integers. We apply it to computing the value at 0 of a partial zeta function on a real quadratic field k. We summarize the Shintani method with focusing on this purpose.

The $\boldsymbol{R}$-algebra $\boldsymbol{R} \otimes_{\boldsymbol{Q}} k$ is called the Minkowski space over k in [3]. This algebra is commutative and semisimple, and hence it is a direct sum of subalgebras which are fields. In fact, it is isomorphic to $\boldsymbol{R} \oplus \boldsymbol{R}$, which is regarded as a two dimensional space over $\boldsymbol{R}$. Here, the isomorphism is obtained from two archimedian places of k. We denote the conjugate element of a in the field k by a' in the sequel. Then,

$$t \otimes a \longrightarrow (ta, ta') \in \boldsymbol{R} \times \boldsymbol{R}$$

gives the isomorphism. We work on the first quadrant in the Minkowski space. Let (x_1, x_2) be a point in the first quadrant. The totally positive unit u of k acts on it by $u(x_1, x_2) = (ux_1, u'x_2)$. Abbreviate $E \cap U(\mathfrak{m}\infty)$ to $E^+(\mathfrak{m})$. The first quadrant in the Minkowski space becomes an $E^+(\mathfrak{m})$-set. A fundamental domain by this action is a union of two connected sets C_1 and C_2. C_1 is a half line $\{(x,x) : x > 0\}$ and C_2 is an open sector of infinite radius:

$$C_2 = \{(x_1, x_2) : 0 < (\varepsilon')^2 x_1 < x_2 < x_1\}$$

where ε is a generator of $E^+(\mathfrak{m})$. For a positive integer m contained in $\mathfrak{m}$, we see $v_1 = (m, m) \in C_1$ and $v_2 = (m\varepsilon, m\varepsilon')$, which exist on the boundary of C_2. We see

$$C_1 = \{xv_1 : x > 0\},$$
$$C_2 = \{x_1v_1 + x_2v_2 : x_i > 0\}.$$

Let σ_1 and σ_2 be coordinate functions on C_1 and C_2 relative to the basis v_1 and $[v_1, v_2]$, respectively. We define bounded subsets S_i in C_i to be

$$S_1 = \{xv_1 : 0 < x \leq 1\},$$
$$S_2 = \{x_1v_1 + x_2v_2 : 0 < x_i \leq 1\}.$$

Let $\mathfrak{a}$ be an integral ideal of k. We may regard each totally positive number α of k as a point (α, α') in the first quadrant of the Minkowski space, simultaneously. Following to notation in the book [3], we define

$$R(\mathfrak{a}, C_i) = (1 + \mathfrak{a}^{-1}\mathfrak{m}) \cap S_i.$$

In [4], the set of coordinates of points belonging to this set is written as

$$R(i, \mathfrak{a}^{-1}\mathfrak{m} + 1) = \{\sigma_i(y) : y \in R(\mathfrak{a}, C_i)\}.$$

We abbreviate notation and write this set as $R(i)$.

The partial zeta-function concerning a ray class $(\mathfrak{a})$, where $\mathfrak{a}$ is an integral ideal which is prime to $\mathfrak{m}$, is defined to be

$$\zeta(\mathfrak{a}, \mathfrak{m}, s) = \sum_{\mathfrak{g}} \frac{1}{N(\mathfrak{g})^s},$$

where $\mathfrak{g}$ runs throughout over the set of integral ideals contained in the ray class. The sum in the right is absolutely convergent in an open set $\Re s > 1$ and meromorphically continued to the whole complex plain with a pole at $s = 1$. We state an explicit formula of values of $\zeta(\mathfrak{a}, \mathfrak{m}, 1-k)$ for $k = 1, 2, \cdots$ following to [4].

