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HARDY'S INEQUALITIES FOR HERMITE AND LAGUERRE EXPANSIONS

Dedicated to Professor Satoru Igari on his 60th birthday

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ABSTRACT

The well-known inequality of Hardy for Fourier coefficients of functions $f(t) \sim \sum_{n=-\infty}^{\infty} b_n e^{int}$ in the real Hardy space is $\sum_{n=-\infty}^{\infty} |b_n|/(|n|+1) < \infty$. We shall establish analogues of this inequality for the Hermite function expansions and also for the Laguerre function expansions.

1. Introduction

Hardy's inequality says that if $f(t) \sim \sum_{n=-\infty}^{\infty} b_n e^{int}$ is in $\text{Re } H^1$, then

$$\sum_{n=-\infty}^{\infty} \frac{|b_n|}{|n|+1} < \infty,$$

where $\text{Re } H^1$ is the real Hardy space consisting of the boundary values of the real parts of functions in the Hardy space H^1 on the unit disk in the plane. The aim of this paper is to establish analogues of this inequality for the Hermite function expansions and also for the Laguerre function expansions.

Let $\mathcal{H}_n(x)$ be the Hermite function defined by

$$\mathcal{H}_n(x) = \{\pi^{1/2} 2^n \Gamma(n+1)\}^{-1/2} H_n(x) e^{-x^2/2},$$

where $H_n(x)$ is the Hermite polynomial of degree n given by

$$H_n(x) = (-1)^n \exp(x^2) (d/dx)^n \exp(-x^2).$$

Then the system $\{\mathcal{H}_n\}_{n=0}^{\infty}$ is complete orthonormal on the real line $\mathbb{R}$ with respect to the ordinary Lebesgue measure dx (compare [6, 5.7]). This system leads to the formal expansion of a function $f(x)$ on $\mathbb{R}$:

$$f(x) \sim \sum_{n=0}^{\infty} c_n(f) \mathcal{H}_n(x),$$

where $c_n(f) = \int_{-\infty}^{\infty} f(x) \mathcal{H}_n(x) dx$ is the n th Hermite–Fourier coefficient of $f(x)$.

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Let $\mathcal{L}_n^\alpha(x)$, $\alpha > -1$, be the Laguerre function defined by

$$\mathcal{L}_n^\alpha(x) = \left\{ \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \right\}^{1/2} L_n^\alpha(x) e^{-x/2} x^{\alpha/2},$$

where $L_n^\alpha(x) = (n!)^{-1} x^{-\alpha} e^x (d/dx)^n \{x^{n+\alpha} e^{-x}\}$ is the Laguerre polynomial of degree n and of order α . Then $\{\mathcal{L}_n^\alpha\}_{n=0}^\infty$ is a complete orthonormal system on the interval $(0, \infty)$ with respect to dx (compare [6, 5.7]). We have the formal expansion

$$g(x) \sim \sum_{n=0}^{\infty} c_n^\alpha(g) \mathcal{L}_n^\alpha(x)$$

of a function $g(x)$ on $(0, \infty)$, where $c_n^\alpha(g) = \int_0^\infty g(x) \mathcal{L}_n^\alpha(x) dx$ is the n th Laguerre–Fourier coefficient.

Let $H^1(\mathbb{R})$ be the real Hardy space consisting of the boundary values of the real parts of functions in the Hardy space $H^1(\mathbb{R}_+^2)$ on the upper half plane $\mathbb{R}_+^2$. In $H^1(\mathbb{R})$, we consider the norm induced by $H^1(\mathbb{R}_+^2)$. For a function $g(x)$ on $(0, \infty)$, we define the extension $\tilde{g}(x)$ of $g(x)$ to $\mathbb{R}$ by $\tilde{g}(x) = g(x)$ if $0 < x$, and $\tilde{g}(x) = 0$ if $x \leq 0$. We define

$$H^1(0, \infty) = \{g(x) \text{ on } (0, \infty); \tilde{g}(x) \in H^1(\mathbb{R})\},$$

$$\|g\|_{H^1(0, \infty)} = \|\tilde{g}\|_{H^1(\mathbb{R})}.$$

Our main results of this paper are as follows.

