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WEIGHTED ESTIMATES FOR MAXIMAL FUNCTIONS ASSOCIATED WITH FOURIER MULTIPLIERS

SHUICHI SATO

Abstract. We prove some weighted estimates for maximal functions associated with certain Fourier multipliers of Bochner-Riesz type.

1. Introduction

Let \( \gamma(t, \xi) \) be a continuous function on \( (0, \infty) \times \mathbb{R}^n \) such that \( \gamma(t, 0) = 0 \) and \( \gamma(t, \xi) > 0 \) for all \( \xi \neq 0 \) and \( t > 0 \). Also, we assume the following:
\[
\lim_{t \to \infty} \gamma(t, \xi) = 0 \quad \text{for all} \quad \xi \in \mathbb{R}^n, \quad \lim_{|\xi| \to \infty} \gamma(t, \xi) = \infty \quad \text{for all} \quad t > 0;
\]
\[
\{ \xi \in \mathbb{R}^n : 1/2 \leq \gamma(t, \xi) \leq 1 \} \subset \{ \xi \in \mathbb{R}^n : c_1 t < |\xi| < c_2 t \}
\]
for all \( t > 0 \) with some constants \( 0 < c_1 < c_2 \);
\[
|\{ \xi \in \mathbb{R}^n : \gamma(t, t\xi) \in [1 - \delta, 1] \}| \leq c \delta
\]
for all \( \delta \in (0, 1/2] \) and \( t > 0 \), where \( |E| \) denotes the Lebesgue measure of a measurable set \( E \).

Let \( \hat{f}(\xi) = \int f(x) e^{-2\pi i \langle x, \xi \rangle} \, dx \) be the Fourier transform, where \( \langle x, \xi \rangle \) denotes the inner product in \( \mathbb{R}^n \). We also write \( \hat{f} = \mathcal{F}(f) \). Throughout this note we assume that \( n \geq 2 \). We consider the Bochner-Riesz mean of order \( \lambda \) with respect to \( \gamma \) defined by
\[
S_{\lambda}^\gamma(f)(x) = \int_{\mathbb{R}^n} \left( 1 - \gamma(t, \xi) \right)^\lambda_+ \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} \, d\xi,
\]
where \( s^\lambda_+ = s^\lambda \) if \( s > 0 \), \( s^\lambda_+ = 0 \) if \( s \leq 0 \). When \( \gamma(t, \xi) = (|\xi|/t)^2 \), this is the ordinary Bochner-Riesz mean. Define the maximal function
\[
S_{\lambda}^\gamma(f)(x) = \sup_{t > 0} |S_{\lambda}^\gamma(f)(x)|.
\]

In this note we generalize some known results on weighted estimates for the maximal functions associated with the ordinary Bochner-Riesz means by considering the generalized Bochner-Riesz means \( S_{\lambda}^\gamma(f) \).
particular, we shall prove some weighted inequalities for $S^\lambda_\gamma$ in the cases when $\gamma(t, \xi) = t^{-1}|\Phi(\xi)|$ and $\gamma(t, \xi) = |\Phi(t^{-1}\xi)|$, where $\Phi$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying certain regularity conditions. It will be shown that if $h$ is a positive homogeneous function of degree $1$ which is infinitely differentiable away from the origin, we can find a suitable $\Phi$ such that $|\Phi(\xi)| = h(\xi)$.

Now, we further assume that $\gamma(t, \cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ for all $t > 0$ and that there exists $\epsilon_0 > 0$ such that

$$ \frac{((\partial \xi)^\alpha \gamma(t, t\xi))}{|\xi|^{|\alpha|}} \leq C_\alpha \epsilon_0^{-|\alpha|} \quad \text{in } U_{c_2} \setminus \{0\} $$

for all $t > 0$ and multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $(\partial \xi)^\alpha = (\partial / \partial x_1)^{\alpha_1} \cdots (\partial / \partial x_n)^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $U_r = \{\xi \in \mathbb{R}^n : |\xi| < r\}$ ($c_2$ is as in (1.2)). Then we have the following:

**Theorem 1.** Suppose that $\gamma$ satisfies the conditions (1.1)–(1.4). Let $\lambda > (n - 1)/2$ (the critical index). Then

$$ \|S^\lambda_\gamma(f)\|_{L^2(w)} \leq C_{\lambda,w} \|f\|_{L^2(w)} \quad (f \in S(\mathbb{R}^n)) $$

for $w \in A_1(\mathbb{R}^n)$ (the Muckenhoupt class), where $S(\mathbb{R}^n)$ denotes the Schwartz space on $\mathbb{R}^n$ and $\|f\|_{L^r(w)} = (\int |f(x)|^r w(x) \, dx)^{1/r}$.

This is a particular case of the following result.

