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The first Chern class and conformal area for a twistor holomorphic immersion

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Abstract We obtain an inequality involving the first Chern class of the normal bundle and the conformal area for a twistor holomorphic surface. Using this inequality, we can improve an inequality obtained by T. Friedrich for the Euler class of the normal bundle of a twistor holomorphic surface in the four-dimensional space form. Moreover, as a corollary, we see that the area of a superminimal surface in the unit sphere is an integer multiple of 2π , which is essentially proved by E. Calabi.

Keywords conformal area · first Chern class · twistor lift · twistor holomorphic surface

Mathematics Subject Classification (2000) 53C42 · 53A30

1 Introduction

The twistor space is important and useful in the study of conformal geometry, because it allows the translation of certain conformal objects into complex ones. It is also important for the study of surfaces in the unit sphere, particularly, superminimal surfaces (see [1] and [2]). Moreover, in [4], the notion of a twistor holomorphic surface in a four-dimensional manifold is introduced, which contains that of a superminimal surface. In fact, a superminimal surface is minimal and twistor holomorphic. Because the property that a surface is twistor holomorphic is invariant under conformal transformations of the ambient space, it is interesting to study such surfaces from the view point of

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conformal geometry. By definition, a twistor holomorphic surface has a holomorphic twistor lift. Hence, we can also study twistor holomorphic surfaces using complex geometry. Therefore, twistor holomorphic surfaces have interesting features. For example, twistor holomorphic surfaces in the four-dimensional space form of constant curvature have the lowest Willmore energy in each regular homotopy class and these values are given by the Euler classes (see [4] and [7] for example). In particular, the areas of superminimal surfaces take discrete values.

The twistor lifts of surfaces with codimension two have been extensively studied (see for example [1], [4], [5] and [7]). We study the case of surfaces with higher codimension, as these are also interesting from the view point of conformal and complex geometry. In this paper, we obtain an inequality involving the first Chern class of the normal bundle and the conformal area for a twistor holomorphic surface. Note that the conformal area, introduced in [9], is an important conformal invariant. Using this inequality, we can improve an inequality, obtained in [4], for the Euler class of the normal bundle of a twistor holomorphic surface in the four-dimensional space form. Moreover, as a corollary, it can be shown that the area of a compact superminimal surface in the unit sphere is an integer multiple of 2π , which is essentially proved in [2].

In Section 2, we study complex vector bundles and the decomposition of connections. We then recall the definitions of the twistor space and twistor lift in Section 3. In Section 4, we obtain an equality involving the first Chern classes for a twistor holomorphic submanifold, and in the last section, derive an inequality involving the first Chern class of the normal bundle and the conformal area.

2 Complex vector bundles.

Throughout this paper, all manifolds and maps are assumed to be smooth. Let E be a vector bundle over a manifold M and E_x the fiber of E over $x \in M$. We write TM for the tangent bundle of M and $\text{End}(E)$ for the vector bundle whose fiber $\text{End}(E)_x$ over $x \in M$ is the space of all linear maps from E_x into itself. Let $\varphi : N \rightarrow M$ be a smooth map and F a fiber bundle over M . The pull back bundle of F over N by φ is denoted by φ^*F . The set of all connections of a vector bundle E is denoted by $\mathcal{C}(E)$. The space of all sections of a fiber bundle F is denoted by $\Gamma(F)$. Let $\Lambda^k(E)$ be the set of all E -valued k -forms on M .

Let M be an almost complex manifold with an almost complex structure J and E a vector bundle over M with $I \in \Gamma(\text{End}(E))$ satisfying $I^2 = -id$ and $D \in \mathcal{C}(E)$. In general, we do not assume that I is parallel with respect to D . We define

$$D'_X \zeta := \frac{1}{2}(D_X \zeta - ID_{JX} \zeta), \quad D'' \zeta := \frac{1}{2}(D_X \zeta + ID_{JX} \zeta)$$

for $X \in TM$ and $\zeta \in \Gamma(E)$. It is easy to see $D = D' + D''$. The connection D on E induces the connection $\bar{D}$ on $\text{End}(E)$, which is given as follows. For $S \in \Gamma(\text{End}(E))$, $\bar{D}_X S \in \Gamma(\text{End}(E))$ is defined by

$$(\bar{D}_X S)(\zeta) := [D_X, S](\zeta) = D_X S(\zeta) - S(D_X \zeta)$$

for $X \in TM$ and $\zeta \in \Gamma(E)$. Let $\nabla \in \mathcal{C}(TM)$ be a torsion free connection on M . We can also define the connection (we use the same letter $\bar{D}$) on $T^*M \otimes \text{End}(E)$, that is,

$$\begin{aligned} (\bar{D}_X T)_Y \zeta &:= D_X(T_Y \zeta) - T_{\nabla_X Y} \zeta - T_Y(D_X \zeta) = [D_X, T_Y](\zeta) - T_{\nabla_X Y} \zeta \\ &= (\bar{D}_X T_Y)(\zeta) - T_{\nabla_X Y} \zeta \end{aligned}$$

for $T \in \Gamma(T^*M \otimes \text{End}(E))$. We define $D^I \in \mathcal{C}(E)$ by

