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Estimates for Littlewood-Paley functions and extrapolation

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Abstract. We prove certain L^p -estimates for Littlewood-Paley functions arising from rough kernels. The estimates are useful for extrapolation to prove L^p -boundedness of the Littlewood-Paley functions under a sharp kernel condition.

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1. Introduction

We consider the Littlewood-Paley function on $\mathbb{R}^n$ defined by

$$S_\psi(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where ψ is in $L^1(\mathbb{R}^n)$ and $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. We always assume that

$$\int_{\mathbb{R}^n} \psi(x) dx = 0.$$

For the theory of Littlewood-Paley functions one can see [14], [15] and [16]. Also, recent developments can be found in [1], [4], [5], [7], [8], [9], [11], [12]. One of the well-known sufficient conditions for L^p boundedness of S_ψ is the following:

Theorem A. *Suppose that ψ satisfies*

$$\begin{aligned} |\psi(x)| &\leq C(1 + |x|)^{-n-\epsilon} \quad \text{for some } \epsilon > 0, \\ \int_{\mathbb{R}^n} |\psi(x - y) - \psi(x)| dx &\leq C|y|^\epsilon \quad \text{for some } \epsilon > 0. \end{aligned}$$

Then the operator S_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.

This is due to Benedek, Calderón and Panzone [3]. Fan-Sato [9] proved the following result, which substantially relaxes the conditions imposed on ψ in Theorem A.

Theorem B. *Suppose that the function ψ satisfies the following conditions:*

- (1) $B_\epsilon(\psi) < \infty$ for some $\epsilon > 0$, where $B_\epsilon(\psi) = \int_{|x| \geq 1} |\psi(x)| |x|^\epsilon dx$;
- (2) $D_u(\psi) < \infty$ for some $u > 1$, where $D_u(\psi) = \left(\int_{|x| \leq 1} |\psi(x)|^u dx \right)^{1/u}$;
- (3) $|\psi(x)| \leq h(|x|)\Omega(x')$ for all $x \in \mathbb{R}^n \setminus \{0\}$, where $x' = x/|x|$, for some non-negative functions h on $(0, \infty)$ and Ω on S^{n-1} (the unit sphere in $\mathbb{R}^n$) such that
 - (a) $h(r)$ is non-increasing on $(0, \infty)$ and $h(|x|) \in L^1(\mathbb{R}^n)$,
 - (b) $\Omega \in L^q(S^{n-1})$ for some q , $1 < q \leq \infty$.

Then S_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.

For the rest of this note we assume that ψ is compactly supported and the support is contained in the unit ball $\{|x| \leq 1\}$, for the sake of simplicity. We shall prove L^p estimates for S_ψ that are useful in extrapolation arguments to obtain a minimum condition on ψ for L^p boundedness of S_ψ .

Theorem 1. *Suppose that $|\psi(x)| \leq h(|x|)\Omega(x')$ for all $x \in \mathbb{R}^n \setminus \{0\}$, where h is a non-negative, non-increasing function on $(0, \infty)$ with support in $(0, 1]$ and Ω is a non-negative function on S^{n-1} . We assume that $h(|x|) \in L^1(\mathbb{R}^n)$, $\Omega \in L^1(S^{n-1})$ and $\psi \in L^q(\mathbb{R}^n)$ for some $q > 1$. Put $m_\psi(x) = h(|x|)\Omega(x')$. Then, we have*

$$\|S_\psi(f)\|_p \leq C_p (q/(q-1))^{1/2} (\|\psi\|_q + \|m_\psi\|_1) \|f\|_p$$

for all $p \in (1, \infty)$, where the constant C_p is independent of q , ψ , h and Ω ; and $q/(q-1) = 1$ when $q = \infty$.

Theorem 2. *Suppose that $\psi \in L^q(\mathbb{R}^n)$ for some $q > 1$. Then, we have*

$$\|S_\psi(f)\|_p \leq C_p (q/(q-1))^{1/2} \|\psi\|_q \|f\|_p$$

for all $p \in [2, \infty)$ with a constant C_p independent of q and ψ .

We are interested in the L^p estimates for $S_\psi(f)$ of Theorems 1 and 2 when q is near 1.