In general, when an $r \times n$ matrix A $(r \leq n)$ is given, we define the generalized Bernoulli polynomial $B_m(A, x)$ of r variables $x = (x_1, \cdots, x_r)$ for $m \geq 0$ from a system of linear forms

$$A \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} L_1(t_1, \cdots, t_n) \\ \vdots \\ L_t(t_1, \cdots, t_n) \end{pmatrix}.$$

We write t for $(t_1, \cdots, t_n)$, shortly. We regard a product

$$\left. \prod_{j=1}^{r} \frac{exp(ux_jL_j(t))}{exp(uL_j(t)) - 1} \right]_{t_k=1} \tag{8}$$

as a function on variables $u, t_1, \cdots, t_n$, and expand it into a Laurent series. Let

$$\frac{1}{(m!)^n} B_m^{(k)}(A; x_1, \cdots, x_r), \quad m = 1, 2, \cdots; \quad 1 \leq k \leq n$$

be the coefficients of terms concerning $u^{n(m-1)}(t_1 \cdots t_{k-1}t_{k+1} \cdots t_n)^{m-1}$. We define the mth generalized Bernoulli polynomial $B_m(A; x_1, \cdots, x_r)$ to be

$$\frac{1}{n} \sum_{k=1}^{n} B_m^{(k)}(A; x_1, \cdots, x_r).$$

To apply this to a real quadratic field, we define matrices A_1 and A_2 from v_1 and v_2. Let A_1 be the 1×2 matrix (m, m) corresponding to v_1 and A_2 be the square matrix of degree two corresponding to the basis $[v_1, v_2]$:

$$A_2 = \begin{pmatrix} m & m \\ m\varepsilon & m\varepsilon' \end{pmatrix}.$$

From Theorem 1 in [4], we obtain

THEOREM 3. Let m be positive integers. Then, we have

$$\zeta(\mathfrak{a},\mathfrak{m},1-m)= -\frac{N\mathfrak{a}^{m-1}}{m^2}\sum_{x\in R(1)} B_m(A_1;x) + \frac{N\mathfrak{a}^{m-1}}{m^2}\sum_{(x_1,x_2)\in R(2)} B_m(A_2;x_1,x_2).$$

Therefore, to know the value of $\zeta(\mathfrak{a},\mathfrak{m},0)$, we need the generalized first Bernoulli polynomials $B_1(A_1;x)$ and $B_1(A_2;x_1,x_2)$.

(1) $B_1(A;x)$. The product (8) is reduced to

$$\left.\frac{exp(ux(m(t_1+t_2)))}{exp(um(t_1+t_2))-1}\right]_{t_k=1}$$

in this case. $B_1^{(k)}(A_1;x)$ is equal to the first Bernoulli polynomial $B_1(x)$. Thus

$$B_1(A_1;x)=B_1(x)=x-\frac{1}{2}.$$

(2) $B_1(A_2;x_1,x_2)$. $B_1^{(k)}(A_2;x_1,x_2)$ is the constant term of the Laurent series of the following function:

$$\left.\frac{exp(ux_1m(t_1+t_2))}{exp(um(t_1+t_2))-1}\right]_{t_k=1}\times \left.\frac{exp(ux_2m(\varepsilon t_1+\varepsilon' t_2))}{exp(um(\varepsilon t_1+\varepsilon' t_2))-1}\right]_{t_k=1}$$

We obtain

$$B_1^{(1)}(A_2;x_1,x_2)= \frac{\varepsilon'}{2}B_2(x_1)+\frac{\varepsilon}{2}B_2(x_2)+B_1(x_1)B_1(x_2)$$

$$B_1^{(2)}(A_2;x_1,x_2)= \frac{\varepsilon}{2}B_2(x_1)+\frac{\varepsilon'}{2}B_2(x_2)+B_1(x_1)B_1(x_2)$$

respectively, where $B_2(x)$ is the second Bernoulli polynomial. Thus, $B_1(A_2;x_1,x_2)$ equals

$$\frac{Tr(\varepsilon)}{4}(B_2(x_1)+B_2(x_2))+B_1(x_1)B_2(x_2).$$

COROLLARY 4. The value of $\zeta(\mathfrak{a},\mathfrak{m},0)$ is equal to

$$\frac{Tr(\varepsilon)}{4}\sum_{(x_1,x_2)\in R(2)}(B_2(x_1)+B_2(x_2)) + \sum_{(x_1,x_2)\in R(2)} B_1(x_1)B_1(x_2) - \sum_{x\in R(1)} B_1(x),$$

where $B_1(x)=x-\frac{1}{2}$ and $B_2(x)=x^2-x+\frac{1}{6}$.