$$D_X^I \zeta = D_X \zeta - \frac{1}{2} I(\bar{D}_X I)(\zeta)$$

for $X \in TM$ and $\zeta \in \Gamma(E)$. Then we have

$$D_X^{I'} \zeta = \frac{1}{2} (D'_X \zeta - ID'_X I \zeta), \quad D_X^{I''} \zeta = \frac{1}{2} (D''_X \zeta - ID''_X I \zeta)$$

for $X \in TM$ and $\zeta \in \Gamma(E)$. Set

$$A_X^{D'} \zeta := \frac{1}{2} (D'_X \zeta + ID'_X I \zeta), \quad A_X^{D''} \zeta := \frac{1}{2} (D''_X \zeta + ID''_X I \zeta),$$

for $X \in TM$ and $\zeta \in \Gamma(E)$. We see that $D' = D^{I'} + A^{D'}$ and $D'' = D^{I''} + A^{D''}$, and hence $D = D^{I'} + D^{I''} + A^{D'} + A^{D''}$. We can obtain the following lemmas from straightforward calculations.

Lemma 1 *We have $A^{D'}, A^{D''} \in \Lambda^1(\text{End}(E))$.*

Lemma 2 *We have $[D^I, I](= D^I I - I D^I) = 0$, that is, I is parallel with respect to D^I .*

Lemma 3 *We have*

$$\begin{aligned} D_{JX}^{I'} &= ID_X^{I'}, \quad A_{JX}^{D'} = IA_X^{D'}, \quad D_{JX}^{I''} = -ID_X^{I''}, \quad A_{JX}^{D''} = -IA_X^{D''} \\ D_X^{I'} I &= ID_X^{I'}, \quad A_X^{D'} I = -IA_X^{D'}, \quad D_X^{I''} I = ID_X^{I''}, \quad A_X^{D''} I = -IA_X^{D''} \end{aligned}$$

for $X \in TM$.

The operators $A^{D'}$ and $A^{D''}$ are explicitly given as follows.

Lemma 4 *We have*

$$\begin{aligned} A_X^{D'} &= \frac{1}{4} (I(\bar{D}_X I) + (\bar{D}_{JX} I)), \\ A_X^{D''} &= \frac{1}{4} (I(\bar{D}_X I) - (\bar{D}_{JX} I)) \end{aligned}$$

for $X \in TM$.

Proof By the definitions of D^I , $A^{D'}$ and $A^{D''}$, we have

$$A_X^{D'} + A_X^{D''} = \frac{1}{2}I(\bar{D}_X I)$$

for $X \in TM$. Moreover, by Lemma 3, it holds that

$$\frac{1}{2}I(\bar{D}_{JX} I) = A_{JX}^{D'} + A_{JX}^{D''} = I(A_X^{D'} - A_X^{D''}),$$

and hence,

$$A_X^{D'} - A_X^{D''} = \frac{1}{2}(\bar{D}_{JX} I)$$

for $X \in TM$. Therefore, the desired equations are obtained. $\square$

If $A^{D''} = 0$, then we have $A_X^{D'} = (1/2)(\bar{D}_{JX} I)$. Hence, we see that

$$(\bar{D}_X A^{D'})_Y = \bar{D}_X A_Y^{D'} - A_{\nabla_X Y}^{D'} = \frac{1}{2}\bar{D}_X \bar{D}_{JY} I - \frac{1}{2}\bar{D}_{J\nabla_X Y} I. \quad (1)$$

Let R^D be the curvature form of $D \in \mathcal{C}(E)$. Using (1), we have the following lemma.

Lemma 5 *If $A^{D''} = 0$, then we have*

$$R_{X,Y}^D = R_{X,Y}^{D'} + \frac{1}{4}(\bar{D}_X I)(\bar{D}_Y I) - \frac{1}{4}(\bar{D}_Y I)(\bar{D}_X I) + \frac{1}{2}I(R_{X,Y}^{\bar{D}})$$

for all $X, Y \in TM$.

Proof Because the difference tensor between D and D^I is given by

$$A_X^{D'} = \frac{1}{2}I(\bar{D}_X I) = \frac{1}{2}(\bar{D}_{JX} I)$$

for $X \in TM$, we have

$$\begin{aligned} R_{X,Y}^D &= R_{X,Y}^{D'} + (\bar{D}_X A^{D'})_Y - (\bar{D}_Y A^{D'})_X - [A_X^{D'}, A_Y^{D'}] \\ &= R_{X,Y}^{D'} + \frac{1}{2}\bar{D}_X \bar{D}_{JY} I - \frac{1}{2}\bar{D}_{J(\nabla_X Y)} I \\ &\quad - \frac{1}{2}\bar{D}_Y \bar{D}_{JX} I + \frac{1}{2}\bar{D}_{J(\nabla_Y X)} I \\ &\quad - \frac{1}{4}(\bar{D}_{JX} I)(\bar{D}_{JY} I) + \frac{1}{4}(\bar{D}_{JY} I)(\bar{D}_{JX} I) \\ &= R_{X,Y}^{D'} + \frac{1}{2}(\bar{D}_X I)(\bar{D}_Y I) + \frac{1}{2}I(\bar{D}_X \bar{D}_Y I) - \frac{1}{2}\bar{D}_{J(\nabla_X Y)} I \\ &\quad - \frac{1}{2}(\bar{D}_Y I)(\bar{D}_X I) - \frac{1}{2}I(\bar{D}_Y \bar{D}_X I) + \frac{1}{2}\bar{D}_{J(\nabla_Y X)} I \\ &\quad - \frac{1}{4}(\bar{D}_{JX} I)(\bar{D}_{JY} I) + \frac{1}{4}(\bar{D}_{JY} I)(\bar{D}_{JX} I) \\ &= R_{X,Y}^{D'} + \frac{1}{4}(\bar{D}_X I)(\bar{D}_Y I) - \frac{1}{4}(\bar{D}_Y I)(\bar{D}_X I) + \frac{1}{2}I(R_{X,Y}^{\bar{D}}) \end{aligned}$$