Let $L(\log L)^{1/2}(S^{n-1})$ be the class of the functions Ω on S^{n-1} satisfying

$$\int_{S^{n-1}} |\Omega(\theta)| [\log(2 + |\Omega(\theta)|)]^{1/2} d\sigma(\theta) < \infty,$$

where $d\sigma$ denotes the Lebesgue surface measure on S^{n-1} . The class $L(\log L)^{1/2}(\mathbb{R}^n)$ of functions on $\mathbb{R}^n$ is defined similarly. Using Theorems 1 and 2 and applying extrapolation, we can prove the following two results.

Corollary 1. *Let $\Omega \in L(\log L)^{1/2}(S^{n-1})$ and $\Omega \geq 0$. Suppose that $|\psi(x)| \leq h(|x|)\Omega(x')$ for all $x \in \mathbb{R}^n \setminus \{0\}$, where h is as in Theorem 1. We further assume that $h(|x|) \in L^q(\mathbb{R}^n)$ for some $q > 1$. Then, S_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.*

Corollary 2. *Suppose that ψ is in $L(\log L)^{1/2}(\mathbb{R}^n)$. Then S_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in [2, \infty)$.*

When $\psi \in L^q(\mathbb{R}^n)$ for some $q > 1$, L^p boundedness of S_ψ for $p \in [2, \infty)$ was proved in [9]. Corollary 2 is an improvement over the result. See [4] and also Remark 2 of [9] for L^p boundedness of S_ψ for $p < 2$.

Let $\psi(x) = |1 - |x|^2|^{\alpha-1} \Omega(x')$ if $0 < |x| < 1$, and $\psi(x) = 0$ if $|x| > 1$, where we assume that $\alpha > 0$, $\Omega \in L(\log L)^{1/2}(S^{n-1})$ and $\int_{S^{n-1}} \Omega d\sigma = 0$. Then, we can see that Corollary 2 applies to the operator S_ψ .

Let

$$\psi(x) = |x|^{-n+1} \Omega(x') \chi_{(0,1]}(|x|) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\},$$

where $\Omega \in L^1(S^{n-1})$, $\int_{S^{n-1}} \Omega d\sigma = 0$. Here χ_E denotes the characteristic function of a set E . Then, the Littlewood-Paley function $S_\psi(f)$ coincides with the Marcinkiewicz integral $\mu_\Omega(f)$ in Stein [14] (see also Hörmander [10]). As an application of Theorem 1 we have the following result:

Corollary 3. *If $\Omega \in L^q(S^{n-1})$ for some $q > 1$, then*

$$\|\mu_\Omega(f)\|_p \leq C_p (q/(q-1))^{1/2} \|\Omega\|_q \|f\|_p$$

for $p \in (1, \infty)$, where the constant C_p is independent of q and Ω . Here $\|\Omega\|_q$ denotes the norm of Ω in $L^q(S^{n-1})$.

Also, as a consequence of Corollary 1, we have the following result of Al-Salman, Al-Qassem, Cheng and Pan [1].

Theorem C. *If $\Omega \in L(\log L)^{1/2}(S^{n-1})$, then μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.*

The case $p = 2$ of Theorem C is due to Walsh [17]. In Section 2 we shall prove Theorem 1. We adapt the method of [6] involving the Littlewood-Paley theory for the case of square function operators. We apply a Littlewood-Paley decomposition adapted to a suitable lacunary sequence depending on q for which $\psi \in L^q(\mathbb{R}^n)$. See [2] and also [13] for the method of appropriately choosing a lacunary sequence. Theorem 2 is proved in Section 3 by applying the Littlewood-Paley theory in like manner. In proving Theorems 1 and 2, basic estimates and key observations of [11], [9] will be used. Also, to prove Theorem 2, we apply an induction argument similar to the one used in [13] to get sharp L^p estimates for singular Radon transforms.

Applying an extrapolation method (see, e.g., Zygmund [18, Chap. XII, pp. 119–120]), we shall prove Corollary 1 by Theorem 1 in Section 4. Similarly, Corollary 2 follows from Theorem 2. Throughout this note, the letter C will be used to denote non-negative constants which may be different in different occurrences.