**Theorem 2.** Let $\gamma$ be as in Theorem 1. Suppose that $\lambda > (n - 1)/2$, $(n - 1)/\lambda < p \leq 2$, $1 < q \leq \lambda < r \leq p$. Then

$$ \|S^\lambda_\gamma(f)\|_{L^r(w)} \leq C_{\lambda,w} \|f\|_{\dot{F}^{0,r}_p(w)} \quad (f \in S(\mathbb{R}^n)) $$

for all $w \in A_1(\mathbb{R}^n)$, where $\dot{F}^{0,r}_p(w)$ is the weighted (homogeneous) Triebel-Lizorkin space.

See [4] for the Triebel-Lizorkin space $\dot{F}^{s,r}_p$ (see also [14]). The definition of the norm for the weighted Triebel-Lizorkin space $\dot{F}^{s,r}_p(w)$ is the same as that for $\dot{F}^{s,r}_p$ except that the weighted measure $w(x) \, dx$ is used in place of the Lebesgue measure (see [1]). Note that, if $1 < r \leq p \leq 2$, $w \in A_p$, and $f \in S(\mathbb{R}^n)$,

$$ \|f\|_{L^p(w)} \approx \|f\|_{\dot{F}^{0,2}_p(w)} \leq c \|f\|_{\dot{F}^{0,r}_p(w)} \leq c \|f\|_{\dot{F}^{s,r}_p(w)} \quad \text{for } s \geq 0. $$

Thus Theorem 1 follows from Theorem 2 with $p = r = 2$.

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a bijection. We define a space $BL$ to be the space of all those bijections $\Phi$ which satisfy $\Phi(0) = 0$ and

$$ c|\xi - \eta| \leq |\Phi(\xi) - \Phi(\eta)| \leq C|\xi - \eta| \quad \text{for all } \xi, \eta \in \mathbb{R}^n $$

with some constants $0 < c < C$. Note that if $\Phi \in BL$, $|\Phi(\xi)| \approx |\xi|$ and $|\Phi(E)| \approx |E|$ for a measurable set $E$. 

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping with the components $F_1, F_2, \ldots, F_n$. We define a subspace $D$ of $BL$. Let $F \in BL$. We say $F \in D$ if $F_j \in C^\infty(\mathbb{R}^n \setminus \{0\})$ ($j = 1, 2, \ldots, n$) and there exists a neighborhood $U$ of the origin such that

$$\max_{1 \leq j \leq n} |(\partial \xi)^n F_j(\xi)| \leq C_\alpha |\xi|^{1-h} \quad \text{in } U \setminus \{0\}$$

for all multi-indices $\alpha$.

For a mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, we consider $\gamma(t, \xi)$ defined by either of the following two equations:

$$\gamma(t, \xi) = t^{-1}|\Phi(\xi)|, \quad \gamma(t, \xi) = |\Phi(t^{-1}\xi)|.$$

Then we have the following:

**Corollary 1.** Suppose that $\Phi \in D$ and let $\gamma(t, \xi)$ be as above. Suppose that $\lambda > (n-1)/2$. Then

$$\|S^1_w(f)\|_{L^2(w)} \leq C_{\lambda,w} \|f\|_{L^2(w)} \quad (f \in S(\mathbb{R}^n))$$

for $w \in A_1(\mathbb{R}^n)$.

This follows from Theorem 1, since under the hypotheses of Corollary 1 $\gamma(t, \xi)$ satisfies the conditions (1.1)-(1.4) with $\epsilon_0 = 1$ in (1.4).

Let $\hat{h}$ be a positive homogeneous function of degree 1. By this we mean that $h(t\xi) = t^\lambda h(\xi)$ for all $t > 0$ and $\xi \in \mathbb{R}^n$, $h(0) = 0$ and $h(\xi) > 0$ for $\xi \neq 0$. Then, in fact, Corollary 1 is equivalent to the following:

**Corollary 2.** Suppose that $\Phi \in D$ and $h \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Let $\gamma(t, \xi) = t^{-1}(h \circ \Phi)(\xi) = t^{-1}h(\Phi(\xi))$ or $\gamma(t, \xi) = (h \circ \Phi)(t^{-1}\xi)$. Suppose that $\lambda > (n-1)/2$. Then

$$\|S^1_w(f)\|_{L^2(w)} \leq C_{\lambda,w} \|f\|_{L^2(w)} \quad (f \in S(\mathbb{R}^n))$$

for $w \in A_1(\mathbb{R}^n)$.