for all $X, Y \in TM$. $\square$

Using Lemma 5, we can state the following.

Lemma 6 *If $A^{D''} = 0$, then we have*

$$\mathrm{Tr}(R_{X,Y}^D \circ I) = \mathrm{Tr}(R_{X,Y}^{D^I} \circ I) + \frac{1}{2} \mathrm{Tr}(\bar{D}_{JX} I) \circ (\bar{D}_Y I)$$

for $X, Y \in TM$.

Proof From $I(R_{X,Y}^{\bar{D}} I) = I(R_{X,Y}^D I - IR_{X,Y}^D) = IR_{X,Y}^D I + R_{X,Y}^D$, we see that $\mathrm{Tr} I(R_{X,Y}^{\bar{D}} I) I = \mathrm{Tr}(-IR_{X,Y}^D + R_{X,Y}^D I) = 0$ for all $X, Y \in TM$. $\square$

Let $c_1(E, D^I)$ be the first Chern form of (E, D^I) . We see that

$$4\pi c_1(E, D^I)(X, Y) = \mathrm{Tr}(R_{X,Y}^{D^I} \circ I)$$

for $X, Y \in TM$. Hence, if $A^{D''} = 0$, Lemma 6 implies that

$$\mathrm{Tr}(R_{X,Y}^D \circ I) = 4\pi c_1(E, D^I)(X, Y) + \frac{1}{2} \mathrm{Tr}(\bar{D}_{JX} I) \circ (\bar{D}_Y I). \quad (2)$$

3 Twistor spaces and twistor lifts.

Consider the space of all complex structures compatible with the metric and orientation on an oriented $2n$ -dimensional vector space V with the inner product. For more details, see [11]. A complex structure on V is an endomorphism $J : V \rightarrow V$ such that $J^2 = -id$. If a complex structure J satisfies ${}^t J = J^{-1}$, where t is the transpose of J , then we say that J is compatible with the inner product. A complex structure J is said to be compatible with the specified orientation of V if the orientation of V determined by J agrees with the specified orientation. We denote the set of all skew-symmetric endomorphisms of V by $\mathfrak{so}(V)$. Let $W(V)$ be the subset of $\mathfrak{so}(V)$ consisting of all complex structures on V compatible with the inner product and orientation of V . If we choose $J_0 \in W(V)$, then we have the identification $W(V) \cong \mathrm{SO}(V)/U(V, J_0)$, where $\mathrm{SO}(V)$ is the special orthogonal group acting on V and $U(V, J_0) = \{A \in \mathrm{SO}(V) \mid AJ_0 = J_0 A\}$. The tangent space of $W(V)$ at J can be identified with the vector subspace $\{a \in \mathfrak{so}(V) \mid aJ = -Ja\}$. We define an almost complex structure $\mathcal{J}$ on $W(V)$ by

$$\mathcal{J}(a) = \frac{1}{2}[J, a]$$

for $a \in T_J W(V)$. Note that we have $\mathcal{J}(a) = (1/2)[J, a] = (1/2)(Ja - aJ) = Ja = -aJ$ for all $a \in T_J W(V)$. We define an $\mathrm{SO}(V)$ -invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{so}(V)$ by $\langle a, b \rangle = \mathrm{Tr}(a^t b)$ for all $a, b \in \mathfrak{so}(V)$. Such an inner product restricts to a Riemannian metric on $W(V)$. Let p_J be the projection from $T_J \mathfrak{so}(V) (\cong \mathfrak{so}(V))$ onto $T_J W(V)$ at $J \in W(V)$. We can explicitly write p_J as

$$p_J(a) = \frac{1}{2}(a + JaJ) \quad (3)$$

for all $a \in T_{J\text{so}}(V)$. The space $(W(V), \langle \cdot, \cdot \rangle, \mathcal{J})$ is a Kähler manifold (see [11]).