2. Proof of Theorem 1

For $k \in \mathbb{Z}$ (the set of integers) and $\rho \geq 2$, let T_k be an operator mapping functions on $\mathbb{R}^n$ to $\mathcal{H}$ -valued functions on $\mathbb{R}^n$ defined as

$$(T_k(f)(x))(t) = (\psi_t * f)(x) \chi_{[1, \rho)}(\rho^{-k}t),$$

where $\mathcal{H}$ is the Hilbert space $L^2((0, \infty), dt/t)$. Note that

$$|T_k(f)(x)|_{\mathcal{H}} = \left(\int_{\rho^k}^{\rho^{k+1}} |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

Lemma 1. *Let ψ be as in Theorem 1. Then we have*

$$\left\| \left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_s \leq C_s \|\psi\|_1^{1/2} \|m_\psi\|_1^{1/2} (\log \rho)^{1/2} \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_s$$

for all $s \in (1, \infty)$, where m_ψ is as in Theorem 1 and the constant C_s is independent of ρ , ψ , h and Ω .

Proof. Define a maximal function

$$M_\psi(f)(x) = \sup_{t>0} |\psi_t * f(x)|.$$

Since the method of rotations implies $\|M_\psi(f)\|_r \leq C_r \|m_\psi\|_1 \|f\|_r$ for all $r > 1$, arguing as in the proofs of Lemmas 1 and 2 of [9] and checking the constants in the arguments, we can obtain Lemma 1. $\square$

Lemma 2. *Let ψ be as in Theorem 2. Then*

$$\int_{\rho^k}^{\rho^{k+1}} \left| \hat{\psi}(t\xi) \right|^2 \frac{dt}{t} \leq C(\log \rho) \|\psi\|_q^2 \min(1, |\rho^{k+1}\xi|, |\rho^k\xi|^{-1})^{1/(2q')},$$

where the constant C is independent of $\rho \geq 2$, $q > 1$ and ψ . Here $q' = q/(q-1)$, and $\hat{g}$ denotes the Fourier transform of g ; we also write $\hat{g} = \mathcal{F}(g)$.

Proof. First, a direct computation implies that

$$\int_1^\rho \left| \hat{\psi}(t\rho^k\xi) \right|^2 dt/t \leq (\log \rho) \|\psi\|_1^2 \leq C(\log \rho) \|\psi\|_q^2. \quad (2.1)$$

Next, using Lemmas 2 and 3 of [11], we see that

$$\begin{aligned} \int_1^\rho \left| \hat{\psi}(t\rho^k\xi) \right|^2 dt/t &\leq \sum_{0 \leq m \leq (\log \rho)/\log 2} \int_1^2 \left| \hat{\psi}(t2^m\rho^k\xi) \right|^2 dt/t \\ &\leq \sum_{0 \leq m \leq (\log \rho)/\log 2} C \|\psi\|_q^2 |2^m\rho^k\xi|^{-1/(2q')} \\ &\leq C(\log \rho) \|\psi\|_q^2 |\rho^k\xi|^{-1/(2q')}. \end{aligned} \quad (2.2)$$

Finally, by the proof of Lemma 1 of [11], we have $|\hat{\psi}(\xi)| \leq C|\xi||\psi|_1$, and hence

$$\begin{aligned} \int_1^\rho \left| \hat{\psi}(t\rho^k \xi) \right|^2 dt/t &\leq C\|\psi\|_1^2 \int_1^\rho |t\rho^k \xi|^2 dt/t \leq C\|\psi\|_1^2 \rho^{k+1} \xi^2 \\ &\leq C(\log \rho) \|\psi\|_q^2 |\rho^{k+1} \xi|^2. \end{aligned} \quad (2.3)$$

Combining (2.1)–(2.3), we have the conclusion, since

$$\min(1, |\rho^{k+1} \xi|^2, |\rho^k \xi|^{-1/(2q')}) \leq \min(1, |\rho^{k+1} \xi|, |\rho^k \xi|^{-1/(2q')}).$$

□

We choose a sequence $\{\Psi_k\}_{k=-\infty}^\infty$ of non-negative functions in $C^\infty(\mathbb{R})$ such that

$$\text{supp}(\Psi_k) \subset [\rho^{-k-1}, \rho^{-k+1}], \quad \sum_k \Psi_k(t) = 1 \quad \text{for } t > 0.$$

We further assume that

$$|(d/dt)^j \Psi_k(t)| \leq c_j |t|^{-j} \quad (j = 1, 2, \dots),$$

where the constants c_j are independent of ρ and k . This is feasible since $\rho \geq 2$. Define Fourier multiplier operators Δ_j as

$$\mathcal{F}(\Delta_j(f))(\xi) = \Psi_j(|\xi|) \hat{f}(\xi) \quad \text{for } j \in \mathbb{Z}$$

and decompose $(\psi_t * f)(x)$ as $(\psi_t * f)(x) = \sum_{j \in \mathbb{Z}} F_j(x, t)$, where

$$F_j(x, t) = \sum_{k \in \mathbb{Z}} \Delta_{j+k}(\psi_t * f)(x) \chi_{[\rho^k, \rho^{k+1})}(t).$$