We can derive Corollary 2 from Corollary 1 as follows. Define $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Lambda(\xi) = \begin{cases} |\xi|h(\xi)^{-1}\xi & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0. \end{cases}$$

Note that $\Lambda^{-1}(\eta) = h(\eta)|\eta|^{-1}$ ($\eta \neq 0$), $\Lambda^{-1}(0) = 0$. We can easily see that $\Lambda \in D$. Define $\Gamma = \Lambda^{-1} \circ \Phi \in D$. Since $|\Gamma| = h \circ \Phi$, by applying Corollary 1 to $\gamma(t, \xi) = t^{-1}|\Gamma(\xi)|$ and $\gamma(t, \xi) = |\Gamma(t^{-1}\xi)|$ we get Corollary 2.

When $\lambda$ is near 0, we have the following estimates with power weights:
Theorem 3. Let $\gamma(t, \xi) = t^{-1}|\Phi(\xi)|$, $\Phi \in D$. Suppose that $\lambda > 0$ and $-1 < \alpha \leq 0$. Then
\[
\int_{\mathbb{R}^n} \left| S^\lambda_* (f)(x) \right|^2 |x|^{\alpha} dx \leq C_{\lambda, \alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{\alpha} dx \quad (f \in \mathcal{S}(\mathbb{R}^n)).
\]

When $\gamma(t, \xi) = (|\xi|/t)^2$, this is due to Carbery-Rubio de Francia-Vega [2]. A complex interpolation between Theorem 3 and Corollary 1 with $w(x) = |x|^\alpha (-n < \alpha \leq 0)$ gives the following (see [2], [8]):

Corollary 3. Let $\gamma(t, \xi)$ be as in Theorem 3. Suppose that $0 < \lambda \leq (n-1)/2$ and $-2\lambda - 1 < \alpha \leq 0$. Then
\[
\int_{\mathbb{R}^n} \left| S^\lambda_* (f)(x) \right|^2 |x|^{\alpha} dx \leq C_{\lambda, \alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{\alpha} dx.
\]

This result can be used to get the following:

Corollary 4. Let $\gamma(t, \xi)$ be as in Theorem 3. Suppose that $0 < \lambda \leq (n-1)/2$, $2 \leq p < 2n/(n-1-2\lambda)$ and $n(1-2/p) < -\alpha < 1 + 2\lambda$. Put $w_\alpha(x) = \min(1, |x|^\alpha)$. Then
\[
\| S^\lambda_* (f) \|_{L^2(w_\alpha)} \leq c \| f \|_{L^2} \leq c \| f \|_{L^p}.
\]

The second inequality of the conclusion of Corollary 4 follows by Hölder’s inequality. As in [2], by Corollary 4 we can see that
\[
\lim_{t \to \infty} S^\lambda_* (f)(x) = f(x) \quad \text{a.e.}
\]
for $0 < \lambda \leq (n-1)/2$ and $f \in L^p(\mathbb{R}^n)$ provided $2 \leq p < 2n/(n-1-2\lambda)$.

Remark 1. When $\gamma(t, \xi) = t^{-1}h(\xi)$, where $h$ is a certain positive homogeneous function of degree 1, the $L^2(w)$ boundedness of $S^\lambda_*$ for $\lambda > (n-1)/2$ and $w \in A_1$ can be derived from the estimates of Seeger for the Littlewood-Paley functions (see [10, 11]). The case where $h \in C^\infty(\mathbb{R}^n \setminus \{0\})$ follows from Corollary 2.

Remark 2. Let $a$ be a non-negative, continuous function on $[0, \infty)$. We assume that $a \in C^\infty((0, \infty))$, $a(0) = 0$, $a(1) = 1$, $a'(s) > 0$ for $s > 0$, $a(s) \to \infty$ as $s \to \infty$ and
\[
|((d/ds)^{\ell} a(s))| \leq cs^{\ell - \ell}
\]
for all $s \in (0, \gamma)$ and $\ell \geq 0$ with some positive constants $\gamma$, $\epsilon_1$. Then Theorem 3 and Corollaries 1–4 stated above still hold with $\gamma(t, \xi) = a(t^{-1}(h \circ \Phi)(\xi))$ and also Corollaries 1, 2 remain true with $\gamma(t, \xi) = a((h \circ \Phi)(t^{-1} \xi))$, where $h$ is a positive homogeneous function of degree 1 in $C^\infty(\mathbb{R}^n \setminus \{0\})$ and $\Phi \in D$. In particular, this remark applies to the function $a(s) = s^m$, $m > 0$. In this case, $\gamma(t, \xi) = t^{-m}(H \circ \Phi)(\xi)$ or $r(t, \xi) = (H \circ \Phi)(t^{-1} \xi)$, where $H$ is a homogeneous function of degree $m$ (see [3], [6], [7], [12] for related results).
In Section 2, we shall prove Theorem 2. Suppose that $h$ is a positive homogeneous function of degree 1 such that $h \in C^\infty(\mathbb{R}^n \setminus \{0\})$, $\nabla h(\xi) \neq 0$ for $\xi \in \mathbb{R}^n \setminus \{0\}$. Put $\Sigma_h = \{\xi \in \mathbb{R}^n : h(\xi) = 1\}$. If the hypersurface $\Sigma_h$ has non-vanishing Gaussian curvature and if $\lambda > (n - 1)/2$, then
\[
|\mathcal{F}((1 - h)^\lambda)(x)| \leq c(1 + |x|)^{-n-\varepsilon}
\]
for some $\varepsilon > 0$ (see Sogge [13]). Therefore, if $\gamma(t, \xi) = t^{-1}h(\xi)$, we have $S^1_t(f) \leq cM(f)$, where $M$ denotes the Hardy-Littlewood maximal operator, and hence $S^1_t$ is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p$. Although pointwise estimates similar to those given above are not available in the present situation, we have the weighted $L^q$ estimates for the kernels arising from a decomposition of the operator $S^1_t$ defined by the general functions $\gamma(t, \xi)$ (Lemma 2), which can be applied to prove Theorem 2.

