Design of Optimum M -Phase Spreading Sequences of Markov Chains

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SUMMARY We design $M(\geq 3)$ -phase spreading sequences of Markov chains optimal in terms of bit error probabilities in asynchronous SSMA (spread spectrum multiple access) communication systems. To this end, we obtain the distributions of the normalized MAI (multiple access interference) for such systems and find a necessary and sufficient condition that the distributions become independent of the phase shifts.

key words: M -phase spreading sequences, Markov chains, the central limit theorem (CLT), spread spectrum multiple access (SSMA), multiple access interference (MAI), bit error probabilities

1. Introduction

Designing spreading sequences is one of the essential elements of spread spectrum techniques. Spreading sequences are a kind of pseudo-random numbers. Normally pseudo-random numbers are designed in order that their auto-correlations are to be like delta functions of time and that their cross-correlations are to be always small. Actually, Kasami sequences and Gold sequences, both examples of linear feedback shift register (LFSR) sequences, are well known to have very good *even* cross-correlation [1]. However, note that both do not have good *odd* cross-correlation. These criteria for the design of spreading sequences are based only on the sequences themselves without taking into account the performance of the whole system.

There are other approaches to the design of spreading sequences such as designing from the viewpoint of the performance of the system in which the sequences are used. To evaluate the performance of asynchronous spread-spectrum multiple-access (SSMA) communication systems, the average interference parameter (AIP) was introduced by Pursley as a measure of the average signal-to-noise ratio [2]. For the problem of designing spreading sequences, it is natural to find spreading sequences that minimize the AIP. In fact the property of Kasami sequences was examined with respect to the AIP [3]. This problem has been often discussed (see [4] or [5] for example), but the optimum spreading sequences of LFSR in terms of the AIP were never determined. A breakthrough was made only after the introduction of chaotic spreading sequences, and the problem of characterizing spreading sequences that minimize the mean value of the AIP was considered and was successfully

solved in [6]*. However, we note here that the result in [6] only gave correlational properties of the optimum spreading sequences. It did not tell us how to design the optimum $M(\geq 3)$ -phase spreading sequences based on the mean value of the AIP except binary sequences.

Chaotic spreading sequences are sequences of pseudo-random numbers generated by one-dimensional ergodic transformations. Since the two tutorial works [7] and [8] have already described in detail the generation of chaotic spreading sequences and its applications, in the following we only give a brief review of the history of chaotic spreading sequences.

The first *real-valued* sequences generated by logistic maps were proposed to be used as spreading sequences in SSMA communication systems [9]. At the same time, *binary* sequences generated by Chebyshev maps were proposed [10]. Then M -phase, *complex-valued* spreading sequences generated by PL (piecewise linear) Markov maps were proposed [11]. We note here that the auto-correlations of these last sequences are exponentially vanishing while the ones of the former two sequences are like delta functions. The appearance of the latter sequences whose auto-correlations exponentially decay is a breakthrough in the design from the viewpoint of the performance of the system, and it is a revolutionary change in the sequence design because sequences whose auto-correlations are like delta functions were commonly regarded as good sequences. In fact, it was shown in [11] that the optimum sequences in terms of the mean value of the AIP were sequences whose auto-correlations exponentially decay but not those sequences whose auto-correlations are like delta functions.

Since the chaotic sequences proposed in [11] are equivalent to sequences generated by a class of irreducible and aperiodic Markov chains, we need not use the word *chaotic* any more. It suffices to consider M -phase spreading sequences of Markov chains. Under the assumption that the chains are irreducible and aperiodic, we have shown in [12] that the central limit theorem (CLT) holds for the normalized multiple access interference (MAI) for SSMA communication systems using spreading sequences of Markov chains. By virtue of this result, we can explicitly obtain the theoretical expressions of bit error probabilities originated from the MAI for such systems. We use these theoretical expressions

Manuscript received November 29, 2006.

Final manuscript received March 27, 2007.

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DOI: 10.1093/ietfec/e90-a.10.2055

*The mean value of the AIP is averaged over spreading sequences as random variables while the original AIP defined in [2] is a random variable of spreading sequences.

as objective functions and design the optimum M -phase spreading sequences of Markov chains in terms of bit error probabilities. To this end, we find the stochastic matrices P that minimize the bit error probabilities since $M \times M$ matrices P determine stationary M -state Markov chains. Then the optimum M -phase spreading sequences of Markov chains with such P can be easily obtained by Kalman's method which gives one-dimensional piecewise linear transformation generating sequences of Markov chains with an arbitrarily prescribed transition matrix [13].

This report is composed of eight sections. In Sect. 2, we review correlation functions and the theoretical expression of the MAI for asynchronous SSMA systems using M -phase spreading sequences. In Sect. 3, we consider 3-phase spreading sequences generated by general 3-state Markov chains characterized by transition matrix P and obtain the distributions of the normalized MAI for the systems using such 3-phase sequences. It is found that the limit of variance of the normalized MAI for such systems generally depends on the phase shifts, and we give a necessary and sufficient condition that it becomes independent of the phase shifts. We also show that for any fixed time delay, the distributions of the normalized MAI for such systems are Gaussian if and only if P is a circulant matrix. In Sect. 4, we obtain the optimum 3-phase spreading sequences of Markov chains. In Sect. 5, we consider M -phase spreading sequences of Markov chains characterized by transition matrix P and generalize the necessary and sufficient condition obtained in Sect. 3 to the case $M \geq 3$. Namely, for the systems using such M -phase sequences, we give a necessary and sufficient condition that the distributions become independent of the phase shifts for any fixed time delay, which implies the condition that P is a circulant matrix generally guarantees only the sufficiency. In Sect. 6, we obtain the optimum M -phase spreading sequences of Markov chains characterized by circulant matrices. Based on this result, we construct as a concrete example the optimum Q (quadratic)-phase spreading sequences of Markov chains[†]. In Sect. 7, it is found that the correlational properties of M -phase spreading sequences of Markov chains are generally independent of the mixing property for the associated M -state Markov chains. We also give a necessary and sufficient condition for these two properties to be independent. The report concludes in Sect. 8.

2. Preliminaries

To evaluate the MAI for SSMA communication systems, we use correlation functions for complex-valued PN (pseudo-noise) sequences introduced in [14]. We write the conjugate of the complex number α by $\bar{\alpha}$.

Definition 1: The *aperiodic cross-correlation function* of time delay ℓ for the complex-valued sequences $X = (X_n)_{n=0}^{N-1}$ and $Y = (Y_n)_{n=0}^{N-1}$ is defined by

$$R_N^A(\ell; X, Y) = \sum_{n=0}^{N-1-\ell} X_n \overline{Y_{n+\ell}},$$

where $\ell = 0, 1, 2, \dots, N-1$. This gives the *even* and *odd cross-correlation functions* of time delay ℓ for the complex-valued sequences X and Y of period N which are respectively given by

$$R_N^E(\ell; X, Y) = R_N^A(\ell; X, Y) + \overline{R_N^A(N-\ell; Y, X)}$$

and

$$R_N^O(\ell; X, Y) = R_N^A(\ell; X, Y) - \overline{R_N^A(N-\ell; Y, X)}.$$

The complex representation of the MAI for asynchronous SSMA communication systems is given by

$$I_{J,p} = \sum_{j=1}^{J-1} \left\{ \frac{d_p^{(j)} + d_{p+1}^{(j)}}{2} S(\tau_j, R_N^E(\ell_j; X, Y^{(j)})) + \frac{d_p^{(j)} - d_{p+1}^{(j)}}{2} S(\tau_j, R_N^O(\ell_j; X, Y^{(j)})) \right\} e^{\sqrt{-1}\phi_j},$$

where $(d_p^{(j)})_{p=-\infty}^{\infty} \in \{-1, 1\}^{\mathbb{Z}}$ are sequences of the j -th user's transmitted data bits, and $0 \leq \tau_j < 1$ and $0 \leq \phi_j < 2\pi$ are the fractional part of the time delay and the phase shift in the j -th channel, respectively. We use $\mathbb{Z}$ to denote the set of integers. For a sequence of complex numbers $(\alpha(\ell))_{\ell=0}^{N-1}$, we define

$$S(\tau; \alpha(\ell)) = (1 - \tau)\alpha(\ell) + \tau\alpha(\ell + 1), \quad 0 \leq \tau < 1.$$