Then, we have $S_\psi(f)(x) \leq \sum_{j \in \mathbb{Z}} U_j(f)(x)$, where

$$U_j(f)(x) = \left(\int_0^\infty |F_j(x, t)|^2 \frac{dt}{t} \right)^{1/2} = \left(\sum_k |T_k(\Delta_{j+k}(f))(x)|_{\mathcal{H}}^2 \right)^{1/2}.$$

Let $E_j = \{\rho^{-1-j} \leq |\xi| \leq \rho^{1-j}\}$. Then by the Plancherel theorem and Lemma 2 we have

$$\begin{aligned} \|U_j(f)\|_2^2 &= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_{\rho^k}^{\rho^{k+1}} |\Delta_{j+k}(\psi_t * f)(x)|^2 \frac{dt}{t} dx \\ &\leq \sum_{k \in \mathbb{Z}} C \int_{E_{j+k}} \left(\int_{\rho^k}^{\rho^{k+1}} \left| \hat{\psi}(t\xi) \right|^2 \frac{dt}{t} \right) \left| \hat{f}(\xi) \right|^2 d\xi \\ &\leq \sum_{k \in \mathbb{Z}} C(\log \rho) \|\psi\|_q^2 \int_{E_{j+k}} \min(1, |\rho^{k+1} \xi|, |\rho^k \xi|^{-1/(2q')}) \left| \hat{f}(\xi) \right|^2 d\xi \\ &\leq C(\log \rho) \|\psi\|_q^2 \min(1, \rho^{-|j|+2})^{1/(2q')} \sum_{k \in \mathbb{Z}} \int_{E_{j+k}} \left| \hat{f}(\xi) \right|^2 d\xi \\ &\leq C(\log \rho) \|\psi\|_q^2 \min(1, \rho^{-|j|+2})^{1/(2q')} \|f\|_2^2, \end{aligned}$$

where the last inequality is valid since the sets E_j are finitely overlapping. Thus we have

$$\|U_j(f)\|_2 \leq C(\log \rho)^{1/2} \|\psi\|_q \min(1, \rho^{-|j|+2})^{1/(4q')} \|f\|_2. \quad (2.4)$$

We now use the Littlewood-Paley inequality

$$\left\| \left(\sum_k |\Delta_k(f)|^2 \right)^{1/2} \right\|_p \leq c_p \|f\|_p, \quad (2.5)$$

where $1 < p < \infty$ and the constant c_p is independent of ρ . By (2.5) and Lemma 1 we see that

$$\begin{aligned} \|U_j(f)\|_s &= \left\| \left(\sum_{k \in \mathbb{Z}} |T_k(\Delta_{j+k}(f))|_{\mathfrak{H}}^2 \right)^{1/2} \right\|_s \\ &\leq C \|\psi\|_1^{1/2} \|m_\psi\|_1^{1/2} (\log \rho)^{1/2} \left\| \left(\sum_{k \in \mathbb{Z}} |\Delta_{j+k}(f)|^2 \right)^{1/2} \right\|_s \\ &\leq C \|\psi\|_1^{1/2} \|m_\psi\|_1^{1/2} (\log \rho)^{1/2} \|f\|_s \end{aligned} \quad (2.6)$$

for $s \in (1, \infty)$. Interpolating between (2.4) and (2.6), we get

$$\begin{aligned} \|U_j(f)\|_p &\leq C(\log \rho)^{1/2} \left(\|\psi\|_q \min(1, \rho^{-|j|+2})^{1/(4q')} \right)^\eta \left(\|\psi\|_1^{1/2} \|m_\psi\|_1^{1/2} \right)^{1-\eta} \|f\|_p \\ &\leq C(\log \rho)^{1/2} \|\psi\|_q^{(1+\eta)/2} \|m_\psi\|_1^{(1-\eta)/2} \min(1, \rho^{-|j|+2})^{\eta/(4q')} \|f\|_p \end{aligned}$$