THEOREM. (i) *There exists a constant C such that*

$$\sum_{n=0}^{\infty} \frac{|c_n(f)|}{(n+1)^{29/36}} \leq C \|f\|_{H^1(\mathbb{R})} \quad (1)$$

for $f(x) \sim \sum_{n=0}^{\infty} c_n(f) \mathcal{H}_n(x)$ in $H^1(\mathbb{R})$.

(ii) *Let $\alpha \geq 0$. Then there exists a constant C such that*

$$\sum_{n=0}^{\infty} \frac{|c_n^\alpha(g)|}{n+1} \leq C \|g\|_{H^1(0, \infty)} \quad (2)$$

for $g(x) \sim \sum_{n=0}^{\infty} c_n^\alpha(g) \mathcal{L}_n^\alpha(x)$ in $H^1(0, \infty)$.

The proof of the Theorem will be given in the next section. The atomic decomposition characterization of Hardy spaces will play an essential role in the proof. Also, the transplantation theorem (see (9) below) of R. Askey [1, p. 401, line 14] for Laguerre coefficients will be of use in the proof of part (ii) of the Theorem. The deduction of Paley's theorem from the Theorem will be discussed in Section 3 with a remark.

For a historical survey on Hardy's inequality and Paley's theorem, we may refer to [2, p. 398, Comments]. The exposition of the role of the atomic decomposition characterization in the classical harmonic analysis is in R. R. Coifman and G. Weiss [3]. We shall also consult J. Garcia-Cuerva and J. L. Rubio de Francia [4] for the Hardy space theory.

2. Proof of the Theorem

An H^1 atom is a function $a(x)$, $x \in \mathbb{R}$, supported in an interval $(b, b+h)$ satisfying $|a(x)| \leq h^{-1}$ and $\int a(x) dx = 0$. The space $H^1(\mathbb{R})$ is characterized in terms of atoms (compare [4, Chapter 3]): $f(x)$ belongs to $H^1(\mathbb{R})$ if and only if $f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x)$,

where every $a_j(x)$ is an H^1 atom and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$. The $H^1(\mathbb{R})$ norm of $f(x)$ is equivalent to $\inf \sum_{j=0}^{\infty} |\lambda_j|$, the infimum being taken over all decompositions. The space $H^1(0, \infty)$ is also characterized as follows [4, Lemma 7.40]: $g(x)$ belongs to $H^1(0, \infty)$ if and only if $g(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x)$, where every $a_j(x)$ is an H^1 atom satisfying $\text{supp } a(x) \subset (0, \infty)$ and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$.

We shall first give a lemma. Let $\{\phi_n\}_{n=0}^{\infty}$ be an orthonormal system on an interval (c, d) . For a function $f(x)$ on (c, d) , we consider the formal expansion

$$f(x) \sim \sum_{n=0}^{\infty} b_n(f) \phi_n(x),$$

where $b_n(f) = \int_c^d f(x) \phi_n(x) dx$ is the n th Fourier coefficient with respect to the system $\{\phi_n\}$.

LEMMA. Suppose that

$$\max_{c < x < d} \left| \frac{d}{dx} \phi_n(x) \right| \leq K n^{\delta}, \quad \delta > -\frac{1}{2},$$

with a constant K independent of n . Then there exist constants C_1 and C_2 satisfying the following.

(i) For an H^1 atom $a(x)$ with $\text{supp } a(x) \subset (c, d)$,

$$|b_n(a)| \leq C_1 n^{\delta} \|a\|_2^{-2} \quad (3)$$

and

$$\sum_{n=0}^{\infty} \frac{|b_n(a)|}{(n+1)^{(\delta+2)/3}} \leq C_2. \quad (4)$$

(ii) Let $f(x)$ be a function of the form $f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x)$, where every $a_j(x)$ is an H^1 atom with $\text{supp } a(x) \subset (c, d)$ and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$. Then

$$\sum_{n=0}^{\infty} \frac{|b_n(f)|}{(n+1)^{(\delta+2)/3}} \leq C_2 \sum_{j=0}^{\infty} |\lambda_j|. \quad (5)$$