3. Distributions of the Normalized MAI for SSMA Communication Systems Using 3-Phase Spreading Sequences of Markov Chains

Our goal is to construct the optimum $M(\geq 3)$ -phase spreading sequences of Markov chains in terms of bit error probabilities. To this end, we need the distributions of the normalized MAI $\Re(I_{J,p}/\sqrt{N})$ for SSMA communication systems using M -phase spreading sequences of Markov chains. We use $\Re(\alpha)$ to denote the real part of the complex number α . The imaginary part of α is denoted by $\Im(\alpha)$. In this section, we consider 3-phase spreading sequences of Markov chains and obtain the limit of variance of the normalized MAI.

Although $M = 3$ is the simplest case with respect to $M \geq 3$, it is not so easy to find the optimum chains in general 3-state Markov chains. While a 2×2 transition matrix has generally only 2 parameters, a 3×3 one has generally 6 parameters. Moreover, correlation values of 3-phase sequences generally take complex numbers although the ones of binary sequences always take real numbers [15]. Because of the difference of correlational properties, we see in the following that the limit of the normalized variance of the MAI depends differently on phase shifts between

[†]In particular, 4-phase sequences are called Q -phase sequences.

asynchronous SSMA communication systems using binary spreading sequences and the systems using 3-phase spreading sequences. Thus the main results of this section are significant and not straightforward from our previous ones on binary spreading sequences of Markov chains in [16]–[18].

Now let $X = (X_n)_{n=0}^{\infty}$ and $Y = (Y_n)_{n=0}^{\infty}$ be sequences of $\{1, e^{\frac{2}{3}\pi\sqrt{-1}}, e^{-\frac{2}{3}\pi\sqrt{-1}}\}$ -valued stationary random variables. Suppose that X and Y are stationary 3-state Markov chains with a 3×3 transition matrix P and are mutually independent. We can write

$$P = \begin{pmatrix} 1-a-b & a & b \\ c & 1-c-d & d \\ e & f & 1-e-f \end{pmatrix},$$

where $0 \leq a \leq 1$, $0 \leq b \leq 1$, $0 \leq a+b \leq 1$, $0 \leq c \leq 1$, $0 \leq d \leq 1$, $0 \leq c+d \leq 1$, $0 \leq e \leq 1$, $0 \leq f \leq 1$ and $0 \leq e+f \leq 1$.

Let their probabilities be

$$P(X_n = e^{i\frac{2}{3}\pi\sqrt{-1}}) = P(Y_n = e^{i\frac{2}{3}\pi\sqrt{-1}}) = 1/3$$

for $i = 0, 1, 2$, which lead to P being a doubly stochastic matrix. That is, $e = a + b - c$ and $f = c + d - a$.

If for simplicity we suppose irreducible, aperiodic chains, then we have

$$E[X_n] = E[Y_n] = 0$$

and

$$E[X_n \overline{Y_{n+\ell}}] = E[X_n]E[\overline{Y_n}] = 0 \quad \text{for } \ell \geq 0,$$

which immediately implies by definition

$$E\left[\Re\left(S\left(\tau, R_N^{E/O}(\ell; X, Y)/\sqrt{N}\right)e^{\sqrt{-1}\phi}\right)\right] = 0 \quad \text{for } \ell \geq 0, \quad (1)$$

where the superscript E/O denotes either even or odd cross-correlation functions and $E[Z]$ denotes the expected value of the random variable Z .

After some elementary but lengthy calculations, we obtain

Lemma 1:

$$\begin{aligned} \Re(E[X_n \overline{X_{n+\ell}}]) &= \Re(E[Y_n \overline{Y_{n+\ell}}]) \\ &= \frac{1}{2}(\lambda_1^\ell + \lambda_2^\ell) \quad \text{for } \ell \geq 0 \end{aligned}$$

and

$$\begin{aligned} \Im(E[X_n \overline{X_{n+\ell}}]) &= \Im(E[Y_n \overline{Y_{n+\ell}}]) \\ &= \frac{\sqrt{3}}{2}(-a+c)\frac{\lambda_1^\ell - \lambda_2^\ell}{\lambda_1 - \lambda_2} \quad \text{for } \ell \geq 0, \end{aligned}$$

which immediately leads to

$$\begin{aligned} |E[X_n \overline{X_{n+\ell}}]|^2 &= |E[Y_n \overline{Y_{n+\ell}}]|^2 \\ &= \frac{1}{4}(\lambda_1^\ell + \lambda_2^\ell)^2 + \frac{3}{4}(-a+c)^2 \left(\frac{\lambda_1^\ell - \lambda_2^\ell}{\lambda_1 - \lambda_2}\right)^2 \quad \text{for } \ell \geq 0. \end{aligned}$$

In addition we have

Lemma 2:

$$\begin{aligned} \Re(E[X_n X_{n+\ell}]) &= \Re(E[Y_n Y_{n+\ell}]) \\ &= -\frac{1}{2}\{2(a-d) + (b-c)\}\frac{\lambda_1^\ell - \lambda_2^\ell}{\lambda_1 - \lambda_2} \quad \text{for } \ell \geq 0 \end{aligned}$$

and

$$\begin{aligned} \Im(E[X_n X_{n+\ell}]) &= \Im(E[Y_n Y_{n+\ell}]) \\ &= -\frac{\sqrt{3}}{2}(b-c)\frac{\lambda_1^\ell - \lambda_2^\ell}{\lambda_1 - \lambda_2} \quad \text{for } \ell \geq 0, \end{aligned}$$

which immediately leads to

$$\begin{aligned} |E[X_n X_{n+\ell}]|^2 &= |E[Y_n Y_{n+\ell}]|^2 \\ &= \{(a-d)^2 + (a-d)(b-c) + (b-c)^2\} \left(\frac{\lambda_1^\ell - \lambda_2^\ell}{\lambda_1 - \lambda_2}\right)^2 \quad \text{for } \ell \geq 0. \end{aligned}$$

We use λ_1 and λ_2 to denote eigenvalues of P except the eigenvalue 1.