for some $\eta \in (0, 1]$ depending on p , where $1 < p < \infty$. Thus

$$\begin{aligned} \|S_\psi(f)\|_p &\leq \sum_{j \in \mathbb{Z}} \|U_j(f)\|_p \\ &\leq C \left(\sum_{j \in \mathbb{Z}} \min(1, \rho^{-|j|+2})^{\eta/(4q')} \right) (\log \rho)^{1/2} \|\psi\|_q^{(1+\eta)/2} \|m_\psi\|_1^{(1-\eta)/2} \|f\|_p \\ &\leq C(1 - \rho^{-\eta/(4q')})^{-1} (\log \rho)^{1/2} \|\psi\|_q^{(1+\eta)/2} \|m_\psi\|_1^{(1-\eta)/2} \|f\|_p \\ &\leq C(1 - \rho^{-\eta/(4q')})^{-1} (\log \rho)^{1/2} (\|\psi\|_q + \|m_\psi\|_1) \|f\|_p, \end{aligned}$$

where the last inequality follows from Young's inequality. Taking $\rho = 2^{q'}$, we get the conclusion of Theorem 1, since

$$(1 - \rho^{-\eta/(4q')})^{-1} (\log \rho)^{1/2} = (1 - 2^{-\eta/4})^{-1} ((q \log 2)/(q - 1))^{1/2}$$

if $\rho = 2^{q'}$.

3. Proof of Theorem 2

Let ψ be as in Theorem 2 and $\rho \geq 2$.

Lemma 3. *Let $\theta \in (0, 1)$. Suppose $2 \leq s < 2(1 + \theta)/\theta$ and put $r = (s/2)'$. Then we have*

$$\left\| \left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_s \leq C_s (\log \rho)^{1/2} \|\psi\|_q \left(1 - \rho^{-\theta/(4q')} \right)^{-1/r} \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_s,$$

where the constant C_s is independent of ψ , $q > 1$ and ρ .

To prove this we need the following:

Lemma 4. *Let $p(x) = \int_1^\rho |\psi_t(x)| dt/t$. Define a maximal function $N_\psi^{(\rho)}(f)(x) = \sup_{k \in \mathbb{Z}} |(f * p_{\rho^k})(x)|$. Let $\theta \in (0, 1)$. Then, for $p > 1 + \theta$ we have*

$$\|N_\psi^{(\rho)}(f)\|_p \leq C_p (\log \rho) \|\psi\|_q \left(1 - \rho^{-\theta/(4q')} \right)^{-2/p} \|f\|_p,$$

where the constant C_p is independent of ψ , $q > 1$ and ρ .

Proof. Let $\varphi \in C^\infty(\mathbb{R}^n)$ be compactly supported and satisfy $\hat{\varphi}(0) = 1$, $\varphi \geq 0$. Define $\Phi(x) = p(x) - (\log \rho) \|\psi\|_1 \varphi(x)$. Then

$$\|\Phi_{\rho^k}\|_1 \leq C (\log \rho) \|\psi\|_q, \quad (3.1)$$

$$|\mathcal{F}(\Phi_{\rho^k})(\xi)| \leq C (\log \rho) \|\psi\|_q \min(|\rho^{k+1}\xi|, |\rho^k\xi|^{-1})^{1/(4q')}. \quad (3.2)$$

Note that $\|p\|_1 = (\log \rho) \|\psi\|_1$ and

$$|\hat{p}(\xi)| \leq (\log \rho)^{1/2} \left(\int_1^\rho |\mathcal{F}(|\psi|)(t\xi)|^2 dt/t \right)^{1/2}.$$

Using these observations and arguing as in the proof of Lemma 2, we can prove the estimates (3.1), (3.2).

We see that $\|N_\psi^{(\rho)}(f)\|_\infty \leq C (\log \rho) \|\psi\|_q \|f\|_\infty$, since $\|p_{\rho^k}\|_1 \leq C (\log \rho) \|\psi\|_q$. So, Lemma 4 follows by interpolation if we prove it for $p \in (1 + \theta, 2]$. Define

$$g(f)(x) = \left(\sum_{k=-\infty}^{\infty} |\Phi_{\rho^k} * f(x)|^2 \right)^{1/2}.$$

Then

$$\begin{aligned} N_\psi^{(\rho)}(f) &\leq g(f) + (\log \rho) \|\psi\|_1 \sup_{t>0} |\varphi_t * f| \\ &\leq g(f) + C (\log \rho) \|\psi\|_1 M(f), \end{aligned} \quad (3.3)$$

where $M(f)$ denotes the Hardy-Littlewood maximal function. Thus, it suffices to prove $\|g(f)\|_p \leq CAB^{2/p} \|f\|_p$ for $p \in (1 + \theta, 2]$, where $A = (\log \rho) \|\psi\|_q$ and $B =$