Let $(\tilde{M}, \tilde{g})$ be a $2n$ -dimensional oriented Riemannian manifold. We define the twistor space $\mathcal{Z}(\tilde{M})$ of $\tilde{M}$ by

$$\mathcal{Z}(\tilde{M}) := \bigcup_{x \in \tilde{M}} W(T_x \tilde{M}).$$

The bundle projection $p : \mathcal{Z}(\tilde{M}) \rightarrow \tilde{M}$ and the Levi-Civita connection induce the vertical and horizontal subbundles of $T\mathcal{Z}(\tilde{M})$. The almost complex structure $J^{\mathcal{Z}}$ on the twistor space is defined by $J^{\mathcal{Z}}(X) = (J(p_*(X)))^h_J$ for all horizontal vectors X and $J^{\mathcal{Z}}(Y) = \mathcal{J}(Y)$ for all vertical vectors Y at $J \in \mathcal{Z}(\tilde{M})$, where $(\cdot)^h$ stands for the horizontal lift and $\mathcal{J}$ is the complex structure defined above. It is well-known that $J^{\mathcal{Z}}$ is integrable if and only if $\tilde{M}$ is anti-self-dual when $\dim \tilde{M} = 4$, and $\tilde{M}$ is conformally flat when $\dim \tilde{M} > 4$.

Next we define the twistor lift for an almost Hermite submanifold in an even-dimensional Riemannian manifold. Let M be a $2k$ -dimensional almost Hermite manifold with an almost complex structure I and Hermite metric g , and $\tilde{M}$ an oriented $2n$ -dimensional Riemannian manifold. Let $f : M \rightarrow \tilde{M}$ be an isometric immersion from M into $\tilde{M}$. The normal bundle of f is denoted by $T^\perp M$. We assume that there exists a complex structure $I^\perp$ on the normal bundle $T^\perp M$ that is compatible with the metric. We can then define $\tilde{I}$ along f by $\tilde{I}(X) = I(X)$ for all $X \in TM$ and $\tilde{I}(\xi) = I^\perp(\xi)$ for all $\xi \in T^\perp M$. If $\tilde{I}(x)$ is compatible with the orientation of $T_{f(x)}\tilde{M}$ for all $x \in M$, then the section $\tilde{I} \in \Gamma(f^*(\mathcal{Z}(\tilde{M})))$ is called a *twistor lift* of M . In [8], $\tilde{I}$ is called an admissible twistor lift. In the case of $\dim M = 2$ and $\dim \tilde{M} = 4$, we can define the twistor lift uniquely.

Let ∇ (resp. $\tilde{\nabla}$) be the Levi-Civita connection of g (resp. $\tilde{g}$) and the connection on the normal bundle $T^\perp M$ is denoted by $\nabla^\perp$. The second fundamental form (resp. the shape operator) is denoted by α (resp. S). The mean curvature vector field H is defined by

$$H = \frac{1}{2k} \sum_{i=1}^{2k} \alpha(e_i, e_i),$$

where $(e_1, \dots, e_{2k})$ is an orthonormal frame of M . We often omit the symbol $f^\#$ for the induced objects of f if there is no confusion. For example, we use the same letter $\tilde{\nabla}$ for the connection on $\text{End}(f^*(T\tilde{M}))$. It is easy to obtain the following lemma.

Lemma 7 *We have*

$$(\tilde{\nabla}_X \tilde{I})(Y) = (\tilde{\nabla}_X I)(Y) + \alpha(X, IY) - I^\perp \alpha(X, Y)$$

and

$$(\tilde{\nabla}_X \tilde{I})(\xi) = (\tilde{\nabla}_X^\perp I)(\xi) - S_{I^\perp \xi} X + I S_\xi X$$

for all $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$.

If the twistor lift satisfies $\tilde{\nabla}_X \tilde{I} = 0$ for all $X \in TM$, then M is called a *superminimal* submanifold (or f is called a superminimal immersion). If M is superminimal, then it is minimal, that is, $H = 0$. By Lemma 7, we have the following proposition.

Proposition 1 *Let M be a submanifold in $\tilde{M}$ with twistor lift $\tilde{I}$. Then M is superminimal if and only if the following equations hold :*

- (1) $\tilde{\nabla} I = 0$,
- (2) $\tilde{\nabla}^\perp I^\perp = 0$,
- (3) $\alpha(X, IY) = I^\perp \alpha(X, Y)$ for all $X, Y \in TM$.

We study the influence of conformal changes on geometric objects of submanifolds. On $\tilde{M}$, we take two Riemannian metrics $\tilde{g}_1$ and $\tilde{g}_2$ satisfying $\tilde{g}_2 = \varphi^2 \tilde{g}_1$ for a smooth function $\varphi > 0$ on $\tilde{M}$, that is, $\tilde{g}_1$ and $\tilde{g}_2$ are conformally equivalent. The Levi-Civita connections $\tilde{\nabla}^{\tilde{g}_1}$, $\tilde{\nabla}^{\tilde{g}_2}$ of $\tilde{g}_1$, $\tilde{g}_2$ are related as follows :

$$\tilde{\nabla}_X^{\tilde{g}_2} Y = \tilde{\nabla}_X^{\tilde{g}_1} Y + (XF)Y + (YF)X - \tilde{g}_1(X, Y) \text{grad}_{\tilde{g}_1} F$$

for $X, Y \in \Gamma(T\tilde{M})$, where $F = \log \varphi$. Thus, we see that

$$\begin{aligned} \tilde{\nabla}_X^{\tilde{g}_2} \tilde{I}' Y &= \tilde{\nabla}_X^{\tilde{g}_1} \tilde{I}' Y + \frac{1}{2}((XF)Y + (YF)X \\ &\quad - (\tilde{I}XF)\tilde{I}Y - (\tilde{I}YF)\tilde{I}X), \end{aligned} \quad (4)$$