To obtain optimum 3-phase spreading sequences of Markov chains, we need the limit of the variance of the normalized MAI for such systems. Thus, by an elementary but a bit tedious calculation, we obtain

Lemma 3:

$$\begin{aligned} \lim_{N \rightarrow \infty} V\left[\Re\left(S\left(\tau, R_N^E(\ell; X, Y)/\sqrt{N}\right)e^{\sqrt{-1}\phi}\right)\right] &= \lim_{N \rightarrow \infty} V\left[\Re\left(S\left(\tau, R_N^O(\ell; X, Y)/\sqrt{N}\right)e^{\sqrt{-1}\phi}\right)\right] \\ &= \{(1-\tau)^2 + \tau^2\} \\ &\quad \times \frac{1}{2}\left\{1 + \frac{1}{2}\left(\frac{\lambda_1^2}{1-\lambda_1^2} + 2\frac{\lambda_1\lambda_2}{1-\lambda_1\lambda_2} + \frac{\lambda_2^2}{1-\lambda_2^2}\right)\right. \\ &\quad \left.+ 2\left(\frac{3}{4}(-a+c)^2 + |E[X_n X_{n+1}]|^2 \cos 2\phi\right)\right. \\ &\quad \left.\times \frac{1+\lambda_1\lambda_2}{(1-\lambda_1\lambda_2)(1-\lambda_1^2)(1-\lambda_2^2)}\right\} \\ &\quad + 2(1-\tau)\tau \\ &\quad \times \frac{1}{2}\left\{\lambda_1 + \lambda_2 + \frac{1}{2}\left(\frac{\lambda_1^3}{1-\lambda_1^2} + \frac{\lambda_1\lambda_2(\lambda_1+\lambda_2)}{1-\lambda_1\lambda_2} + \frac{\lambda_2^3}{1-\lambda_2^2}\right)\right. \\ &\quad \left.+ 2\left(\frac{3}{4}(-a+c)^2 + |E[X_n X_{n+1}]|^2 \cos 2\phi\right)\right. \\ &\quad \left.\times \frac{\lambda_1 + \lambda_2}{(1-\lambda_1\lambda_2)(1-\lambda_1^2)(1-\lambda_2^2)}\right\}, \quad (2) \end{aligned}$$

where $V[Z]$ denotes the variance of random variable Z .

Since these limits coincide with each other and are independent of ℓ , we denote these limits by $\sigma_3(\tau, \phi)^2$. Thus the CLT (central limit theorem) holds for $\Re(I_{J,p}/\sqrt{N})$ (see [12] for details).

On the other hand, for any fixed τ and ϕ , the limit of

variance of the normalized MAI for SSMA communication systems using binary spreading sequences of Markov chains is given by

$$\sigma_2(\tau, \phi)^2 := \left\{ \left\{ (1-\tau)^2 + \tau^2 \right\} \left(1 + \frac{2\lambda^2}{1-\lambda^2} \right) + 2(1-\tau)\tau \frac{2\lambda}{1-\lambda^2} \right\} \cos^2 \phi. \quad (3)$$

At this point, we call attention to an important fact that the limit of variance of the normalized MAI (2) inseparably depends on the phase shift ϕ for SSMA communication systems using 3-phase spreading sequences of Markov chains. On the other hand, (3) separately depends on the phase shift ϕ . This is the most significant difference between the SSMA communication systems using 3-phase sequences of Markov chains and the systems using binary sequences of Markov chains.

If we regard $\phi_1, \dots, \phi_{J-1}$ as the outcomes of mutually independent random variables $\Phi_1, \dots, \Phi_{J-1}$ with uniform distributions on $[0, 2\pi)$, then we have

Corollary 1: Consider SSMA communication systems of J users using 3-phase spreading sequences of Markov chains. For any fixed $\tau_1, \dots, \tau_{J-1}$, the density distribution of the normalized MAI for such systems is given by

$$\frac{1}{(2\pi)^{J-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \varphi_{0, \sum_{j=1}^{J-1} \sigma_3(\tau_j, \phi_j)^2}(z) d\phi_1 \cdots d\phi_{J-1}, \quad z \in \mathbb{R}, \quad (4)$$

where $\mathbb{R}$ denotes the set of real numbers, and

$$\varphi_{\mu, \sigma^2}(z) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right), \quad z \in \mathbb{R}$$

is the Gaussian density with mean μ and variance σ^2 .

Corollary 1 leads to

Theorem 1: Consider SSMA communication systems using 3-phase spreading sequences of Markov chains characterized by transition matrix P . For any fixed $\tau_1, \dots, \tau_{J-1}$, the distributions of the normalized MAI for such systems are Gaussian if and only if P is a circulant matrix.

Proof: See Appendix A.

We remark here that circulant matrices include P with $a = b = c = d = 1/3$, which implies 3-phase spreading sequences generated by a Markov chain characterized by such P are sequences of independent and identically distributed (i.i.d.) random variables with uniform distribution.

We note here that Theorem 1 can never be obtained by using the mean value of the AIP. In fact, taking the mean value of the AIP as an objective function, the globally optimum 3-state Markov chains with uniform distributions are characterized by transition matrix P with its eigenvalues $\{1, -2 + \sqrt{3}, 0\}$. On the other hand, if P is circulant, then the eigenvalues λ_1 and λ_2 of P are the solutions of the following quadratic equation:

$$x^2 - \{2 - 3(a+b)\}x + \{1 - 3(a+b) + 3(a^2 + ab + b^2)\} = 0. \quad (5)$$

Since its discriminant D satisfies

$$D = -3(a-b)^2 \leq 0,$$

the eigenvalues of P , λ_1 and λ_2 are double real or complex roots. In either case, we can regard λ_1 and λ_2 as the complex conjugate of each other because the coefficients of Eq. (5) are all real numbers. Therefore, a 3×3 transition matrix P with its eigenvalues $\{1, -2 + \sqrt{3}, 0\}$ can never be circulant.

Now, for sequences generated by a class of 3-state Markov chains characterized by a circulant transition matrix, taking $c = b$ and $|E[X_n X_{n+1}]|^2 = 0$ (see Proof of Theorem 1) in (2) shows that

$$\begin{aligned} & \lim_{N \rightarrow \infty} V \left[\Re \left(S(\tau, R_N^E(\ell; X, Y)/\sqrt{N}) e^{\sqrt{-1}\phi} \right) \right] \\ &= \lim_{N \rightarrow \infty} V \left[\Re \left(S(\tau, R_N^O(\ell; X, Y)/\sqrt{N}) e^{\sqrt{-1}\phi} \right) \right] \\ &= \{(1-\tau)^2 + \tau^2\} \left\{ \frac{1}{4} \left(\frac{1}{1-\lambda_1^2} + 2 \frac{\lambda_1 \lambda_2}{1-\lambda_1 \lambda_2} + \frac{1}{1-\lambda_2^2} \right) \right. \\ &\quad \left. + \frac{3}{4} (-a+b)^2 \frac{1 + \lambda_1 \lambda_2}{(1-\lambda_1 \lambda_2)(1-\lambda_1^2)(1-\lambda_2^2)} \right\} \\ &\quad + 2(1-\tau)\tau \left\{ \frac{1}{2} (\lambda_1 + \lambda_2) \right. \\ &\quad \left. + \frac{1}{4} \left(\frac{\lambda_1^3}{1-\lambda_1^2} + \frac{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)}{1-\lambda_1 \lambda_2} + \frac{\lambda_2^3}{1-\lambda_2^2} \right) \right. \\ &\quad \left. + \frac{3}{4} (-a+b)^2 \frac{\lambda_1 + \lambda_2}{(1-\lambda_1 \lambda_2)(1-\lambda_1^2)(1-\lambda_2^2)} \right\}, \end{aligned} \quad (6)$$

where $\lambda_1 + \lambda_2 = 2 - 3(a+b)$ and $\lambda_1 \lambda_2 = \{1 - 3(a+b) + 3(a^2 + ab + b^2)\}$. Since these limits are independent of ϕ , we denote these limits by $\sigma_3(\tau)^2$.