$\left(1 - \rho^{-\theta/(4q')}\right)^{-1}$. Recalling a well-known property of Rademacher's functions, we can see that this follows from

$$\|V_\epsilon(f)\|_p \leq CAB^{2/p} \|f\|_p \quad (3.4)$$

for $p \in (1 + \theta, 2]$ with a constant C independent of ϵ , where $V_\epsilon(f) = \sum_k \epsilon_k \Phi_{\rho^k} * f$, $\epsilon = \{\epsilon_k\}$, $\epsilon_k = 1$ or -1 .

To prove (3.4) we need the following.

Lemma 5. *We define a sequence $\{p_j\}_1^\infty$ by $p_1 = 2$ and $1/p_{j+1} = 1/2 + (1 - \theta)/(2p_j)$ for $j \geq 1$. Then, for $j \geq 1$ we have*

$$\|V_\epsilon(f)\|_{p_j} \leq C_j AB^{2/p_j} \|f\|_{p_j}, \quad (3.5)$$

where C_j is independent of ψ , $q > 1$, ρ and ϵ .

We note that $\{p_j\}$ is decreasing and converges to $1 + \theta$.

Proof of Lemma 5. Let

$$V_j(f) = \sum_{k=-\infty}^{\infty} \epsilon_k (\Phi_{\rho^k} * \Delta_{j+k}(f)).$$

where the operators Δ_j are as in the proof of Theorem 1. Then, $V_\epsilon(f) = \sum_j V_j(f)$. Using Plancherel's theorem and the estimates (3.1), (3.2) as in the proof of (2.4), we have

$$\|V_j(f)\|_2 \leq CA \min\left(1, \rho^{-|j|+2}\right)^{1/(4q')} \|f\|_2. \quad (3.6)$$

It follows that $\|V_\epsilon(f)\|_2 \leq \sum_j \|V_j(f)\|_2 \leq CAB \|f\|_2$. This proves (3.5) for $j = 1$.

We now assume (3.5) for $j = s$ and prove it for $j = s + 1$. It will complete the proof of Lemma 5 by induction. First, by (3.3) we see that

$$\begin{aligned} \sup_k \left| \Phi_{\rho^k} * f \right| &\leq \sup_k (p_{\rho^k} * |f|) + (\log \rho) \|\psi\|_1 \sup_{t>0} (\varphi_t * |f|) \\ &\leq g(|f|) + 2(\log \rho) \|\psi\|_1 \sup_{t>0} (\varphi_t * |f|) \\ &\leq g(|f|) + C(\log \rho) \|\psi\|_1 M(f). \end{aligned}$$

From this we see that

$$\left\| \sup_k \left| \Phi_{\rho^k} * f \right| \right\|_{p_s} \leq CAB^{2/p_s} \|f\|_{p_s}, \quad (3.7)$$

since our assumption implies $\|g(f)\|_{p_s} \leq CAB^{2/p_s} \|f\|_{p_s}$.

Let $1/v - 1/2 = 1/(2p_s)$. Then, by the proof of Lemma in [6, p. 544] and the estimates (3.1), (3.7), we have the vector valued inequality

$$\left\| \left(\sum_k |\Phi_{\rho^k} * g_k|^2 \right)^{1/2} \right\|_v \leq CAB^{1/p_s} \left\| \left(\sum_k |g_k|^2 \right)^{1/2} \right\|_v. \quad (3.8)$$

For the sake of completeness, here we give a proof of (3.8). By duality it suffices to prove (3.8) with v' in place of v . Note that $p_s = (v'/2)'$. Take a non-negative $u \in L^{p_s}$ such that $\|u\|_{p_s} \leq 1$ and

$$I = \int \left(\sum_k |\Phi_{\rho^k} * g_k|^2 \right) u \, dx,$$

where

$$I = \left\| \left(\sum_k |\Phi_{\rho^k} * g_k|^2 \right)^{1/2} \right\|_{v'}^2.$$