$$\begin{aligned} \tilde{\nabla}_X^{\tilde{g}_2} \tilde{I}'' Y &= \tilde{\nabla}_X^{\tilde{g}_1} \tilde{I}'' Y + \frac{1}{2}((XF)Y + (\tilde{I}XF)\tilde{I}Y \\ &\quad - \tilde{g}_1(X, Y) \text{grad}_{\tilde{g}_1} F - \tilde{g}_1(\tilde{I}X, Y) \tilde{I} \text{grad}_{\tilde{g}_1} F), \end{aligned} \quad (5)$$

$$\begin{aligned} A_X^{\tilde{\nabla}^{\tilde{g}_2}} Y &= A_X^{\tilde{\nabla}^{\tilde{g}_1}} Y + \frac{1}{2}((YF)X + (\tilde{I}YF)\tilde{I}X \\ &\quad - \tilde{g}_1(X, Y) \text{grad}_{\tilde{g}_1} F + \tilde{g}_1(\tilde{I}X, Y) \tilde{I} \text{grad}_{\tilde{g}_1} F), \end{aligned} \quad (6)$$

$$A_X^{\tilde{\nabla}^{\tilde{g}_2}} Y = A_X^{\tilde{\nabla}^{\tilde{g}_1}} Y \quad (7)$$

for $X, Y \in \Gamma(T\tilde{M})$. We define $\alpha^{(2,0)}$ and $\alpha^{(0,2)}$ by

$$\begin{aligned} \alpha^{(2,0)}(X, Y) &:= \alpha(X, Y) - I^\perp \alpha(IX, Y) - I^\perp \alpha(X, IY) - \alpha(IX, IY), \\ \alpha^{(0,2)}(X, Y) &:= \alpha(X, Y) + I^\perp \alpha(IX, Y) + I^\perp \alpha(X, IY) - \alpha(IX, IY) \end{aligned}$$

for X and $Y \in TM$. To find conformal invariants for submanifolds, equations (4)-(7) can be used. For example, decompose $A_X^{f^\# \tilde{\nabla}''} Y$ and $A_X^{f^\# \tilde{\nabla}''} \xi$ ($X, Y \in TM$ and $\xi \in T^\perp M$) into their tangential and normal components. We have the following lemma.

Lemma 8 *Let M be a submanifold in $\tilde{M}$ with twistor lift $\tilde{I}$. The following objects are invariant under conformal transformations of $\tilde{M}$: $\alpha^{(2,0)}$, $A^{\nabla''}$, $\alpha^{(0,2)}$ and $\tilde{\nabla}^\perp I^\perp$. In particular, if $\dim M = 2$ and M is compact,*

$$\int_M \|\tilde{\nabla}^\perp I^\perp\|^2 dv_g$$

is invariant under conformal transformations of $\tilde{M}$.

Proof Because $A^{\nabla^\perp'}$ and $A^{\nabla^\perp''}$ are invariant under conformal transformations of $\tilde{M}$, so is $A^{\nabla^\perp'} + A^{\nabla^\perp''}$. Hence, $\bar{\nabla}^\perp I^\perp$ is invariant. It is clear that the others are also invariant under conformal transformations. $\square$

If the twistor lift satisfies $\tilde{I}_* \circ I = J^\mathcal{Z} \circ \tilde{I}_*$, then M is called a *twistor holomorphic* submanifold (or f is called a twistor holomorphic immersion). By the definition of $J^\mathcal{Z}$, f is twistor holomorphic if and only if it holds that

$$\bar{\nabla}_{IX} \tilde{I} = \tilde{I} \bar{\nabla}_X \tilde{I} \quad (8)$$

for all $X \in TM$, which is equivalent to $A^{f^\# \bar{\nabla}''} = 0$. In fact, taking the vertical components of $(\tilde{I}_* \circ I)(X) = (J^\mathcal{Z} \circ \tilde{I}_*)(X)$ for $X \in TM$, we see that $p_{\tilde{I}}(\bar{\nabla}_{IX} \tilde{I}) = \tilde{I}(p_{\tilde{I}}(\bar{\nabla}_X \tilde{I}))$. Using (3), we obtain equation (8), from which we can state the following.

Proposition 2 *Let M be a submanifold in $\tilde{M}$ with twistor lift $\tilde{I}$. Then M is twistor holomorphic if and only if the following conditions hold :*

- (1) $A^{\nabla''} = 0$, that is, the almost complex structure I is integrable,
- (2) $A^{\nabla^\perp''} = 0$, that is, $\bar{\nabla}_{IX}^\perp I^\perp = I^\perp(\bar{\nabla}_X^\perp I^\perp)$ holds for all $X \in TM$,
- (3) $\alpha^{(0,2)} = 0$.

By Lemma 8 and Proposition 2, we have the following corollary.

Corollary 1 *The property that f is twistor holomorphic is invariant under conformal transformations of $\tilde{M}$.*

If M is superminimal, then M is twistor holomorphic. Let $Q^l(c)$ be the space form of dimension l and constant curvature c , and set $S^l := Q^l(1)$. We now give examples of superminimal and twistor holomorphic immersions.