In Section 3, we shall prove Theorem 3. The proof is based on the weighted $L^2$ estimates of [2] and [8] for certain Littlewood-Paley functions.

2. Proof of Theorem 2

To handle the singularity of $\gamma(t, \xi)$ at $\xi = 0$, we need the following pointwise estimates for Fourier transform.

Lemma 1. Let $g : \mathbb{R}^n \to \mathbb{R}$ be continuous and $g(0) = 0$. Let $\varphi \in C_0^\infty(\mathbb{R})$. Suppose that $g^{-1}(\text{supp}(\varphi)) \subset U_\varepsilon$ for some $\varepsilon > 0$, where $U_\varepsilon = \{x \in \mathbb{R}^n : |x| < \varepsilon\}$. We further assume that $g \in C^{n+1}(U_\varepsilon \setminus \{0\})$ and there exists $m > 0$ such that
\[
|\partial^\alpha g(\xi)| \leq c|\xi|^{m - |\alpha|}
\]
in $U_\varepsilon \setminus \{0\}$ for $|\alpha| \leq n + 1$.

Then
\[
|\mathcal{F}(\varphi \circ g)(x)| \leq c(1 + |x|)^{-n-\delta}
\]
for some $\delta > 0$.

Proof. Take $\tilde{\varphi}(\xi) \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi(g(\xi)) = \tilde{\varphi}(\xi)\varphi(g(\xi))$. Write $\varphi(g(\xi)) = \tilde{\varphi}(\xi)(\varphi(g(\xi)) - \varphi(0)) + \varphi(0)\varphi(\xi)$. Then it suffices to estimate the Fourier transform of $\Psi(\xi) := \tilde{\varphi}(\xi)(\varphi(g(\xi)) - \varphi(0))$. We have
\[
|\partial^\alpha \Psi(\xi)| \leq c|\xi|^{m - |\alpha|}
\]
in $\mathbb{R}^n \setminus \{0\}$ for $|\alpha| \leq n + 1$.

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be such that $\text{supp}(\psi) \subset \{1/2 \leq |\xi| \leq 2\}$, $\sum_{j \in \mathbb{Z}} \psi(2^{-j}\xi) = 1$ for $\xi \neq 0$, where $\mathbb{Z}$ denotes the set of all integers. Write
\[
\hat{\psi}(x) = \sum_{j \leq M} \int \psi(2^{-j}\xi)\Psi(\xi)e^{-2\pi i(x, \xi)}\,d\xi
\]
for some $M \geq 0$. We split the sum on the right hand side into two pieces: $\hat{\Psi}(x) = I + II$, where

$$I = \sum_{j \leq N} \int \psi(2^{-j} \xi) \hat{\Psi}(\xi) e^{-2\pi i (x, \xi)} \, d\xi, \quad II = \sum_{N < j \leq M} \int \psi(2^{-j} \xi) \hat{\Psi}(\xi) e^{-2\pi i (x, \xi)} \, d\xi,$$

for $N \leq 0$, which will be specified below. We may assume $|x| > 2$. Applying integration by parts $k$ times ($1 \leq k \leq n+1$) and using (2.1), we have

$$\int |\psi(2^{-i} \xi) \hat{\Psi}(\xi) e^{-2\pi i (x, \xi)}| \, d\xi \leq c|x|^{-k} 2^{jn} 2^{j(m-k)}.$$

To estimate $I$ we use (2.2) with $k = n$ and to estimate $II$ with $k = n+1$. Finally, choosing $N = \log_2(|x|^{-1})$, we can get the conclusion. 