If we regard $\tau_1, \tau_2, \dots, \tau_{J-1}$ as the outcomes of mutually independent random variables $T_1, T_2, \dots, T_{J-1}$ with uniform distributions on $[0, 1)$, then we have

Corollary 2: Consider SSMA communication systems of J users using 3-phase spreading sequences of Markov chains with a circulant transition matrix. The density distribution of the normalized MAI for such systems is given by

$$\int_0^1 \cdots \int_0^1 \varphi_{0, \sum_{j=1}^{J-1} \sigma_3(\tau_j)^2}(z) d\tau_1 \cdots d\tau_{J-1}, \quad z \in \mathbb{R}. \quad (7)$$

Moreover, Corollary 2 leads to

Theorem 2: Consider asynchronous SSMA communication systems using 3-phase spreading sequences of Markov chains with circulant transition matrices. If the chains are irreducible and aperiodic, the distributions of the normalized MAI for such systems can never be Gaussian.

Proof: The assertion is verified in a similar way as the proof of Theorem 1.

Remark 1: If the system is chip-synchronous, i.e. $\tau \equiv 0$, then the distributions of the normalized MAI are Gaussian.

4. Optimum 3-Phase Spreading Sequences of Markov Chains

An important feature we have established in the previous section for SSMA communication systems using 3-phase spreading sequences of Markov chains is that the limit of variance of the normalized MAI inseparably depends on the phase shift. On the other hand, for the systems using binary spreading sequences of Markov chains, the limit separately depends on the phase shift. Because of the difference, we cannot directly apply the method of designing optimum binary sequences to find the optimum 3-phase sequences.

Before we get under way, let us recall the method of designing optimum binary spreading sequences of Markov chains.

In [12], for SSMA communication systems using binary spreading sequences of Markov chains, we have shown that the standard Gaussian approximations (SGA) are the 0-th order approximations for evaluations of bit error probabilities based on the CLT. The SGA is given by

$$\int_{\sqrt{N}}^{\infty} \varphi_{0,(J-1)E[\sigma_2(T,\Phi)^2]}(z) dz.$$

In terms of $E[\sigma_2(T, \Phi)^2]$, we know a property of the SGA:

Remark 2: For $x \geq 0$, $\int_{\sqrt{N}}^{\infty} \varphi_{0,x}(z) dz$ is a strictly convex function of x .

By virtue of this remark, we can take $E[\sigma_2(T, \Phi)^2]$ as an objective function. The minimizer is given by $\lambda = -2 + \sqrt{3}$.

We now turn our attention to finding the optimum M -phase sequences of Markov chains.

Theorem 1 obtained in the previous section enables us to define a novel class of spreading sequences as follows.

Definition 2: For asynchronous SSMA communication systems using M -phase spreading sequences of Markov chains, if the limit of variance of the MAI is independent of the phase shifts, then the sequences used in such systems are referred to as the *phase-shift-free M -phase spreading sequences of Markov chains*. Otherwise, the sequences are referred to as being *phase-shift-dependent*.

For M -phase spreading sequences of Markov chains, we denote the limit of variance of the MAI by $\sigma_M(\tau, \phi)^2$. In particular, for the phase-shift-free sequences, we denote the limit of the variance by $\sigma_M(\tau)^2$.

In a separate research [19], for a class of M -phase spreading sequences of Markov chains that minimize $E[\sigma_M(T, \Phi)^2] = E[\sigma_M(T)^2]$, we obtain

Theorem 3: The optimum phase-shift-free $M (\geq 3)$ -phase spreading sequences of Markov chains are superior to the optimum phase-shift dependent $M (\geq 3)$ -phase spreading sequences of Markov chains, if both exist, in terms of bit error probabilities based on the CLT for $N > 5$.

By virtue of Theorem 3, to design the optimum $M (\geq 3)$ -phase spreading sequences of Markov chains, we first find the phase-shift-free M -phase spreading sequences of Markov chains that minimize $E[\sigma_M(T)^2]$. Then, denoting the minimum value of $E[\sigma_M(T)^2]$ by $\min\{E[\sigma_M(T)^2]\}$, it suffices to check $\min\{E[\sigma_M(T)^2]\} \leq E[\sigma_M(T, \Phi)^2]$ for any phase-shift-dependent M -phase spreading sequences of Markov chains.

Now we are in a position to design the optimum 3-phase spreading sequences of Markov chains in terms of bit error probabilities. To this end, we first consider a class of 3-state Markov chains characterized by circulant transition matrices and find the chains that minimize $E[\sigma_3(T)^2]$ in such a class.

In view of (6), the objective function $E[\sigma_3(T)^2]$ is computed as

$$\begin{aligned} E[\sigma_3(T)^2] = & \frac{1}{3} \left\{ 1 + \frac{1}{2}(\lambda_1 + \lambda_2) \right. \\ & + \frac{1}{4} \left(\frac{2\lambda_1^2 + \lambda_1^3}{1 - \lambda_1^2} + \frac{4\lambda_1\lambda_2 + \lambda_1\lambda_2(\lambda_1 + \lambda_2)}{1 - \lambda_1\lambda_2} + \frac{2\lambda_2^2 + \lambda_2^3}{1 - \lambda_2^2} \right) \\ & \left. + \frac{3}{4}(-a + b)^2 \frac{2 + 2\lambda_1\lambda_2 + \lambda_1 + \lambda_2}{(1 - \lambda_1\lambda_2)(1 - \lambda_1^2)(1 - \lambda_2^2)} \right\}. \end{aligned}$$

Noting that both λ_1 and λ_2 are functions of the two variables a and b , we denote $E[\sigma_3(T)^2]$ by $\sigma_3(a, b)^2$. Let us consider the domain in $\mathbb{R}^2$ that contains a and b . Since a and b are elements of a circulant transition matrix, we already have the condition: $0 \leq a \leq 1$, $0 \leq b \leq 1$, $0 \leq a + b \leq 1$. In addition, assuming irreducible and aperiodic chains, we obtain $(a, b) \neq (0, 0), (0, 1), (1, 0)$. Thus our goal is to find the minimum value of $\sigma_3(a, b)^2$ on the domain

$$\mathcal{D} = \{(a, b) | 0 \leq a \leq 1, 0 \leq b \leq 1, 0 \leq a + b \leq 1\} \setminus \{(0, 0), (0, 1), (1, 0)\}.$$

Denote a 2-dimensional open ball with center (x, y) and radius $r > 0$ by $B_r(x, y)$. For small $\epsilon > 0$, define $\mathcal{D}_\epsilon = \mathcal{D} \setminus (B_\epsilon(0, 0) \cup B_\epsilon(0, 1) \cup B_\epsilon(1, 0))$. Since $\mathcal{D}_\epsilon$ is a bounded closed set in $\mathbb{R}^2$ and σ_3^2 is a continuous function on $\mathcal{D}_\epsilon$, it follows by the elements of calculus (see [20] for example) that σ_3^2 has the maximum and minimum values on $\mathcal{D}_\epsilon$. Taking into account that

$$\begin{aligned} \lim_{(a,b) \rightarrow (0,0)} \sigma_3(a, b)^2 &= \lim_{(a,b) \rightarrow (0,1)} \sigma_3(a, b)^2 \\ &= \lim_{(a,b) \rightarrow (1,0)} \sigma_3(a, b)^2 = +\infty, \end{aligned}$$

we have

Lemma 4: σ_3^2 has the minimum value $\min \sigma_3^2$ on $\mathcal{D}$.