Example 1 Linearly full minimal immersions from compact surfaces of genus 0 into S^{2n} are superminimal (see [2]). A typical example of a superminimal immersion is the standard immersion from 2-sphere, for example, the Veronese immersion into S^4 .

Example 2 Any holomorphic immersions from a Kähler manifold into a Kähler manifold are superminimal.

Example 3 Simply connected totally umbilic surfaces in $Q^{2n}(c)$ are twistor holomorphic. The normal connection of a totally umbilic surface is flat. Hence, we can take an orthonormal parallel frame of the normal bundle, because the surface is simply connected. A parallel complex structure of the normal bundle can be defined using the frame. Note that a twistor holomorphic surface in $Q^{2n}(c)$ with a flat normal connection must be totally umbilic (see Lemma 13).

Example 4 Any holomorphic immersions from an Hermite manifold into an Hermite manifold are twistor holomorphic. Consider a holomorphic immersion f from a complex submanifold M into the Calabi-Eckmann manifold $S^{2p_1+1} \times$

S^{2p_2+1} . See [6] for differential geometric properties of the Calabi-Eckmann manifold. The immersion f need not be superminimal. In fact, for example, $M = S^{2p_1+1} \times S^1$ ($p_1 \geq 1$) is a complex submanifold in $S^{2p_1+1} \times S^{2p_2+1}$ and the induced structure on M is never Kählerian for topological reasons, where S^1 is a fiber of the Hopf fibration of the second factor. Therefore M is not superminimal.

Other examples can be found in [4] and [5].

Remark 1 In the case of $\dim M = 2$ and $\dim \tilde{M} = 4$, totally geodesic surfaces ($\alpha = 0$) are superminimal. However, the example above shows that this is not true in general for the higher (co)dimensional cases. In fact, the submanifold $S^{2p_1+1} \times S^1$ in $S^{2p_1+1} \times S^{2p_2+1}$ is totally geodesic but not superminimal.

Remark 2 We can define another almost complex structure $J^{\mathcal{Z}'}$ on the twistor space $\mathcal{Z}(\tilde{M})$ as follows : $J^{\mathcal{Z}'}(X) = (J(p_*(X)))_J^h$ for all horizontal vectors X at $J \in \mathcal{Z}(\tilde{M})$ and $J^{\mathcal{Z}'}(Y) = -\mathcal{J}(Y)$ for all vertical vectors Y . Note that $J^{\mathcal{Z}'}$ is never integrable. We can also study an almost Hermite submanifold M in $\tilde{M}$ isometrically immersed by f whose twistor lift is holomorphic with respect to $J^{\mathcal{Z}'}$. Such a submanifold satisfies $A^{\nabla'} = 0$, $A^{\nabla'^{\perp}} = 0$ and $\alpha(X, X) + \alpha(IX, IX) = 0$ for all $X \in TM$. Thus M is $(1, 2)$ -symplectic and f is pluriminimal. See [3] and [10] for example. Note that $J^{\mathcal{Z}'}$ -holomorphicity of the twistor lift is not invariant under conformal transformations of $\tilde{M}$.

4 The first Chern classes for twistor holomorphic submanifolds.

Let (M, g, I) be an almost Hermite submanifold in $(\tilde{M}, \tilde{g})$ with $\dim M = 2k$ and $\dim \tilde{M} = 2n$, and $\tilde{I}$ a twistor lift of M . We define β by

$$\beta(X, Y) = \alpha(X, IY) - I^{\perp}\alpha(X, Y)$$

for $X, Y \in TM$. It is easy to see that $\beta = 0$ if M is superminimal. We obtain $\alpha^{(0,2)}(X, IY) = \beta(X, Y) - \beta(IX, IY)$ for all $X, Y \in TM$. The curvature forms of $\tilde{\nabla}$, ∇ , $\nabla^{\perp}$ are denoted by $\tilde{R}$, R , $R^{\perp}$, respectively. Set $\mathcal{R}^{\perp} \in \Gamma(\text{End}(T^{\perp}M))$ (resp. $\mathcal{R}^{\top} \in \Gamma(\text{End}(TM))$) by

$$\mathcal{R}^{\perp}(\xi) := \sum_{i=1}^{2k} R_{e_i, Ie_i}^{\perp}(I^{\perp}\xi) \quad (\text{resp. } \mathcal{R}^{\top}(X) := \sum_{i=1}^{2k} R_{e_i, Ie_i}(IX))$$

for $\xi \in \Gamma(T^{\perp}M)$ (resp. $X \in \Gamma(TM)$), where $(e_1, \dots, e_{2k})$ is an orthonormal frame of M .

Lemma 9 *Let M be a submanifold in $\tilde{M}$ with twistor lift $\tilde{I}$. We have*

$$\text{Tr} \mathcal{R}^{\perp} = 2 \sum_{i,j=1}^{2k} \tilde{g}(\alpha(Ie_i, e_j), I^{\perp}\alpha(e_i, e_j)) + \sum_{i=1}^{2k} \sum_{l=1}^{2(n-k)} \tilde{g}(\tilde{R}_{e_i, Ie_i} I^{\perp}\xi_l, \xi_l),$$

where $(\xi_1, \dots, \xi_{2(n-k)})$ is an orthonormal frame of $T^{\perp}M$.