Since $|\Phi_{\rho^k} * g_k|^2 \leq \|\Phi\|_1 (|\Phi_{\rho^k}| * |g_k|^2)$,

$$I \leq \|\Phi\|_1 \sum_k \int (|\Phi_{\rho^k}| * |g_k|^2) u \, dx \leq \|\Phi\|_1 \sum_k \int |g_k|^2 \sup_k |\tilde{\Phi}_{\rho^k}| * u \, dx,$$

where $\tilde{\Phi}(x) = \Phi(-x)$. Thus, applying Hölder's inequality, by (3.1), (3.7) we have

$$I \leq \|\Phi\|_1 \left\| \left(\sum_k |g_k|^2 \right)^{1/2} \right\|_{v'}^2 \left\| \sup_k |\tilde{\Phi}_{\rho^k}| * u \right\|_{p_s} \leq CA^2 B^{2/p_s} \left\| \left(\sum_k |g_k|^2 \right)^{1/2} \right\|_{v'}^2$$

as claimed.

Also, from the Littlewood-Paley theory it follows that

$$\|V_j(f)\|_p \leq c_p \left\| \left(\sum_k |\Phi_{\rho^k} * \Delta_{j+k}(f)|^2 \right)^{1/2} \right\|_p, \quad (3.9)$$

where $1 < p < \infty$ and c_p is independent of ρ . By (2.5), (3.8) and (3.9) we have

$$\|V_j(f)\|_v \leq CAB^{1/p_s} \|f\|_v. \quad (3.10)$$

Since $1/p_{s+1} = \theta/2 + (1-\theta)/v$, interpolating between (3.6) and (3.10), we have

$$\|V_j(f)\|_{p_{s+1}} \leq CAB^{(1-\theta)/p_s} \min \left(1, \rho^{-\theta(|j|-2)/(4q')} \right) \|f\|_{p_{s+1}}.$$

Therefore,

$$\begin{aligned} \|V_\epsilon(f)\|_{p_{s+1}} &\leq \sum_j \|V_j(f)\|_{p_{s+1}} \leq CAB^{(1-\theta)/p_s} (1 - \rho^{-\theta/(4q')})^{-1} \|f\|_{p_{s+1}} \\ &\leq CAB^{2/p_{s+1}} \|f\|_{p_{s+1}}. \end{aligned}$$

This proves (3.5) for $j = s+1$, which completes the proof of Lemma 5. $\square$

We now prove (3.4) for $p \in (1+\theta, 2]$. Let $\{p_j\}_1^\infty$ be as in Lemma 5. Then we have $p_{N+1} < p \leq p_N$ for some N . Thus, interpolating between the estimates in (3.5) for $j = N$ and $j = N+1$, we have (3.4). This completes the proof of Lemma 4. $\square$

Proof of Lemma 3. From the proof of Lemma 1 in [9], we see that

$$\left\| \left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_s^2 \leq C \|\psi\|_1 \sum_k \int |f_k|^2 \tilde{N}_\psi^{(\rho)}(g) dx, \quad (3.11)$$

where $\tilde{N}_\psi^{(\rho)}(f) = \sup_{k \in \mathbb{Z}} |\tilde{p}_{\rho^k} * f|$ with $\tilde{p}(x) = p(-x)$, and g is a non-negative function in L^r satisfying $\|g\|_r \leq 1$, $r = (s/2)'$. The range of s in Lemma 3 implies that $r > 1 + \theta$. Therefore, by Hölder's inequality and Lemma 4 it follows that

$$\begin{aligned} \sum_k \int |f_k|^2 \tilde{N}_\psi^{(\rho)}(g) dx &\leq C \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_s^2 \|\tilde{N}_\psi^{(\rho)}(g)\|_r \\ &\leq C(\log \rho) \|\psi\|_q \left(1 - \rho^{-\theta/(4q')} \right)^{-2/r} \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_s^2. \end{aligned}$$