4. Determining $R(i)$.

We suppose the integral ideal $\mathfrak{m}$ is a principal ideal generated by a positive integer m. We set $\mathfrak{m}=(m)$ in this section. Let ε be a generator of $E^+(\mathfrak{m})$. Let a and b be the coefficients of ε with respect to the integral basis $[1,w]$, where we choose w so that its signature equals $(+1,-1)$ and choose ε so that $a>0$ and $b>0$. Since $\varepsilon\equiv 1 \mod \mathfrak{m}$, we see there is integers a_1 and b_1 such that $a=1+ma_1$ and $b=mb_1$. Let α be an integer contained in $\mathcal{O}_{\mathfrak{m}}^{\times}$. We decompose it as $\alpha=c\beta$ and $\beta=f_1+f_2w$ such that $c>0$ and $(f_1,f_2)=1$. Let r be the minimal non-negative residue of c by m. Since $cr^{-1}\in 1+\mathfrak{m}\mathcal{O}_{\mathfrak{m}}$, r and c belong to a same ray class. We choose α from each ray class so that $0 < c < m$ holds. Namely,

$$(9)\quad \begin{array}{l} \alpha = c\beta, \quad 0 < c < m, \ (c, m) = 1, \\ \beta = f_1 + f_2 w, \quad (f_1, f_2) = 1. \end{array}$$

Set $n = cN(\beta)$, where $N(\beta)$ is norm $f_1^2 + f_1 f_2 Tr(w) + f_2^2 N(w)$. To determine $R(1)$, we need to solve an equation for two variables z and x

$$1 + \frac{mz}{\alpha} = x(m, m)$$

where z takes values in $\mathcal{O}$ and $0 < x \leqq 1$. Since $x(m, m) = (xm, xm)$ and since the left hand side expresses a number of k, we observe x is a rational number and $\beta' z$ is a rational integer. Let $(\beta') = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_s^{e_s}$ be a primary decomposition. By (9), we see prime ideals $\mathfrak{p}_i$ are decomposed or ramified in $k/\boldsymbol{Q}$ and $r_i = 1$ when $\mathfrak{p}_i$ are ramified. Moreover, $\mathfrak{p}_i$ and $\mathfrak{p}_j$ for a pair $\{i, j\}$ are not conjugate to each other. Hence, $(\beta) \mid (z)$, because $\beta' z$ is an integer. There is an integer l such that $z = l\beta$. We obtain

$$(10)\qquad x = \frac{1}{m} + \frac{l}{c}.$$

and x belongs to $R(1, \alpha^{-1} m\mathcal{O} + 1)$ if and only if

$$0 < \frac{1}{m} + \frac{l}{c} \leq 1.$$

Thus,

$$-\frac{c}{m} < l \leq c - \frac{c}{m}.$$

Since $c < m$, we have $0 \leq l \leq c - 1$.

The Bernoulli polynomial satisfies the distribution relation:

$$\sum_{j=0}^{N-1} B_k\left(x + \frac{j}{N}\right) = N^{1-k} B_k(Nx)$$

c.f. §2, [2]. We have the following theorem:

Theorem 5.

$$\sum_{x \in R(1)} B_1(A_1, x) = B_1\left(\frac{c}{m}\right).$$

We begin to determine $R(2)$. It is a set of coordinates (x_1, x_2) which satisfy the following equation:

$$\begin{cases} 1 + \dfrac{mz}{\alpha} & = & mx_1 + mx_2\varepsilon \\ 1 + \dfrac{mz'}{\alpha'} & = & mx_1 + mx_2\varepsilon' \end{cases}$$

where z takes values in $\mathcal{O}$ and real numbers x_1 and x_2 belong to the interval $(0, 1]$. Taking account of (9), we modify them to

$$(11)\quad \begin{cases} \dfrac{1}{m} + \dfrac{\beta' z}{n} & = & x_1 + x_2\varepsilon \\ \dfrac{1}{m} + \dfrac{\beta z'}{n} & = & x_1 + x_2\varepsilon' \end{cases}$$

where $n = cN(\beta)$. Set $z = x + yw$ $(x, y \in \boldsymbol{Z})$. We see

$$(12)\qquad \frac{\beta' z - \beta z'}{n} = x_2(\varepsilon - \varepsilon')$$
$$\frac{-f_2 x + f_1 y}{n}(w - w') = bx_2(w - w').$$

Dividing $w - w'$, we have

$$(13)\qquad x_2 = \frac{-f_2 x + f_1 y}{bn}$$

Put $s_2 = -f_2 x + f_1 y$.