Proof By the Ricci equation, we have

$$\tilde{g}(R_{X,Y}^\perp \xi, \eta) = \tilde{g}(S_\xi S_\eta X, Y) - \tilde{g}(S_\eta S_\xi X, Y) + \tilde{g}(\tilde{R}_{X,Y} \xi, \eta)$$

for $X, Y \in TM$ and $\xi, \eta \in T^\perp M$. We then obtain

$$\begin{aligned} \sum_{i=1}^{2k} \tilde{g}(R_{e_i, Ie_i}^\perp \xi, \eta) &= \sum_{i,j=1}^{2k} \tilde{g}(\alpha(e_i, e_j), \eta) \tilde{g}(\alpha(Ie_i, e_j), \xi) \\ &\quad - \tilde{g}(\alpha(e_i, e_j), \xi) \tilde{g}(\alpha(Ie_i, e_j), \eta) \\ &\quad + \sum_{i=1}^{2k} \tilde{g}(\tilde{R}_{e_i, Ie_i} \xi, \eta) \end{aligned}$$

for $\xi, \eta \in T^\perp M$. Therefore it holds that

$$\begin{aligned} &\sum_{i=1}^{2k} \sum_{l=1}^{2(n-k)} \tilde{g}(R_{e_i, Ie_i}^\perp \xi_l, I^\perp \xi_l) \\ &= \sum_{i,j=1}^{2k} \sum_{l=1}^{2(n-k)} \tilde{g}(\alpha(e_i, e_j), I^\perp \xi_l) \tilde{g}(\alpha(Ie_i, e_j), \xi_l) \\ &\quad - \tilde{g}(\alpha(e_i, e_j), \xi_l) \tilde{g}(\alpha(Ie_i, e_j), I^\perp \xi_l) + \sum_{i=1}^{2k} \sum_{l=1}^{2(n-k)} \tilde{g}(\tilde{R}_{e_i, Ie_i} \xi_l, I^\perp \xi_l) \\ &= -2 \sum_{i,j=1}^{2k} \tilde{g}(I^\perp \alpha(e_i, e_j), \alpha(Ie_i, e_j)) - \sum_{i=1}^{2k} \sum_{l=1}^{2(n-k)} \tilde{g}(\tilde{R}_{e_i, Ie_i} I^\perp \xi_l, \xi_l). \end{aligned}$$

□

Lemma 10 *Let M be a submanifold in $\tilde{M}$ with twistor lift $\tilde{I}$. We have*

$$\|\beta\|^2 = 2\|\alpha\|^2 - \text{Tr} \mathcal{R}^\perp + \sum_{i=1}^{2k} \sum_{l=1}^{2(n-k)} \tilde{g}(\tilde{R}_{e_i, Ie_i} I^\perp \xi_l, \xi_l)$$

and

$$\begin{aligned} \|\alpha^{(0,2)}\|^2 &= 2\|\beta\|^2 - 2\text{Tr} \mathcal{R}^\perp + 4 \sum_{i,j=1}^{2k} \tilde{g}(\alpha(e_i, Ie_j), \alpha(Ie_i, e_j)) \\ &\quad + 2 \sum_{i=1}^{2k} \sum_{l=1}^{2(n-k)} \tilde{g}(\tilde{R}_{e_i, Ie_i} I^\perp \xi_l, \xi_l). \end{aligned}$$

Proof We have

$$\begin{aligned}
\|\beta\|^2 &= \sum_{i,j=1}^{2k} \tilde{g}(\alpha(e_i, Ie_j) - I^\perp \alpha(e_i, e_j), \alpha(e_i, Ie_j) - I^\perp \alpha(e_i, e_j)) \\
&= 2\|\alpha\|^2 - 2 \sum_{i,j=1}^{2k} \tilde{g}(\alpha(e_i, Ie_j), I^\perp \alpha(e_i, e_j)) \\
&= 2\|\alpha\|^2 - \text{Tr} \mathcal{R}^\perp + \sum_{i=1}^{2k} \sum_{l=1}^{2(n-k)} \tilde{g}(\tilde{R}_{e_i, Ie_i} I^\perp \xi_l, \xi_l).
\end{aligned}$$

Moreover we obtain

$$\begin{aligned}
\|\alpha^{(0,2)}\|^2 &= \sum_{i,j=1}^{2k} \tilde{g}(\alpha^{(0,2)}(e_i, Ie_j), \alpha^{(0,2)}(e_i, Ie_j)) \\
&= \sum_{i,j=1}^{2k} \tilde{g}(\beta(e_i, e_j) - \beta(Ie_i, Ie_j), \beta(e_i, e_j) - \beta(Ie_i, Ie_j)) \\
&= 2\|\beta\|^2 - 2 \sum_{i,j=1}^{2k} \tilde{g}(\beta(e_i, e_j), \beta(Ie_i, Ie_j)) \\
&= 2\|\beta\|^2 - 4 \sum_{i,j=1}^{2k} \tilde{g}(I^\perp \alpha(e_i, e_j), \alpha(Ie_i, e_j)) \\
&\quad + 4 \sum_{i,j=1}^{2k} \tilde{g}(\alpha(e_i, Ie_j), \alpha(Ie_i, e_j)).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\|\alpha^{(0,2)}\|^2 &= 2\|\beta\|^2 - 2\text{Tr} \mathcal{R}^\perp + 2 \sum_{i=1}^{2k} \sum_{l=1}^{2(n-k)} \tilde{g}(\tilde{R}_{e_i, Ie_i} I^\perp \xi_l, \xi_l) \\
&\quad + 4 \sum_{i,j=1}^{2k} \tilde{g}(\alpha(e_i, Ie_j), \alpha(Ie_i, e_j))
\end{aligned}$$