Now let us consider how to find this minimum value $\min \sigma_3^2$ and obtain the optimum 3-phase spreading sequences of Markov chains. Noting that $(\lambda_1 - \lambda_2)^2 = -3(-a + b)^2$, we have

$$\begin{aligned} \sigma_3(a, b)^2 = & \frac{1}{3} \left(1 + \frac{2\lambda_1\lambda_2}{1 - \lambda_1\lambda_2} \right) \\ & + \frac{1}{3} \frac{\lambda_1 + \lambda_2}{2} \left(1 + \frac{\lambda_1\lambda_2}{1 - \lambda_1\lambda_2} \right). \end{aligned}$$

If we denote

$$\lambda_1 = re^{\sqrt{-1}\theta}, \quad \lambda_2 = re^{-\sqrt{-1}\theta}, \\ 0 \leq r < 1, \quad 0 \leq \theta < 2\pi,$$

then we can regard σ_3^2 as a function of the two variables r and θ . Thus what we have to do is to minimize

$$\sigma_3(r, \theta)^2 = \frac{1}{3} \left(1 + \frac{2r^2}{1-r^2} \right) + \frac{1}{3} r \cos \theta \left(1 + \frac{r^2}{1-r^2} \right)$$

under the condition: $0 \leq r < 1, \quad 0 \leq \theta < 2\pi$.

First let us consider how to minimize σ_3^2 with respect to θ . For any r with $0 \leq r < 1$,

$$r \left(1 + \frac{r^2}{1-r^2} \right) \geq 0$$

holds. Thus we immediately obtain

Lemma 5: σ_3^2 takes the minimum value $\min \sigma_3^2$ if and only if $\theta = \pi$.

By virtue of this lemma, we obtain

Theorem 4: σ_3^2 takes the minimum value $\min \sigma_3^2 = 1/(2\sqrt{3})$ if and only if $\lambda_1 = \lambda_2 = -2 + \sqrt{3}$.

In view of (2) and noting $E[\cos 2\Phi] = 0$, $\min \sigma_3^2 \leq E[\sigma_3(T, \Phi)^2]$ holds for any phase-shift-dependent 3-phase spreading sequences of Markov chains.

Eventually we obtain

Remark 3: The optimum 3-phase spreading sequences of Markov chains are uniquely characterized by

$$P = \begin{pmatrix} 1-2a & a & a \\ a & 1-2a & a \\ a & a & 1-2a \end{pmatrix}$$

with $a = 1 - 1/\sqrt{3}$. The set of all eigenvalues of P is given by $\Lambda = \{1, -2 + \sqrt{3}, -2 + \sqrt{3}\}$.

For chip-synchronous systems, we obtain

$$\sigma_3(0)^2 = \frac{1}{4} \left(\frac{1}{1-\lambda_1^2} + 2 \frac{\lambda_1 \lambda_2}{1-\lambda_1 \lambda_2} + \frac{1}{1-\lambda_2^2} \right) \\ + \frac{3}{4} (-a+b)^2 \frac{1+\lambda_1 \lambda_2}{(1-\lambda_1 \lambda_2)(1-\lambda_1^2)(1-\lambda_2^2)}.$$

Since λ_1 and λ_2 are complex conjugates of each other, we obtain

$$\frac{1}{1-\lambda_1^2} + 2 \frac{\lambda_1 \lambda_2}{1-\lambda_1 \lambda_2} + \frac{1}{1-\lambda_2^2} \\ \geq \frac{(1-|\lambda_1^2|)^2}{|1-\lambda_1^2|^2} + 2 \frac{|\lambda_1|^2}{1-|\lambda_1|^2} > 1$$

and

$$\frac{1+\lambda_1 \lambda_2}{(1-\lambda_1 \lambda_2)(1-\lambda_1^2)(1-\lambda_2^2)} = \frac{1+|\lambda_1|^2}{(1-|\lambda_1|^2)|1-\lambda_1^2|^2} > 0,$$

which lead to

Remark 4: For chip-synchronous systems, sequences of i.i.d. random variables are optimal in terms of bit error probabilities.

5. Distributions of the Normalized MAI for SSMA Communication Systems Using $M(\geq 3)$ -Phase Spreading Sequences of Markov Chains

In Sect. 3, we have considered SSMA communication systems using 3-phase spreading sequences of Markov chains characterized by a transition matrix P and have shown that for any fixed $\tau_1, \dots, \tau_{J-1}$ the distributions of the normalized MAI for such systems are Gaussian if and only if P is a circulant matrix. We generalize this result as follows.

Now we consider M -phase spreading sequences of Markov chains. Let $X = (X_n)_{n=0}^\infty$ and $Y^{(j)} = (Y_n^{(j)})_{n=0}^\infty$ ($j = 1, 2, \dots, J-1$) be Markov chains on a state space

$$S = \left\{ 1, e^{\frac{1}{M}2\pi\sqrt{-1}}, e^{\frac{2}{M}2\pi\sqrt{-1}}, \dots, e^{\frac{M-1}{M}2\pi\sqrt{-1}} \right\}. \quad (8)$$

We set

$$p_{ij} = P \left(X_{n+1} = e^{\frac{i-1}{M}2\pi\sqrt{-1}} \middle| X_n = e^{\frac{j-1}{M}2\pi\sqrt{-1}} \right) \\ = P \left(Y_{n+1}^{(j)} = e^{\frac{i-1}{M}2\pi\sqrt{-1}} \middle| Y_n^{(j)} = e^{\frac{j-1}{M}2\pi\sqrt{-1}} \right)$$

for $i, j = 1, 2, \dots, M$. We define the transition matrix by $P = (p_{ij})_{i,j=1}^M$.

Suppose that X and $Y^{(j)}$ are stationary Markov chains and are mutually independent. Let their stationary distributions be

$$P \left(X_n = e^{\frac{i-1}{M}2\pi\sqrt{-1}} \right) = P \left(Y_n^{(j)} = e^{\frac{j-1}{M}2\pi\sqrt{-1}} \right) \\ = 1/M \quad (9)$$

for $i = 1, 2, \dots, M$. We also suppose the chains are irreducible and aperiodic. Thus we obtain

Theorem 5: Consider SSMA communication systems using $M(\geq 3)$ -phase spreading sequences of Markov chains. For any fixed τ , the distributions of the normalized MAI for such systems are Gaussian if and only if the sequences of Markov chains $(X_n)_{n=0}^\infty$ satisfy

$$E[X_n X_{n+1}] = E[X_n X_{n+2}] = \dots = E[X_n X_{n+M-2}] = 0.$$

Proof: See Appendix B.

Definition 2 and Proof of Theorem 5 imply

Corollary 3: $M(\geq 3)$ -phase spreading sequences generated by M -state Markov chains are phase-shift-free if and only if the sequences of Markov chains $(X_n)_{n=0}^\infty$ satisfy

$$E[X_n X_{n+1}] = E[X_n X_{n+2}] = \dots = E[X_n X_{n+M-2}] = 0.$$

This result tells us how to design phase-shift-free M -phase spreading sequences of Markov chains.

As practical examples, we consider 3-phase and Q -phase spreading sequences of Markov chains.