Next, we add two equations and obtain

$$(14)\quad \frac{1}{m} + \frac{(f_1 + f_2 Tr(w))x + f_2 N(w) y}{n} + \frac{(-f_2 x + f_1 y) Tr(w)}{2n} = x_1 + \frac{Tr(\varepsilon) x_2}{2}.$$

Set $r_2 = (f_1+f_2Tr(w))x+f_2N(w)y$. Since $\varepsilon = a + bw$, the equality can be simplifies by using (13).

$$\frac{1}{m}+\frac{r_2}{n}+\frac{bTr(w)x_2}{2} = x_1 + ax_2 + \frac{bTr(w)x_2}{2}. \tag{15}$$

Therefore

$$x_1 = \frac{1}{m}+\frac{r_2}{n}-ax_2. \tag{16}$$

Since $b = mb_1$, we see nbx_1 is an integer. Put $s_1 = nbx_1$.

LEMMA 6. If the equation (11) has a solution, there are integers x, y, s_1, s_2 and r_2 satisfying the following three equations

$$\left\{\begin{array}{rl} f_2x - f_1y & = -s_2 \\ (f_1 + f_2Tr(w))x + f_2N(w)y & = r_2 \\ nb_1 + br_2 - as_2 & = s_1. \end{array}\right. \tag{17}$$

Conversely, there is an integral solution (x, y, s_1, s_2, r_2) for these equations satisfying $0 < \frac{s_i}{nb} \leq 1$ for $i = 1, 2$, the equation (11) has a corresponding solution.

Proof. We have only to verify that a set of solution of (17) gives that of (11). Assume there is a solution for (17). We see $x_2 = \frac{s_2}{nb}$ satisfies the equation (12). Since $x_1 = \frac{s_1}{nb}$ implies (16), the equation (15) is valid. Thus, the equation (14) also holds. (11) follows from (12) and (14). □

By virtue of this lemma, we are able to determine x, y, s_1, and r_2 for $x_2 = \frac{s_2}{bn}$ if we give s_2 so that x_2 belongs to the interval $(0, 1]$. We study in cases. Note $x_1 = \frac{s_1}{bn}$.

(i) Suppose $\beta = \pm 1$. We see $\alpha = \pm c$, $n = c$, $f_1 = \pm 1$, $f_2 = 0$. The equation (17) is reduced to

$$\left\{\begin{array}{rcl} y & = & \pm s_2 \\ x & = & \pm r_2 \\ s_1 & = & nb_1 + br_2 - as_2. \end{array}\right.$$

Hence, s_1 and r_2 must be integers satisfying

$$x_1 = \frac{s_1}{nb} = -\left(-\frac{1}{m}+ax_2\right)+\frac{r_2}{c}, \qquad 0 < x_1 \leq 1.$$

(ii) Suppose $\beta = \pm w$. Then, $\alpha = \pm cw$, $n = cN(w) < 0$, $f_1 = 0$, $f_2 = \pm 1$. The equation (16) is reduced to

$$\left\{\begin{array}{rcl} x & = & \mp s_2 \\ Tr(w)x + N(w)y & = & \pm r_2 \\ nb_1 + br_2 - as_2 & = & s_1. \end{array}\right.$$

We solve this equation for x and y:

$$N(w)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mp N(w)s_2 \\ \pm Tr(w)s_2 \pm r_2 \end{pmatrix}.$$

Thus, $Tr(w)s_2 + r_2 \equiv 0 \mod |N(w)|$. There is an integer t such that

$$r_2 = -Tr(w)s_2 + N(w)t.$$

Therefore, s_1 and t must be determined so that

$$x_1 = \frac{s_1}{nb} = -\left\{-\frac{1}{m}+(a+bTr(w))x_2\right\}+\frac{t}{c}, \qquad 0 < x_1 \leq 1.$$

(iii) Suppose $f_1f_2 \neq 0$. Let p be an arbitrary prime number dividing f_2. Since

$$N(\beta) = f_1^2 + f_1f_2Tr(w) + f_2^2N(\beta),$$

f_1^2 is congruent to $N(\beta)$ with modulo p. Thus, $p \nmid N(\beta)$, because f_1 and f_2 are prime to each other. This implies $(f_2, N(\beta)) = 1$. Let g_2 be an integer such that $g_2f_2 \equiv 1 \mod |N(\beta)|$. We choose an integer λ so that

$$\lambda \equiv -g_2(f_1 + f_2Tr(w)) \mod |N(\beta)|.$$

We solve the equation (17) for x and y. The solutions must satisfies

$$N(\beta)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_2N(w) & f_1 \\ -f_1 - f_2Tr(w) & f_2 \end{pmatrix}\begin{pmatrix} -s_2 \\ r_2 \end{pmatrix}.$$