Now, we give a proof of Theorem 2. Decompose

$$(1 - \gamma(t, \xi))_+^\lambda = \sum_{j=0}^\infty 2^{-j\lambda} n_j \gamma(t, \xi),$$

where $n_j \in C_0^\infty(\mathbb{R})$ ($j \geq 0$), $\text{supp} (n_j) \subset [1 - 2^{-j}, 1]$ ($j \geq 1$), $\text{supp} (n_0) \subset (-1, 1)$ and $|(d/dr)^j n_j(r)| \leq c_j 2^{j\ell}$ for $\ell \geq 0$. Let $L_{j,t}^\lambda(x) = \mathcal{F}^{-1}(2^{-j\lambda} n_j \gamma(t, \cdot))(x)$ for $j \geq 0$ and $K_t^\lambda(x) = \mathcal{F}^{-1}((1 - \gamma(t, \cdot))_+^\lambda)(x) - L_{0,t}^\lambda(x)$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Put $G_t(\xi) = \gamma(t, t\xi)$. Note that $G_t^{-1}(\text{supp} (n_0)) \subset U_{c_3}$ for all $t > 0$, where $U_{c_3}$ is as in (1.4). This can be seen by using the second condition of (1.1), (1.2) and the intermediate value theorem. By (1.4) and Lemma 1 with $g = G_t$ and $\varphi = n_0$, we have $\sup_{t>0} \|L_{0,t}^\lambda f\| \leq cMf$. Since

$$\left\| \sup_{t>0} |L_{0,t}^\lambda f| \right\|_{L^p(w)} \leq c \|Mf\|_{L^p(w)} \leq c \|f\|_{L^p(w)} \leq c \|f\|_{L^p_\varphi(w)}$$

(see (1.5)) to prove Theorem 2, it suffices to show

$$\int \sup_{t>0} |K_t^\lambda f(x)| p w(x) \, dx \leq c \left( \|f\|_{L^p_\varphi(w)} \right)^p.$$

Decompose $K_t^\lambda(x) = \sum_{j=1}^\infty L_{j,t}^\lambda(x)$. Then, by Hölder’s inequality we have

$$|K_t^\lambda f(x)|^p \leq \left( \sum_{j=1}^\infty c_j^{-q/p} \right)^{p/q} \left( \sum_{j=1}^\infty c_j |L_{j,t}^\lambda f(x)|^p \right),$$
where $1/p + 1/q = 1$ and $\{c_j\}$ is a sequence of positive numbers such that $\sum_{j=1}^{\infty} c_j^{-q/p} < \infty$. Thus we have
\begin{equation}
\int \sup_{t>0} |K^\lambda_t * f(x)|^p w(x) \, dx \leq c \sum_{j=1}^{\infty} c_j \int \sup_{t>0} |L^\lambda_{j,t} * f(x)|^p w(x) \, dx
\end{equation}
\begin{equation}
\leq c \sum_{j=1}^{\infty} c_j \sum_{k \in \mathbb{Z}} \int_{2^k \leq t \leq 2^{k+1}} \sup_{2^k \leq t \leq 2^{k+1}} |L^\lambda_{j,t} * f(x)|^p w(x) \, dx.
\end{equation}
Note that (1.2) implies
\begin{equation}
\sup_{2^k \leq t \leq 2^{k+1}} |L^\lambda_{j,t} * f(x)| = \sup_{2^k \leq t \leq 2^{k+1}} |L^\lambda_{j,t} \Delta_k f(x)|,
\end{equation}
where
\begin{equation}
(\Delta_k f)(\xi) = \Psi(2^{-k} \xi) \hat{f}(\xi)
\end{equation}
with $\Psi \in C_0^\infty(\mathbb{R}^n)$ satisfying
\begin{equation}
supp(\Psi) \subset \{b_1 \leq |\xi| \leq b_2\}, \quad \Psi(\xi) = 1 \text{ if } a_1 \leq |\xi| \leq a_2
\end{equation}
for some suitable numbers $a_1, a_2, b_1, b_2$ such that $0 < b_1 < a_1 < a_2 < b_2$.
By (2.4) and (2.5), to prove (2.3) it suffices to show that there exists $\varepsilon > 0$ such that
\begin{equation}
\int \sup_{2^k \leq t \leq 2^{k+1}} |L^\lambda_{j,t} * f(x)|^p w(x) \, dx \leq c 2^{-j \varepsilon} \int |f(x)|^p w(x) \, dx,
\end{equation}
where the constant $c$ is independent of $k$ and $j$. Indeed, by (2.6) we have
\begin{equation}
\sum_{k \in \mathbb{Z}} \int \sup_{2^k \leq t \leq 2^{k+1}} |L^\lambda_{j,t} \Delta_k f(x)|^p w(x) \, dx
\leq c 2^{-j \varepsilon} \sum_{k \in \mathbb{Z}} \int |\Delta_k f(x)|^p w(x) \, dx \leq c 2^{-j \varepsilon} \left( \|f\|_{L^p(w)} \right)^p,
\end{equation}
where the last inequality follows by a standard argument (see [1]); thus, using (2.5) and (2.7) in (2.4) and choosing $\{c_j\}$ suitably, we get (2.3).
To prove (2.6), we use the following estimates:

**Lemma 2.** Let $t \in [2^k, 2^{k+1}]$, $k \in \mathbb{Z}$. For any $\lambda$, $p$ and $\delta$ satisfying $\lambda > (n-1)/2$, $(n-1)/\lambda < p \leq 2$, $1 < p$ and $0 < \delta < p\lambda + 1 - n$, there exists $\varepsilon > 0$ such that
\begin{equation}
\int_{\mathbb{R}^n} |(L^\lambda_{j,t} * f(x))|^q (1 + |x|)^{(n+\delta)q/p} \, dx \leq c 2^{-j \varepsilon},
\end{equation}
where $1/p + 1/q = 1$, $(L^\lambda_{j,t} * f)(x) = r^{-n} L^\lambda_{j,t}(x/r)$ ($r > 0$) and the constant $c$ is independent of $t$, $k$ and $j$. 


Proof. Fix $t \in [2^k, 2^{k+1}]$. Since (1.4) holds, integration by parts gives

\begin{equation}
(2.8)
\left| (L_{j,t}^\lambda)_{2^k}(x) \right| \leq C_M 2^{-j \lambda} 2^{-j/2} (1 + 2^{-j} |x|)^{-M} \quad \text{for all } M > 0,
\end{equation}

where $C_M$ is independent of $t$, $k$ and $j$. Also, by (1.3) we have

\begin{equation}
(2.9)
\left| \{ \xi \in \mathbb{R}^n : \gamma(t, 2^k \xi) \in [1 - \delta, 1] \} \right| \leq c \delta,
\end{equation}

where $c$ is independent of $\delta \in (0, 1/2]$, $t$ and $k$.

By (2.9) and the Hausdorff-Young inequality we have

\begin{equation}
(2.10)
\int_{\mathbb{R}^n} \left| (L_{j,t}^\lambda)_{2^k}(x) \right|^q \, dx \leq \left( \int |\mathcal{F}((L_{j,t}^\lambda)_{2^k})(\xi)|^p \, d\xi \right)^{q/p}
\leq c \left( 2^{-bj \lambda} \left| \{ \xi \in \mathbb{R}^n : \gamma(t, 2^k \xi) \in [1 - 2^{-j}, 1] \} \right| \right)^{q/p}
\leq c 2^{-j(q \lambda + |q/p|)}.
\end{equation}

This implies

\begin{equation}
(2.11)
\int_{|x| \leq 2^j} \left| (L_{j,t}^\lambda)_{2^k}(x) \right|^q (1 + |x|)^{(n+\delta)q/p} \, dx
\leq c 2^{j(n+\delta)q/p} 2^{-j(q \lambda + |q/p|)} = c 2^{-j(q-1)(\rho \lambda + 1 - n - \delta)}.
\end{equation}

Let $0 < \tau < 1/2$. Then, by Hölder’s inequality we have

\[
\int_{|x| > 2^j} \left| (L_{j,t}^\lambda)_{2^k}(x) \right|^q (1 + |x|)^{(n+\delta)q/p} \, dx
\leq \left( \int_{|x| > 2^j} \left| (L_{j,t}^\lambda)_{2^k}(x) \right|^q (1 + |x|)^{(n+\delta)q/(pr)} \, dx \right)^{\tau} \left( \int_{\mathbb{R}^n} \left| (L_{j,t}^\lambda)_{2^k}(x) \right|^q \, dx \right)^{1-\tau}.
\]

By (2.8) we see that

\[
\left( \int_{|x| > 2^j} \left| (L_{j,t}^\lambda)_{2^k}(x) \right|^q (1 + |x|)^{(n+\delta)q/(pr)} \, dx \right)^{\tau}
\leq c (2^{-j \lambda} 2^{-j})^q \left( \int_{|x| > 2^j} (2^{-j} |x|)^{-q M |x|^{n+\delta q/(pr)}} \, dx \right)^{\tau}
\leq c (2^{-j \lambda} 2^{-j})^q 2^{j(n+\delta)q/p} 2^{jn \tau} \left( \int_{|x| > 1} |x|^{-q M + (n+\delta) q/(pr)} \, dx \right)^{\tau}
\leq c (2^{-j \lambda} 2^{-j})^q 2^{j(n+\delta)q/p} 2^{jn \tau}.
\]
where $M$ and $\tau$ are chosen so that $n + (n + \delta)q/(p\tau) < qM$. By this and (2.10) we have

\begin{equation}
\int_{|x|>2^j} \left| (L_{j,t}^\lambda)_{2^k}(x) \right|^q (1 + |x|)^{(n+\delta)q/p} \, dx \\
\leq c(2^{-j\lambda} 2^{-j\nu} 2^{j(n+\delta)q/p} 2^{j\nu} 2^{-j(q\lambda+q/p)(1-\tau)} \\
= c2^{-j(q-1)(p\lambda+1-n-\delta-\tau(n-1)/(q-1))},
\end{equation}

where we further assume that $p\lambda+1-n-\delta-\tau(n-1)/(q-1) > 0$. Combining (2.11) and (2.12), we get the conclusion. This completes the proof of Lemma 2.