Proof. Let $a(x)$ be an H^1 atom with support contained in $(b, b+h) \subset (c, d)$ such that $|a(x)| \leq h^{-1}$. Since

$$b_n(a) = \int_c^d a(x) \phi_n(x) dx - \phi_n(b) \int_c^d a(x) dx = \int_b^{b+h} a(x) \{\phi_n(x) - \phi_n(b)\} dx,$$

it follows from Schwarz's inequality and the mean value theorem that

$$\begin{aligned} |b_n(a)| &\leq \|a\|_2 \left\{ \int_b^{b+h} (x-b)^2 dx \right\}^{1/2} \max_{c < x < d} \left| \frac{d}{dx} \phi_n(x) \right| \\ &\leq C_1 n^{\delta} \|a\|_2 h^{3/2}, \end{aligned}$$

where $C_1 = K/3^{1/2}$. By the fact $h \leq \|a\|_2^{-2}$, we have the inequality (3).

For simplicity, we put $\gamma = \|a\|_2^{6/(2\delta+1)}$ and $\sigma = (\delta+2)/3$. It follows from (3) that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{|b_n(a)|}{(n+1)^{\sigma}} &= \left\{ \sum_{n \leq \gamma} + \sum_{n > \gamma} \right\} \frac{|b_n(a)|}{(n+1)^{\sigma}} \\ &\leq C_1 \|a\|_2^{-2} \sum_{n \leq \gamma} \frac{n^{\delta}}{(n+1)^{\sigma}} + \|a\|_2 \left\{ \sum_{n > \gamma} \frac{1}{(n+1)^{2\sigma}} \right\}^{1/2} \\ &\leq C'_1 (\|a\|_2^{-2} \gamma^{(2\delta+1)/3} + \|a\|_2 \gamma^{-(2\delta+1)/6}) \leq C_2, \end{aligned}$$

where C'_1 and C_2 are positive constants independent of $a(x)$.

Since $\|a\|_1 \leq 1$, the series $\sum_{j=0}^{\infty} \lambda_j a_j(x)$ converges in $L^1(c, d)$. We see that (4) implies (5).

We now come to the proof of the Theorem. By virtue of the Lemma and the atomic decomposition characterization, it is enough to show, for part (i) of the Theorem, that

$$\left| \frac{d}{dx} \mathcal{H}_n(x) \right| \leq Cn^{5/12} \quad (6)$$

for $x \in \mathbb{R}$. Here and below, C denotes a positive constant which may differ at each different occurrence. It follows from the equations

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x) \quad \text{and} \quad (d/dx)H_n(x) = 2nH_{n-1}(x)$$

(compare [6, (5.5.8), (5.5.10)]) that

$$(d/dx)\mathcal{H}_n(x) = (n/2)^{1/2}\mathcal{H}_{n-1}(x) + ((n+1)/2)^{1/2}\mathcal{H}_{n+1}(x).$$

By the inequality

$$|\mathcal{H}_n(x)| \leq Cn^{-1/12} \quad (7)$$

(compare [5, (2.23)]), we have (6), which completes the proof of part (i) of the Theorem.

We turn to the proof of part (ii) of the Theorem. Similarly, it suffices to show that

$$\left| \frac{d}{dx} \mathcal{L}_n^\alpha(x) \right| \leq Cn, \quad x > 0. \quad (8)$$

But we shall be able to prove this inequality only in the case that $\alpha \geq 2$ or $\alpha = 0$. The case $0 < \alpha < 2$ will be treated by the transplanted theorem for Laguerre coefficients between α and $\alpha + 2$ [1, p. 401, line 14]:

$$|c_n^\alpha(f)| \leq C \left(|c_{n-1}^{\alpha+2}(f)| + \sum_{j=n}^{\infty} |c_j^{\alpha+2}(f)| \left(\frac{n}{j} \right)^{\alpha/2} j^{-1} \right), \quad n = 1, 2, \dots \quad (9)$$