Hence, s_2 and r_2 need to satisfy

$$(f_1 + f_2Tr(w))s_2 + f_2r_2 \equiv 0 \mod |N(\beta)|.$$

By multiplying g_2, we see this congruent equation holds if and only if

$$r_2 \equiv \lambda s_2 \mod |N(\beta)|.$$

There is an integer t such that

$$r_2 = \lambda s_2 + N(\beta)t. \tag{18}$$

By congruence equations

$$\begin{aligned} -f_2N(w)s_2 + f_1r_2 &\equiv -f_2N(w)s_2 + f_1\lambda s_2 \\ &\equiv (-f_2N(w) + f_1\lambda)s_2 \\ &\equiv -g_2N(\beta)s_2 \\ &\equiv 0 \mod |N(\beta)|, \end{aligned}$$

we see there are integers x and y for s_2 and r_2 satisfying (18). By substitution of the right hand side of (18) for r_2 in $s_1 = nb_1 + br_2 - as_2$, we conclude there are integers satisfying (17) for an arbitrary integer s_2 such that $0 < x_2 \leq 1$ if we determine values of the integer t so that

$$x_1 = \frac{s_1}{bn} = -\left\{-\frac{1}{m} + (a - b\lambda)x_2\right\} + \frac{t}{c}, \quad 0 < x_1 \leq 1$$

holds.

To put these observations together, we introduce a constant defined to be

$$\delta = \begin{cases} a & \text{if } \beta = \pm 1, \\ a + bTr(w) & \text{if } \beta = \pm w, \\ a - b\lambda & \text{if } f_1f_2 \neq 0. \end{cases}$$

Put $N = b_1|N(\beta)|$. We note $|bn| = mcN$. For each $x_2 = \frac{s}{mcN}$, $(s = 1, \cdots, mcN)$, we define a fraction μ_s to be

$$\mu_s = -\frac{1}{m} + \frac{\delta s}{mcN}. \tag{19}$$

Then, $(x_1, x_2) \in R(2)$ if and only if there is an integer σ such that the value of $x_1 = -\mu_s + \frac{\sigma}{c}$ is contained in $(0, 1]$.

LEMMA 7. Let x be a real number of a form $x = -\mu + \frac{\sigma}{c}$ for integers σ. Then, $0 < x \leq 1$ holds if and only if

$$x = \frac{1 - < c\mu >}{c} + \frac{j}{c}, \quad j = 0, 1, \cdots, c-1.$$

Here, $< x >$ denotes the fractional part of x, that is $< x >= x - [x]$.

Proof. Suppose $0 < x \leq 1$. Since $cx = -c\mu + \sigma$, we see $c\mu < \sigma \leq c\mu + c$. Hence, $\sigma = [c\mu] + 1 + j$ for $0 \leq j \leq c - 1$. The converse is clear. □

We conclude that the set $R(2)$ is expressed as

$$\left\{\left(\frac{1-<c\mu_s>}{c}+\frac{j}{c},\frac{s}{mcN}\right):\right.$$
$$\left.0\le j\le c-1, 1\le s\le mcN\right\},$$

explicitly.

5. Summation formulae. We shall compute the values of sums contained in the formula of $\zeta((\alpha),(m),0)$ in Corollary 4 concerning $R(2)=R(2,\alpha^{-1}m\mathcal{O}+1)$ in this section.

The value of $\sum_{(x_1,x_2)\in R(2)} B_2(x_2)$ is easily computed. It equals

$$\sum_{s=1}^{mcN}\sum_{j=0}^{c-1} B_2\left(\frac{s}{mcN}\right)=\sum_{s=1}^{mcN} cB_2\left(\frac{s}{mcN}\right)$$
$$=\frac{1}{6mN}.$$

The rational number μ_s defined in (19) contains a constant δ depending on β.

LEMMA 8. $(\delta, mN)=1$.