We can find in Sogge [13, pp. 70–71] an argument similar to the one used in the proof of Lemma 2. Now we can prove (2.6). By Hölder’s inequality and Lemma 2

\begin{equation}
\sup_{t \in [2^k,2^{k+1}]} \left| (L_{j,t}^\lambda)_{2^k} \ast 2^{kn} f_{2^k}(x) \right|^p \\
\leq \sup_{t \in [2^k,2^{k+1}]} \left( \int \left| (L_{j,t}^\lambda)_{2^k}(x-y) \right|^q (1 + |x-y|)^{(n+\delta)q/p} \, dy \right)^{p/q} \\
\times \int \left| f(2^{-k}y) \right|^p (1 + |x-y|)^{-n-\delta} \, dy \\
\leq c2^{-(p/q)j} \int \left| f(2^{-k}y) \right|^p (1 + |x-y|)^{-n-\delta} \, dy.
\end{equation}

Since $w \in A_1$, we have

\begin{equation}
\int (1 + |x-y|)^{-\delta-n} w(2^{-k}x) \, dx \leq C_w w(2^{-k}y) \quad \text{for a.e. } y \in \mathbb{R}^n.
\end{equation}

By (2.13) and (2.14) we see that

\begin{equation}
\int \sup_{t \in [2^k,2^{k+1}]} \left| L_{j,t}^\lambda \ast f(x) \right|^p w(x) \, dx \\
= \int \sup_{t \in [2^k,2^{k+1}]} \left| (L_{j,t}^\lambda)_{2^k} \ast 2^{kn} f_{2^k}(x) \right|^p 2^{-kn} w(2^{-k}x) \, dx \\
\leq c2^{-(p/q)j} \int 2^{-kn} \left| f(2^{-k}y) \right|^p \left( \int (1 + |x-y|)^{-n-\delta} w(2^{-k}x) \, dx \right) \, dy \\
\leq c2^{-(p/q)j} \int \left| f(y) \right|^p w(y) \, dy,
\end{equation}

which proves (2.6). This completes the proof of Theorem 2.
3. Proof of Theorem 3

To prove Theorem 3, we use the following.

**Proposition 1.** Let \( \gamma(t, \xi) = t^{-1} |\Phi(\xi)| \), \( \Phi \in BL \). Let \( 0 < \delta < 7/8 \) and let \( m_\delta(r) \) be a continuously differentiable function supported in the interval \([1 - \delta, 1]\). Suppose that \( \|(d/dr)m_\delta\|_{L^1(\mathbb{R})} \leq 1 \). Define
\[
(U_\delta^* f)(\xi) = \hat{f}(\xi)m_\delta(\gamma(t, \xi)).
\]

Then
\[
\int_{\mathbb{R}^n} \int_0^\infty |U_\delta^* f(x)|^2 |x|^\alpha \frac{dt}{t} dx \leq c\delta \int_{\mathbb{R}^n} |f(x)|^2 |x|^\alpha dx,
\]
where \(-1 < \alpha \leq 0\) and the constant \(c\) is independent of \(\delta\).

When \( \gamma(t, \xi) = |\xi|/t \), this was proved in Carbery-Rubio de Francia-Vega [2] and Rubio de Francia [8] (see [9] for a related result).

To prove Proposition 1, we use the following result, which can be found in [5, 8].

**Lemma 3.** Let \( 0 < \beta < 2 \). Then
\[
\int_{\mathbb{R}^n} \int_0^\infty |g(t, x)|^2 |x|^{\beta} \frac{dt}{t} dx = c_\beta \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^\infty |\hat{g}(t, \xi) - \hat{g}(t, \eta)|^2 |\xi - \eta|^{-\beta} \frac{dt}{t} \frac{d\xi}{\xi} \frac{d\eta}{\eta},
\]
where \( \hat{g}(t, \xi) = \mathcal{F}(g(t, \cdot))(\xi) \). We also have
\[
\int_{\mathbb{R}^n} |g(x)|^2 |x|^\beta dx = c_\beta \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{g}(\xi) - \hat{g}(\eta)|^2 |\xi - \eta|^{-\beta} \frac{d\xi}{\xi} \frac{d\eta}{\eta}.
\]