To prove (8), we use the equation $(d/dx)L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x)$ (compare [6, (5.1.14)]). It follows that

$$\frac{d}{dx} \mathcal{L}_n^\alpha(x) = \tau_n^\alpha \left(-L_{n-1}^{\alpha+1}(x) e^{-x/2} x^{\alpha/2} + \frac{\alpha}{2} L_n^\alpha(x) e^{-x/2} x^{\alpha/2-1} + \left(-\frac{1}{2}\right) L_n^\alpha(x) e^{-x/2} x^{\alpha/2} \right),$$

where $\tau_n^\alpha = \{\Gamma(n+1)/\Gamma(n+\alpha+1)\}^{1/2}$. By the inequality

$$|\mathcal{L}_n^\alpha(x)| \leq C, \quad x > 0, \quad \alpha \geq 0 \quad (10)$$

(compare [5, (2.9)]), we have

$$\begin{aligned} \left| \frac{d}{dx} \mathcal{L}_n^\alpha(x) \right| &\leq C \left(\tau_n^\alpha |L_{n-1}^{\alpha+1}(x)| e^{-x/2} x^{\alpha/2} + \frac{\alpha}{2} \tau_n^\alpha |L_n^\alpha(x)| e^{-x/2} x^{\alpha/2} x^{-1} + 1 \right) \\ &\leq C(A+B+1), \quad \text{say.} \end{aligned}$$

We divide the matter into two cases, $nx \geq 1$ and $0 < nx < 1$. For $nx \geq 1$, we have, by (10) and $\tau_n^\alpha \leq Cn^{-\alpha/2}$,

$$A \leq \tau_n^\alpha |L_{n-1}^{\alpha+1}(x)| e^{-x/2} x^{(\alpha+1)/2} x^{-1/2} \leq Cn(nx)^{-1/2} \leq Cn,$$

$$B \leq \frac{\alpha}{2} \tau_n^\alpha |L_n^\alpha(x)| e^{-x/2} x^{\alpha/2} x^{-1} \leq Cn(nx)^{-1} \leq Cn.$$

Thus we have (8) for $nx \geq 1$ when $\alpha \geq 0$. For $0 < nx < 1$, we use the inequality $|L_n^\alpha(x)| \leq Cn^\alpha$, $0 < nx < 1$ [6, (7.6.8)]. It follows that $A \leq Cn^{-\alpha/2}n^{\alpha+1}x^{\alpha/2} \leq Cn(nx)^{\alpha/2} \leq Cn$ when $\alpha \geq 0$, and that $B \leq Cn^{-\alpha/2}n^\alpha x^{\alpha/2}x^{-1} \leq Cn(nx)^{\alpha/2-1} \leq Cn$ when $\alpha \geq 2$. Note that the term B does not appear when $\alpha = 0$. We conclude that (8) holds when $\alpha \geq 2$ or $\alpha = 0$. Therefore we have the desired inequality (2) when $\alpha \geq 2$ or $\alpha = 0$.

We shall interpolate between $\alpha = 0$ and $\alpha = 2$ by using (9). Let $0 < \alpha < 2$ and $g(x) \in H^1(0, \infty)$. By (10), we note that $|c_0^\alpha(g)| \leq C\|g\|_{L^1(0, \infty)} \leq C\|g\|_{H^1(0, \infty)}$. It follows from (9) that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{|c_n^\alpha(g)|}{n+1} &= |c_0^\alpha(g)| + \sum_{n=1}^{\infty} \frac{|c_n^\alpha(g)|}{n+1} \\ &\leq C\|g\|_{H^1(0, \infty)} + C \left(\sum_{n=1}^{\infty} \frac{|c_{n-1}^{\alpha+2}(g)|}{n+1} + \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{j=n}^{\infty} |c_j^{\alpha+2}(g)| \left(\frac{n}{j}\right)^{\alpha/2} j^{-1} \right). \end{aligned}$$