Remark 5: In the case that $M = 3$, $E[X_n X_{n+1}] = 0$ implies that the transition matrix of the chain is a circulant matrix.

Applying Corollary 3, in the case that $M = 4$, we obtain

Example 1: Q -phase spreading sequences of Markov chains are phase-shift-free if and only if the sequences of Markov chains $(X_n)_{n=0}^{\infty}$ satisfy

$$E[X_n X_{n+1}] = E[X_n X_{n+2}] = 0.$$

Remark 6: However, the conditions

$$E[X_n X_{n+1}] = E[X_n X_{n+2}] = 0.$$

do not always imply that the transition matrix of the chain is a circulant matrix.

6. Optimum M -Phase Spreading Sequences of Markov Chains with Circulant Transition Matrices

We know that circulant matrices can be diagonalized by Fourier matrices (see Theorem 3.2.2 in [21]). By virtue of this theorem, we can explicitly obtain the limit of the normalized MAI for SSMA communication systems using $M(\geq 3)$ -phase spreading sequences of Markov chains characterized by circulant matrices. Then it is found that, if the $M \times M$ transition matrix P is circulant, M -phase spreading sequences of Markov chains characterized by P are phase-shift-free.

We treat again the stationary and mutually independent M -state Markov chains X and $Y^{(j)}$ discussed in the previous section. To avoid excessive notation, we will use $Y = (Y_n)_{n=0}^{\infty}$ to denote $Y^{(j)}$ in this section.

Consider a class of such Markov chains whose transition matrix P is circulant and set

$$P = \begin{pmatrix} 1 - \sum_{i=1}^{M-1} p_i & p_1 & \cdots & p_{M-1} \\ p_{M-1} & 1 - \sum_{i=1}^{M-1} p_i & \cdots & p_{M-2} \\ \vdots & \vdots & \ddots & \vdots \\ p_1 & p_2 & \cdots & 1 - \sum_{i=1}^{M-1} p_i \end{pmatrix},$$

where $0 \leq p_i \leq 1$ ($i = 1, 2, \dots, M-1$), then we have

Lemma 6:

$$E[X_n \overline{X_{n+\ell}}] = E[Y_n \overline{Y_{n+\ell}}] = \lambda^\ell \quad \text{for } \ell \geq 0,$$

which immediately leads to

$$\left| E[X_n \overline{X_{n+\ell}}] \right|^2 = \left| E[Y_n \overline{Y_{n+\ell}}] \right|^2 = |\lambda|^{2\ell} \quad \text{for } \ell \geq 0.$$

In addition we have

Lemma 7:

$$E[X_n X_{n+\ell}] = E[Y_n Y_{n+\ell}] = 0 \quad \text{for } \ell \geq 0,$$

which immediately leads to

$$|E[X_n X_{n+\ell}]|^2 = |E[Y_n Y_{n+\ell}]|^2 = 0 \quad \text{for } \ell \geq 0,$$

where $\lambda = 1 - \sum_{i=1}^{M-1} p_i + p_1 e^{\frac{M-1}{M} 2\pi \sqrt{-1}} + p_2 e^{\frac{2(M-1)}{M} 2\pi \sqrt{-1}} + \cdots + p_{M-1} e^{\frac{(M-1)^2}{M} 2\pi \sqrt{-1}}$.

Applying Theorem 5 in conjunction with Lemma 7, we have

Corollary 4: Consider SSMA communication systems using M -phase spreading sequences of Markov chains characterized by transition matrix P . For any fixed τ , the distributions of the normalized MAI for such systems are Gaussian if P is a circulant matrix.

Similarly, applying Corollary 3 in conjunction with Lemma 7, we have

Corollary 5: M -phase spreading sequences of Markov chains characterized by transition matrix P are phase-shift-free, if P is a circulant matrix.

Now we are in a position to obtain the distributions of the normalized MAI for SSMA communication systems using M -phase spreading sequences of Markov chains characterized by circulant matrices. We obtain

$$\begin{aligned} \lim_{N \rightarrow \infty} V \left[\Re \left(S \left(\tau, R_N^E(\ell; X, Y) / \sqrt{N} \right) e^{\sqrt{-1}\phi} \right) \right] \\ = \lim_{N \rightarrow \infty} V \left[\Re \left(S \left(\tau, R_N^O(\ell; X, Y) / \sqrt{N} \right) e^{\sqrt{-1}\phi} \right) \right] \\ = \{(1-\tau)^2 + \tau^2\} \frac{1}{2} \left(1 + \frac{2|\lambda|^2}{1-|\lambda|^2} \right) \\ + 2(1-\tau)\tau \frac{1}{2} (\lambda + \bar{\lambda}) \left(1 + \frac{|\lambda|^2}{1-|\lambda|^2} \right). \end{aligned}$$

Since these limits are independent of ϕ , we denote these limits by $\sigma_M(\tau)^2$. Thus the CLT holds for $\Re(I_{J,p}/\sqrt{N})$ (see [12] for details).

If we regard $\tau_1, \tau_2, \dots, \tau_{J-1}$ as the outcomes of mutually independent random variables $T_1, T_2, \dots, T_{J-1}$ with uniform distributions on $[0, 1)$, then we have

Remark 7: Consider SSMA communication systems of J users using M -phase spreading sequences of Markov chains with circulant transition matrices. The density distribution of the normalized MAI for such systems is given by (7) with $\sigma_M(\tau_j)^2$ in place of $\sigma_3(\tau_j)^2$.

The following is a generalization of Theorem 2.

Theorem 6: Consider asynchronous SSMA communication systems using M -phase spreading sequences of Markov chains with circulant transition matrices. If chains are irreducible and aperiodic, the distributions of the normalized MAI for such systems can never be Gaussian.

Remark 8: If the system is chip-synchronous, then the distributions of the normalized MAI are Gaussian.

Now we focus on a class of Markov chains characterized by circulant matrices and design the optimum $M(\geq 3)$ -phase spreading sequences of Markov chains in terms of bit error probabilities. We consider a class of M -state Markov chains characterized by circulant transition matrices and

find the chains that minimize $E[\sigma_M(T)^2]$ in such a class.

The expected value of the variance $\sigma_M(T)^2$ is given by

$$E[\sigma_M(T)^2] = \frac{1}{3} \left(1 + \frac{2|\lambda|^2}{1-|\lambda|^2} \right) + \frac{1}{3} \cdot \frac{1}{2} (\lambda + \bar{\lambda}) \left(1 + \frac{|\lambda|^2}{1-|\lambda|^2} \right).$$

Since $E[\sigma_M(T)^2]$ is a function of λ , we denote it by $\sigma_M(\lambda)^2$.

We can adapt the method of proof of Theorem 6 to prove

Theorem 7: σ_M^2 has the minimum values on $\mathcal{D} = \{z \in \mathbb{C} | |z| < 1\}$, where $\mathbb{C}$ denotes the set of complex numbers.

Since we can write

$$\lambda = re^{\sqrt{-1}\theta}, \quad 0 \leq r < 1, \quad 0 \leq \theta < 2\pi,$$

we can regard σ_M^2 as a function of the two variables r and θ . Thus we have to minimize

$$\sigma_M(r, \theta)^2 = \frac{1}{3} \left(1 + \frac{2r^2}{1-r^2} \right) + \frac{1}{3} r \cos \theta \left(1 + \frac{r^2}{1-r^2} \right)$$

under the conditions: $0 \leq r < 1$, $0 \leq \theta < 2\pi$.