Now, we give a proof of Proposition 1. By duality, to prove Proposition 1 it suffices to show
\[
\int_{\mathbb{R}^n} \left| \int_0^\infty U_\delta^* f_t(x) \frac{dt}{t} \right|^2 |x|^{-\alpha} dx \leq c\delta \int_{\mathbb{R}^n} \int_0^\infty |f_t(x)|^2 |x|^{-\alpha} \frac{dt}{t} dx,
\]
where we write \( f_t(x) = f(t, x) \) for a function \( f \) in \( C_0^\infty(\mathbb{R}^n) \). Define an operator \( L_F \) by
\[
(L_F f)(\xi) = \hat{f}(F\xi),
\]
where \( F \) is a mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). Let
\[
(V_\delta^* f)(\xi) = \hat{f}(\xi)m_\delta(|\xi|/t).
\]
Then, by taking Fourier transform, we can see that
\[
U_\delta^* f(x) = L_\Phi V_\delta^* L_{\Phi^{-1}} f(x).
\]
Since $\Phi \in BL$, by the second part of Lemma 3, (3.2) and a change of variables we have

\begin{equation}
\int_{\mathbb{R}^n} \left| \int_{0}^{\infty} U_t^\delta f_t(x) \frac{dt}{t} \right|^2 |x|^{-\alpha} \, dx \approx \int_{\mathbb{R}^n} \left| \int_{0}^{\infty} V_t^\delta L_{\Phi^{-1}} f_t(x) \frac{dt}{t} \right|^2 |x|^{-\alpha} \, dx
\end{equation}

(see [8] for this argument). Since the estimates in (3.1) are known when $\gamma(t, \xi) = |\xi| / t$, by (3.3) we have

\begin{equation}
\int_{\mathbb{R}^n} \left| \int_{0}^{\infty} U_t^\delta f_t(x) \frac{dt}{t} \right|^2 |x|^{-\alpha} \, dx \leq c \delta \int_{\mathbb{R}^n} \left| \int_{0}^{\infty} |L_{\Phi^{-1}} f_t(x)|^2 |x|^{-\alpha} \frac{dt}{t} \right| \, dx.
\end{equation}

The first part of Lemma 3 implies

\begin{equation}
\int_{\mathbb{R}^n} \left| \int_{0}^{\infty} L_{\Phi^{-1}} f_t(x) \frac{dt}{t} \right|^2 |x|^{-\alpha} \, dx \approx \int_{\mathbb{R}^n} \left| \int_{0}^{\infty} f(t, x) \frac{dt}{t} \right|^2 |x|^{-\alpha} \, dx.
\end{equation}

By (3.4) and (3.5) we get (3.1). This completes the proof of Proposition 1.

Put

\[ S_t^\lambda(f)(x) = \int_{\mathbb{R}^n} \eta(\gamma(t, \xi)) (1 - \gamma(t, \xi))^\lambda \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} \, d\xi, \]

where $\eta \in C_0^\infty(\mathbb{R})$ is such that $\eta(s) = 1$ if $3/4 \leq s \leq 2$ and $\eta(s) = 0$ if $s \leq 1/2$. Define

\[ S_t^\lambda(f)(x) = \sup_{t > 0} |S_t^\lambda(f)(x)|. \]

Then, by applying Proposition 1 we can prove the following result as in [2].

**Proposition 2.** Let $\gamma(t, \xi)$ be as in Proposition 1. Let $\lambda > 0$ and $-1 < \alpha \leq 0$. Then

\[ \int_{\mathbb{R}^n} \left| S_t^\lambda(f)(x) \right|^2 |x|^\alpha \, dx \leq C_\lambda \int_{\mathbb{R}^n} |f(x)|^2 |x|^\alpha \, dx. \]

Now, we can finish the proof of Theorem 3. Let $\zeta \in C_0^\infty(\mathbb{R})$ be such that $\zeta(s) + \eta(s) = 1$ for $s \in [0, 2]$ and $\text{supp}(\zeta) \subset [-1, 3/4]$, where $\eta$ is as in the definition of $S_t^\lambda$. Put

\[ M_t^\lambda f(x) = \int_{\mathbb{R}^n} \zeta(\gamma(t, \xi))(1 - \gamma(t, \xi))^\lambda \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} \, d\xi. \]

Then $S_t^\lambda(f)(x) \leq \sup_{t > 0} |M_t^\lambda f(x)| + S_t^\lambda(f)(x)$. As in the first part of the proof of Theorem 2, by Lemma 1 we have $\sup_{t > 0} |M_t^\lambda f(x)| \leq cM f(x)$. Thus the conclusion follows from Proposition 2 and the $L^2(w)$ boundedness of $M$ for $w \in A_2(\mathbb{R}^n)$. 
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References