We treat the last sum. Since $\{\sum_{n=1}^j n^{\alpha/2}(n+1)^{-1}\} j^{-\alpha/2} \leq C$ for $\alpha > 0$, it follows that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{j=n}^{\infty} |c_j^{\alpha+2}(g)| \left(\frac{n}{j}\right)^{\alpha/2} j^{-1} &= \sum_{j=1}^{\infty} \left\{ \sum_{n=1}^j \frac{n^{\alpha/2}}{n+1} \right\} j^{-\alpha/2} j^{-1} |c_j^{\alpha+2}(g)| \\ &\leq C \sum_{j=1}^{\infty} j^{-1} |c_j^{\alpha+2}(g)|. \end{aligned}$$

Therefore we have

$$\sum_{n=0}^{\infty} \frac{|c_n^\alpha(g)|}{n+1} \leq C \left(\|g\|_{H^1(0, \infty)} + \sum_{n=0}^{\infty} \frac{|c_n^{\alpha+2}(g)|}{n+1} \right)$$

for $\alpha > 0$, which completes the proof of part (ii) of the Theorem since inequality (2) with $\alpha \geq 2$ or $\alpha = 0$ has been proved.

3. Paley type theorem

The following inequalities, (11) and (12), with respect to the Laguerre function system follow from Paley's theorem (compare [7, Vol. II, Theorem (5.1)]) with respect to general systems $\{\phi_n\}$ of functions orthonormal and uniformly bounded, $|\phi_n(x)| \leq C$.

Let $\alpha \geq 0$. For $g(x) \sim \sum_{n=0}^{\infty} c_n^\alpha(g) \mathcal{L}_n^\alpha(x) \in L^p(0, \infty)$, $1 < p \leq 2$,

$$\sum_{n=0}^{\infty} |c_n^\alpha(g)|^p (n+1)^{p-2} \leq C \|g\|_{L^p(0, \infty)}^p. \quad (11)$$

For $\{b_n\}_{n=0}^{\infty}$ with $\sum_{n=0}^{\infty} |b_n|^q (n+1)^{q-2}$, $2 \leq q < \infty$, there exists a function

$$g(x) \sim \sum_{n=0}^{\infty} b_n \mathcal{L}_n^\alpha(x) \in L^q(0, \infty)$$

such that

$$\|g\|_{L^q(0, \infty)}^q \leq C \sum_{n=0}^{\infty} |b_n|^q (n+1)^{q-2}. \quad (12)$$

Also, by applying Paley's theorem to the Hermite function system, we have the same inequalities as (11) and (12), where we substitute $c_n^\alpha(g)$, $g(x) \in L^p(0, \infty)$, by $c_n(f)$, $f(x) \in L^p(\mathbb{R})$, respectively. But, for the Hermite function system, we can obtain sharper inequalities by interpolating Hardy's inequality (1) and Parseval's equation.

PROPOSITION. (i) Let $1 < p \leq 2$. Then there exists a constant C such that

$$\sum_{n=0}^{\infty} |c_n(f)|^p (n+1)^{29(p-2)/36} \leq C \|f\|_{L^p(\mathbb{R})}^p \quad (13)$$

for $f(x) \sim \sum_{n=0}^{\infty} c_n(f) \mathcal{H}_n(x) \in L^p(\mathbb{R})$.

(ii) Let $2 \leq q < \infty$. Then there exists a constant C such that if $\{b_n\}_{n=0}^{\infty}$ satisfies $\sum_{n=0}^{\infty} |b_n|^q (n+1)^{29(q-2)/36} < \infty$, then

$$\|f\|_{L^q(\mathbb{R})}^q \leq C \sum_{n=0}^{\infty} |b_n|^q (n+1)^{29(q-2)/36}, \quad (14)$$

where $f(x) \sim \sum_{n=0}^{\infty} b_n \mathcal{H}_n(x) \in L^q(\mathbb{R})$.