First, let us consider minimizing σ_M^2 with respect to θ . For any r with $0 \leq r < 1$,

$$r \left(1 + \frac{r^2}{1-r^2} \right) \geq 0$$

holds. Thus we immediately obtain

Lemma 8: σ_M^2 takes the minimum value if and only if $\theta = \pi$.

By virtue of this lemma, we obtain

Theorem 8: σ_M^2 takes the minimum value $1/(2\sqrt{3})$ if and only if $\lambda = -2 + \sqrt{3}$.

Interestingly we encounter this very number $-2 + \sqrt{3}$ as we found earlier in Theorem 4 and [16].

Let us suppose that J users communicate through a common channel in the SSMA systems independently and asynchronously. We consider the ideal noiseless channel. Figure 1 shows that the 0-th order approximation based on the SGA for the theoretical estimations of bit error probabilities in asynchronous SSMA communication systems using M -phase spreading sequences of Markov chains as a function of the number of users J for $N = 64$. Points $\times$ indicate the bit error probabilities in the system using i.i.d. M -phase random variables while points $+$ indicate the ones in the system using the optimum M -phase spreading sequences of Markov chains with $\lambda = -2 + \sqrt{3}$.

While the optimization method in [6] does not tell us how to design the optimum spreading sequences based on the mean value of the AIP, the optimization method proposed in this study explicitly gives the optimum 3-phase and Q-phase spreading sequences of Markov chains based on the CLT.

The optimum 3-phase spreading sequences of Markov chains in terms of bit error probabilities have already been given in Remark 3.

Fig. 1 Bit error probabilities in SSMA communication systems using M -phase spreading sequences of Markov chains with J users.

Example 2: In a class of Markov chains whose transition matrix P is circulant, the optimum Q -phase spreading sequences of Markov chains are characterized by

$$P = \begin{pmatrix} 1-2a-c & a & a & c \\ c & 1-2a-c & a & a \\ a & c & 1-2a-c & a \\ a & a & c & 1-2a-c \end{pmatrix} \quad (10)$$

with $a + c = (3 - \sqrt{3})/2$, $0 \leq a \leq 1$, and $0 \leq c \leq 1$. The set of all eigenvalues of P is given by $\Lambda = \{1, -2 + \sqrt{3}, -2a - (1 - \sqrt{3})/2 \pm \sqrt{-1}\{2a - (3 - \sqrt{3})/2\}\}$.

7. Mixing Property for Markov Chains Generating Spreading Sequences

To find the optimum binary sequences in terms of the mean value of the AIP in [6], it was implicitly assumed that the correlational properties of M -phase spreading sequences of Markov chains are equivalent to the mixing property for the associated M -state Markov chains. As far as binary and 3-phase sequences are concerned, this assumption seems to be true.

In this section, we show that the correlational properties of M -phase spreading sequences of Markov chains are generally independent of the mixing property for the associated M -state Markov chains. We also give a necessary and sufficient condition for these two properties to be independent.

In the previous section, we have considered the class of Markov chains with circulant transition matrix. Then we have shown that, in terms of bit error probabilities, the optimum Q -phase spreading sequences of Markov chains in such a class are characterized by (10).

Let us examine the magnitude of the second largest eigenvalue of (10) in absolute value. We write the one of complex conjugate eigenvalues of (10) by

$$\rho(a) = -2a - (1 - \sqrt{3})/2 + \sqrt{-1}\{2a - (3 - \sqrt{3})/2\},$$

where $0 \leq a \leq (\sqrt{3} - 1)/2$. Simple computation leads to

$$|-2 + \sqrt{3}|/\sqrt{2} \leq |\rho(a)| = |\overline{\rho(a)}| \leq \sqrt{3} - 1,$$

and $|\rho((\sqrt{3} - 1)/4)| = |\rho((3 - \sqrt{3})/4)| = |-2 + \sqrt{3}|$. Hence we obtain

$$|-2 + \sqrt{3}| < |\rho(a)| = |\overline{\rho(a)}|$$

for $0 \leq a < (\sqrt{3} - 1)/4$, $(3 - \sqrt{3})/4 < a \leq (\sqrt{3} - 1)/2$, which leads to

Remark 9: Consider the class of Markov chains with circulant transition matrix. Let P be the transition matrix of Markov chains generating the optimum M -phase spreading sequences in such a class, which minimize bit error probabilities in asynchronous SSMA communication systems. The magnitude of the second largest eigenvalue in absolute value of P is not always $|-2 + \sqrt{3}|$.

Interestingly this implies that the correlational properties of M -phase spreading sequences of Markov chains are generally independent of the mixing property for the associated M -state Markov chains. This fact raises very important questions on the designing method in [6] of spreading sequences based on the mean value of the AIP since it is implicitly assumed that these two properties are equivalent. Motivated by Remark 9, we can determine when these two properties become independent in the following.

Let $S = \{1, 2, \dots, M\}$ ($M \geq 2$). We denote its power set by $\mathfrak{P}(S)$. We consider the measurable space $(S, \mathfrak{P}(S))$ and its direct product space $(X, \mathcal{B}) = \prod_1^\infty (S, \mathfrak{P}(S))$. Let $T : X \rightarrow X$ denote the shift transformation defined by $T(x_1 x_2 \dots x_n \dots) = x_2 x_3 \dots x_{n+1} \dots$ for $x = x_1 x_2 \dots x_n \dots \in X$.

A word (or block) over S is a finite sequence of numbers from S . A word of length n is called an n -word. We denote the set of all n -words over S by S^n .

We are given a probability vector $\mathbf{p} = (p_1, \dots, p_M)$ ($\sum_{i=1}^M p_i = 1$) and a stochastic matrix $P = (p_{i,j})_{i,j \in S}$ ($p_{i,j} \geq 0$, $\sum_{j=1}^M p_{i,j} = 1$) such that $\mathbf{p}P = \mathbf{p}$.

For an n -word $a = a_1 \dots a_n$, we define m by

$$m(\{x \in X : x_q = a_1, \dots, x_{q+n-1} = a_n\}) = p_{a_1} \prod_{k=1}^{n-1} p_{a_k, a_{k+1}}$$

for each $q \geq 1$. Thus m can be extended to a probability measure on $(X, \mathcal{B})$ and T preserves the measure m . We call this measure-preserving transformation the $(\mathbf{p}, P)$ -Markov shift.

For simplicity, we suppose $\mathbf{p} = \frac{1}{M}(1, \dots, 1)$ from now on.

For the $(\mathbf{p}, P)$ -Markov shift, it is well known that the following are equivalent: (i) P is irreducible and aperiodic, (ii) T is weak-mixing, and (iii) T is strong-mixing. This implies for any $\mathbf{u}, \mathbf{v} \in \mathbb{C}^M$,

$$\mathbf{u}P^n\mathbf{v}^* = \mathbf{u} \cdot \mathbf{v}^* + O(|\lambda|^n),$$

where $O(|\lambda|^n)$ stands for a quantity that is no more than a constant times $|\lambda|^n$. We use $\mathbf{v}^*$ to denote the conjugate transpose of $\mathbf{v}$ and λ to denote the second largest eigenvalue in absolute value.