Combining this with (3.11), we can get the conclusion of Lemma 3. $\square$

Now we can give a proof of Theorem 2. Given $p \in [2, \infty)$, let $s, \theta \in (0, 1)$ be such that $2 \leq p < s < 2(1 + \theta)/\theta$. Let U_j be as in the proof of Theorem 1. Then, by Lemma 3 and (2.5) we see that

$$\|U_j(f)\|_s \leq C(\log \rho)^{1/2} \|\psi\|_q \left(1 - \rho^{-\theta/(4q')} \right)^{-1/r} \|f\|_s,$$

where $r = (s/2)'$. Interpolating between this estimate and (2.4), we have

$$\|U_j(f)\|_p \leq C(\log \rho)^{1/2} \|\psi\|_q \min(1, \rho^{-|j|+2})^{\eta/(4q')} \left(1 - \rho^{-\theta/(4q')} \right)^{-(1-\eta)/r} \|f\|_p$$

for some $\eta \in (0, 1]$ (we note that (2.4) holds under the assumptions of ψ in Theorem 2). Thus, arguing as in the proof of Theorem 1, we get

$$\|S_\psi(f)\|_p \leq C(\log \rho)^{1/2} \|\psi\|_q \left(1 - \rho^{-\eta/(4q')} \right)^{-1} \left(1 - \rho^{-\theta/(4q')} \right)^{-(1-\eta)/r} \|f\|_p.$$

If we put $\rho = 2^{q'}$, we get the desired estimate in Theorem 2.

4. Proof of Corollary 1

Fix $p \in (1, \infty)$ and f with $\|f\|_p \leq 1$. We write $H(x) = h(|x|)$, where h is as in Corollary 1. Put $R(\psi) = \|S_\psi(f)\|_p$. Let $F_k = \{\theta \in S^{n-1} : 2^{k-1} < |\Omega(\theta)| \leq 2^k\}$ for $k \geq 2$ and $F_1 = \{\theta \in S^{n-1} : |\Omega(\theta)| \leq 2\}$. Let $\Omega_k(\theta) = \Omega(\theta) \chi_{F_k}(\theta)$ for $k \geq 1$. We define $E_k = \{x \in B(0, 1) : x \neq 0, x' \in F_k\}$ for $k = 1, 2, 3, \dots$, where $B(0, 1) = \{|x| \leq 1\}$. We decompose ψ as $\psi = \sum_{k=1}^\infty \psi_k$, where

$$\psi_k = \psi \chi_{E_k} - |B(0, 1)|^{-1} \left(\int_{E_k} \psi dx \right) \chi_{B(0, 1)}.$$

We note that $\int \psi_k dx = 0$ and $|\psi_k(x)| \leq h^*(|x|)\Omega_k^*(x')$ for $x \in \mathbb{R}^n \setminus \{0\}$, where

$$h^*(|x|) = [h(|x|) + C\|H\|_1] \chi_{(0,1]}(|x|), \quad \Omega_k^*(x') = \Omega_k(x') + \|\Omega_k\|_1.$$

We see that $\|\Omega_k^*\|_r \leq C2^k e_k^{1/r}$ for $1 < r < \infty$, where $e_k = \sigma(F_k)$, for $k \geq 1$, and that $\|H^*\|_q \leq C\|H\|_q$, where $H^*(x) = h^*(|x|)$.

Let $m_{\psi_k}(x) = h^*(|x|)\Omega_k^*(x')$ for $k \geq 1$. Then, applying Theorem 1 and using subadditivity of $R(\psi)$, we have

$$\begin{aligned} \|S_\psi(f)\|_p &= R(\psi) \leq \sum_{k \geq 1} R(\psi_k) \\ &\leq C(q/(q-1))^{1/2} \sum_{k < 1/(q-1)} \|m_{\psi_k}\|_q + C \sum_{k \geq 1/(q-1)} k^{1/2} \|m_{\psi_k}\|_{1+1/k} \\ &\leq C_q \|H\|_q + C \|H\|_q \sum_{k \geq 1/(q-1)} k^{1/2} \|\Omega_k^*\|_{1+1/k} \\ &\leq C_q \|H\|_q + C \|H\|_q \sum_{k \geq 1/(q-1)} k^{1/2} 2^k e_k^{k/(k+1)}. \end{aligned}$$

Using Young's inequality, we can see that the sum $\sum_{k \geq 1} k^{1/2} 2^k e_k^{k/(k+1)}$ is bounded by

$$\begin{aligned} &2 \sum_{k \geq 1} (k/(k+1)) \left(k^{(1+1/k)/2} 2^{k(1+1/k)} e_k \right) + 2 \sum_{k \geq 1} 2^{-k-1}/(k+1) \\ &\leq C \sum_{k \geq 1} k^{1/2} 2^k e_k + C \leq C \int_{S^{n-1}} |\Omega(\theta)| (\log(2 + |\Omega(\theta)|))^{1/2} d\sigma(\theta) + C. \end{aligned}$$

Combining results, we get the conclusion of Corollary 1.

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