Proof. Since $(\delta,b)=1$ is clear, we show $(\delta,|N(\beta)|)=1$. When $\beta=\pm1$, it is obvious. When $\beta=\pm w$, we have $(\delta,N(w))=1$, because of

$$a\delta+b^2N(w)=N(\varepsilon)=1.$$

Suppose $f_1f_2\neq0$. We see

$$\delta\equiv a+bg_2(f_1+f_2Tr(w)) \mod |N(\beta)|.$$

Set $\gamma=\beta\varepsilon$. Let f_1' and f_2' be integers such that $\gamma=f_1'+f_2'w$. We note $f_2'\equiv f_1\delta \mod |N(\beta)|$. Since ε is a unit, we have $(f_1,f_2)=(f_1',f_2')=1$. Let p be an arbitrary prime divisor of f_2'. Since

$$N(\gamma)=f_1'^2+f_1'f_2'Tr(w)+f_2'^2N(w),$$

we see $p\nmid N(\gamma)$. Therefore, $(\delta,N(\beta))=1$ follows from $f_2'\equiv f_1\delta \mod |N(\beta)|$. $\square$

Now, we compute $\sum_{(x_1,x_2)\in R(2)} B_2(x_1)$.

$$\sum_{s=1}^{mcN}\sum_{j=0}^{c-1} B_2\left(\frac{1-<c\mu_s>}{c}+\frac{j}{c}\right)$$
$$=\sum_{s=1}^{mcN} c^{-1}B_2(1-<c\mu_s>).$$

By $B_2(1-x)=B_2(x)$, the value of the sum in the right hand side equals

$$\sum_{s=1}^{mcN} c^{-1}B_2(<c\mu_s>).$$

This sum is converted to a double sum

$$\sum_{s=1}^{mN}\sum_{j=0}^{c-1} c^{-1}B_2\left(\left\langle -\frac{c}{m}+\frac{\delta(s+mNj)}{mN}\right\rangle\right)$$
$$=\sum_{s=1}^{mN} B_2\left(\left\langle -\frac{c}{m}+\frac{\delta s}{mN}\right\rangle\right).$$

Furthermore, by Lemma 8,

$$\sum_{s=1}^{mN} B_2\left(\left\langle -\frac{c}{m}+\frac{\delta s}{mN}\right\rangle\right)$$
$$=\sum_{s=1}^{mN} B_2\left(\left\langle -\frac{c}{m}+\frac{s}{mN}\right\rangle\right).$$

The value of $-\frac{c}{m}+\frac{s}{mN}$ is greater than -1 and less than 1. It takes a non-negative value if and only if $s \geq cN$. Hence, the sum equals

$$\sum_{s=1}^{cN-1} B_2\left(1-\frac{c}{m}+\frac{s}{mN}\right)+\sum_{s=cN}^{mN} B_2\left(-\frac{c}{m}+\frac{s}{mN}\right).$$

Using an equality $B_2(1+x)=B_2(x)+2x$, it is modified to be

$$\sum_{s=1}^{mN} B_2\left(-\frac{c}{m}+\frac{s}{mN}\right)+2\sum_{s=1}^{cN-1}\left(-\frac{c}{m}+\frac{s}{mN}\right),$$

which is evaluated to be $\frac{1}{6mN}$. We have calculated every terms in Corollary 4 except of the sum of a product $B_1(x_1)B_1(x_2)$. We modify this remainder sum as follows.

$$\sum_{s=1}^{mcN}\sum_{j=0}^{c-1} B_1\left(\frac{1-<c\mu_s>}{c}+\frac{j}{c}\right)B_1\left(\frac{s}{mcN}\right)$$
$$=\sum_{s=1}^{mcN} B_1(1-<c\mu_s>)B_1\left(\frac{s}{mcN}\right)$$
$$=\sum_{s=1}^{mN}\sum_{j=0}^{c-1} B_1(1-<c\mu_s>)B_1\left(\frac{s+jmN}{mcN}\right)$$
$$=\sum_{s=1}^{mN} B_1(1-<c\mu_s>)B_1\left(\frac{s}{mN}\right)$$
$$=-\sum_{s=1}^{mN} B_1(<c\mu_s>)B_1\left(\frac{s}{mN}\right).$$

THEOREM 9. The value of $\zeta((\alpha),(m),0)$ equals

$$\frac{Tr(\varepsilon)}{12mN}-B_1\left(\frac{c}{m}\right)-\sum_{s=1}^{mN} B_1(<c\mu_s>)B_1\left(\frac{s}{mN}\right).$$

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