Proof. For $\kappa > 0$, we denote by l_{κ}^p , $p \geq 1$, the space of sequences $\{b_n\}_{n=0}^{\infty}$ such that $\|\{b_n\}\|_p = \{\sum_{n=0}^{\infty} |b_n|^p (n+1)^{-2\kappa} \}^{1/p} < \infty$. We define an operator T_{κ} of $L^p(\mathbb{R})$ to l_{κ}^p by $T_{\kappa}f = \{c_n(f)(n+1)^{\kappa}\}_{n=0}^{\infty}$. We consider the case $\kappa = 29/36$. Then by (1), we see that T_{κ} is a bounded linear operator of $H^1(\mathbb{R})$ to l_{κ}^1 , which implies that T_{κ} is of weak type $(H^1(\mathbb{R}), l_{\kappa}^1)$. Parseval's equation says that T_{κ} is an isometry of $L^2(\mathbb{R})$ to l_{κ}^2 ; in particular, T_{κ} is of weak type $(L^2(\mathbb{R}), l_{\kappa}^2)$. We follow line by line the proof [4, pp. 308–310] of the interpolation theorem between H^1 space and L^p space. We conclude that T_{κ} is a bounded operator of $L^p(\mathbb{R})$ to l_{κ}^p , $1 < p \leq 2$, which means (13).

The inequality (14) is obtained by a standard duality argument (compare [7, Vol. II, proof of Theorem (5.1)]).

Lastly, we give a remark. Instead of the interpolation theorem between H^1 space and L^1 space, we may use the interpolation theorem between weak- L^p spaces. But it will become clear in the sequel that the inequality (15) below, obtained by this method, is weaker than our inequality (13).

We now consider l_{κ}^p , T_{κ} with $\kappa = 11/12$. Parseval's equation implies that T_{κ} is of weak type $(L^2(\mathbb{R}), l_{\kappa}^2)$. Moreover, we see that T_{κ} is of weak type $(L^1(\mathbb{R}), l_{\kappa}^1)$. For, we put $\mathcal{O}_t = \{n; |c_n(f)(n+1)^{\kappa}| > t\}$ for $t > 0$, where $f(x) \in L^1(\mathbb{R})$. It follows from (7) that $t < |c_n(f)(n+1)^{\kappa}| < Cn^{-1/12}n^{\kappa}\|f\|_{L^1(\mathbb{R})} = Cn^{10/12}\|f\|_{L^1(\mathbb{R})}$. Thus we have

$$\{t/\|f\|_{L^1(\mathbb{R})}\}^{12/10} < n.$$

This gives $\sum_{n \in \mathcal{O}_t} (n+1)^{-2\kappa} \leq C \{t/\|f\|_{L^1(\mathbb{R})}\}^{12(-2\kappa+1)/10} = C\|f\|_{L^1(\mathbb{R})}/t$, which means that T_{κ} is of weak type $(L^1(\mathbb{R}), l_{\kappa}^1)$. By the Marcinkiewicz interpolation theorem, we see that T_{κ} is a bounded operator of $L^p(\mathbb{R})$ to l_{κ}^p , $1 < p < 2$, that is,

$$\sum_{n=0}^{\infty} |c_n(f)|^p (n+1)^{11(p-2)/12} \leq C \|f\|_{L^p(\mathbb{R})}^p. \quad (15)$$

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References

1. R. ASKEY, 'A transplation theorem for Jacobi coefficients', *Pacific J. Math.* 21 (1967) 393–404.
2. L. S. BOSANQUET *et al.* (eds), *Collected papers of G. H. Hardy*, Vol. III (Oxford University Press, London, 1969).
3. R. R. COIFMAN and G. WEISS, 'Extensions of Hardy spaces and their use in analysis', *Bull. Amer. Math. Soc.* 83 (1977) 569–645.

4. J. GARCIA-CUERVA and J. L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics* (North-Holland, Amsterdam, 1985).
5. C. MARKETT, 'Nikolskii-type inequalities for Laguerre and Hermite expansions', *Functions, series, operators*, Colloq. Math. Soc. János Bolyai 35 (ed. J. Szabados, North-Holland, Amsterdam, 1982) 811–834.
6. G. SZEGÖ, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ. 23 (Amer. Math. Soc., Providence, RI, 4th edn, 1975).
7. A. ZYGMUND, *Trigonometric series*, Vols I, II (Cambridge University Press, London, 2nd edn, 1968).

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