On the other hand, we are now interested in not all the complex-valued M -dimensional vectors, but the specific M -dimensional vector $\mathbf{s} = (1, e^{\frac{1}{M}2\pi\sqrt{-1}}, e^{\frac{2}{M}2\pi\sqrt{-1}}, \dots, e^{\frac{M-1}{M}2\pi\sqrt{-1}})$. In this situation, we obtain

$$E[X_n \overline{X_{n+\ell}}] = \mathbf{s}P^\ell \mathbf{s}^* = \sum_{i=1}^{M-1} c_i \lambda_i^\ell, \quad (11)$$

where c_i is a constant, and $|\lambda| = |\lambda_1| = \dots = |\lambda_q| > |\lambda_{q+1}| \geq \dots \geq |\lambda_{M-1}|$ ($1 \leq q \leq M-1$).

By the elements of matrix theory, we obtain

Theorem 9: In the above-mentioned situation, let $\mathbf{u}_i$ and $\mathbf{v}_i$ be respectively the left and right eigenvectors of P corresponding to λ_i ($1 \leq i \leq q$). The decay rate of the correlation value (11) is independent of the mixing rate of the $(\mathbf{p}, P)$ -Markov shift if and only if $\mathbf{s} \cdot \mathbf{u}_i = 0$ or $\mathbf{v}_i \cdot \mathbf{s}^* = 0$ for all i .

8. Conclusion

In this research, we considered SSMA communication systems using $M(\geq 3)$ -phase spreading sequences of Markov chains and found a necessary and sufficient condition that the distribution of the normalized MAI for such systems becomes independent of the phase shifts. These results enabled us to design M -phase spreading sequences of Markov chains optimal in terms of bit error probabilities. As concrete examples, we constructed the optimum 3-phase and Q (quadratic)-phase spreading sequences of Markov chains. And in passing, we examined mixing property for Markov chains generating spreading sequences.

Acknowledgment

This study was supported by the International Communications Foundation and the Telecommunications Advancement Foundation. The author thanks Dr. Roger R. Anderson, retired Research Scientist in the Department of Physics and Astronomy, the University of Iowa, USA for his helping the author's English writings.

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Appendix A: Proof of Theorem 1

To simplify discussions, we prove Theorem 2 in the case that $J = 2$. Let Z be a real-valued random variable representing the normalized MAI with random variable Φ . For fixed any τ , the density (4) is normal if and only if its Laplace trans-

form $E[e^{sZ}]$ is given by

$$\exp\left(\frac{1}{2}s^2\frac{1}{2\pi}\int_0^{2\pi}\sigma_3(\tau,\phi)^2d\phi\right)=\exp\left(\frac{1}{2}s^2E[\sigma_3(\tau,\Phi)^2]\right),$$

which is in fact the Laplace transform of

$$\varphi_{0,E[\sigma(\tau,\Phi)^2]}(z), \quad z \in \mathbb{R}.$$

On the other hand, the Laplace transform of (4) is actually computed as

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left(\frac{1}{2}s^2\sigma_3(\tau,\phi)^2\right)d\phi=E\left[\exp\left(\frac{1}{2}s^2\sigma_3(\tau,\Phi)^2\right)\right].$$

Since the exponential function is convex, the use of Jensen's inequality gives

$$\begin{aligned} &\exp\left(\frac{1}{2}s^2\frac{1}{2\pi}\int_0^{2\pi}\sigma_3(\tau,\phi)^2d\phi\right) \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\exp\left(\frac{1}{2}s^2\sigma_3(\tau,\phi)^2\right)d\phi \end{aligned}$$

with equality if and only if $\sigma_3(\tau,\phi)^2$ does not depend on ϕ . Namely $|E[X_n X_{n+1}]|^2 = 0$ or $\{(1-\tau)^2 + \tau^2\}(1+\lambda_1\lambda_2) + 2(1-\tau)\tau(\lambda_1 + \lambda_2) = 0$. For any fixed τ , the equality is established by $|E[X_n X_{n+1}]|^2 = 0$, i.e. $a = d$ and $b = c$. This condition is satisfied if and only if the transition matrix is circulant, which completes the proof.

Appendix B: Proof of Theorem 5

Simple computations lead to

$$\begin{aligned} &V\left[\Re\left(S\left(\tau, R_N^{E/O}(\ell; X, Y)/\sqrt{N}\right)e^{\sqrt{-1}\phi}\right)\right] \\ &= \frac{1}{2N}E\left[\left|S\left(\tau, R_N^{E/O}(\ell; X, Y)\right)\right|^2\right] \\ &\quad + \frac{1}{2N}E\left[\Re\left(S\left(\tau, R_N^{E/O}(\ell; X, Y)\right)\right)^2\right]\cos 2\phi \\ &\quad + \frac{1}{2N}E\left[\Im\left(S\left(\tau, R_N^{E/O}(\ell; X, Y)\right)\right)^2\right]\sin 2\phi. \end{aligned}$$

Let $X = (X_n)_{n=0}^\infty$ and $Y = (Y_n)_{n=0}^\infty$ be stationary M -state Markov chains. We suppose that the state space and the stationary distributions are given by (8) and (9) respectively. We also suppose the chains are irreducible and aperiodic. Thus we have

$$\begin{aligned} &\lim_{N \rightarrow \infty} \frac{1}{N}E\left[S\left(\tau, R_N^{E/O}(\ell; X, Y)\right)^2\right] \\ &= 2(1-\tau)(1-2\tau) \\ &\quad \times \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{N-2-\ell} (N-1-\ell-k)|E[X_n X_{n+k}]|^2 \\ &\quad + 2(1-\tau)\tau \\ &\times \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{N-2-\ell} (N-1-\ell-k)|E[X_n X_{n+k}] + E[X_n X_{n+k+1}]|^2 \\ &\quad + 2\tau(2\tau-1) \end{aligned}$$

$$\times \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{N-2-\ell} (N-1-\ell-k) |E[X_n X_{n+k+1}]|^2. \quad (\text{A} \cdot 1)$$

Since this takes a nonnegative real number, we obtain

$$\begin{aligned} & \lim_{N \rightarrow \infty} V \left[\Re \left(S \left(\tau, R_N^{E/O}(\ell; X, Y) / \sqrt{N} \right) e^{\sqrt{-1}\phi} \right) \right] \\ &= \lim_{N \rightarrow \infty} \frac{1}{2N} E \left[\left| S \left(\tau, R_N^{E/O}(\ell; X, Y) \right) \right|^2 \right] \\ &+ \lim_{N \rightarrow \infty} \frac{1}{2N} E \left[S \left(\tau, R_N^{E/O}(\ell; X, Y) \right)^2 \right] \cos 2\phi. \end{aligned}$$

Thus we can adapt the method of proof of Theorem 1 to prove

Lemma 9: Consider SSMA communication systems using M -phase spreading sequences of Markov chains. For any fixed τ , the distributions of normalized MAI for such systems are Gaussian if and only if

$$\lim_{N \rightarrow \infty} \frac{1}{2N} E \left[S \left(\tau, R_N^{E/O}(\ell; X, Y) \right)^2 \right] = 0.$$

Noting $E[X_n^2] = 0$ for $M \geq 3$, we can also adapt the method of proof of Theorem 1 in [15] to prove

Lemma 10: $|E[X_n X_{n+\ell}]|^2 = 0$ for all $\ell = 1, 2, \dots$ if and only if

$$E[X_n X_{n+1}] = E[X_n X_{n+2}] = \dots = E[X_n X_{n+M-2}] = 0.$$

By virtue of these two lemmas and taking account of (A·1), the proof is complete.

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