by Lemma 9. □

We define the $*$ -Ricci tensor Ric^* of M by

$$Ric^*(X, Y) = \frac{1}{2} \text{Tr}(T_x M \ni Z \mapsto R_{X, IY} IZ)$$

for all $X, Y \in T_x M$ at each point $x \in M$. The $*$ -scalar curvature κ^* of M is defined by

$$\kappa^* = \sum_{i=1}^{2k} Ric^*(e_i, e_i) = \frac{1}{2} \text{Tr} \mathcal{R}^\top,$$

where $e_1, \dots, e_{2k}$ is an orthonormal frame of M .

Lemma 11 *Let M be a submanifold in $\tilde{M}$ with twistor lift $\tilde{I}$. We have*

$$\sum_{i,j=1}^{2k} \tilde{g}(\alpha(e_i, Ie_j), \alpha(Ie_i, e_j)) = \frac{1}{2} \sum_{i,j=1}^{2k} \tilde{g}(\tilde{R}(e_i, Ie_i) Ie_j, e_j) - \kappa^*.$$

Proof By the Gauss equation, we have

$$\begin{aligned} & \sum_{i,j=1}^{2k} \tilde{g}(\alpha(e_i, Ie_j), \alpha(Ie_i, e_j)) \\ &= \sum_{i,j=1}^{2k} \tilde{g}(\tilde{R}_{e_i, Ie_i} Ie_j, e_j) - \sum_{i,j=1}^{2k} g(R_{e_i, Ie_i} Ie_j, e_j) \\ & \quad + \sum_{i,j=1}^{2k} \tilde{g}(\alpha(e_i, e_j), \alpha(Ie_i, Ie_j)) \\ &= \sum_{i,j=1}^{2k} \tilde{g}(\tilde{R}_{e_i, Ie_i} Ie_j, e_j) - 2\kappa^* + \sum_{i,j=1}^{2k} \tilde{g}(\alpha(e_i, e_j), \alpha(Ie_i, Ie_j)). \end{aligned}$$

Hence, we have

$$\sum_{i,j=1}^{2k} \tilde{g}(\alpha(e_i, Ie_j), \alpha(Ie_i, e_j)) = \frac{1}{2} \sum_{i,j=1}^{2k} \tilde{g}(\tilde{R}_{e_i, Ie_i} Ie_j, e_j) - \kappa^*.$$

□

Let κ be the scalar curvature of (M, g) .

Lemma 12 *Let M be a submanifold in $\tilde{M}$ with twistor lift $\tilde{I}$. We have*

$$\begin{aligned} \|\alpha^{(0,2)}\|^2 &= 4 \sum_{i,j=1}^{2k} \tilde{g}(\tilde{R}_{e_i, e_j} e_j, e_i) - 4(\kappa + \kappa^*) + 16k^2 \|H\|^2 - 4\text{Tr}\mathcal{R}^\perp \\ & \quad + 4 \sum_{i=1}^{2k} \sum_{l=1}^{2(n-k)} \tilde{g}(\tilde{R}_{e_i, Ie_i} I^\perp \xi_l, \xi_l) + 2 \sum_{i,j=1}^{2k} \tilde{g}(\tilde{R}_{e_i, Ie_i} Ie_j, e_j). \end{aligned}$$

Proof By the Gauss equation, we have

$$\sum_{i,j=1}^{2k} \tilde{g}(\tilde{R}_{e_i, e_j} e_j, e_i) = \kappa + \|\alpha\|^2 - 4k^2 \|H\|^2.$$

Using Lemmas 10 and 11, we obtain the conclusion. □

Lemma 13 *Let M be a submanifold in the space form $Q^{2n}(c)$ with twistor lift $\tilde{I}$. We have*

$$\|\alpha^{(0,2)}\|^2 = 16k^2 c - 4(\kappa + \kappa^*) + 16k^2 \|H\|^2 - 4\text{Tr}\mathcal{R}^\perp.$$

Proof We see that

$$\begin{aligned} \sum_{i,j=1}^{2k} \tilde{g}(\tilde{R}_{e_i, e_j} e_j, e_i) &= 2ck(2k-1), \\ \sum_{i=1}^{2k} \sum_{l=1}^{2(n-k)} \tilde{g}(\tilde{R}_{e_i, Ie_i} I^\perp \xi_l, \xi_l) &= 0, \\ \sum_{i,j=1}^{2k} \tilde{g}(\tilde{R}_{e_i, Ie_i} Ie_j, e_j) &= 4ck. \end{aligned}$$

Using Lemma 12, we can obtain the desired equation. $\square$

Let Ω be the fundamental 2-form of (g, I) . Set $\Omega^t := \Omega \wedge \cdots \wedge \Omega$ (t -times). It is easy to prove the following lemma.

Lemma 14 *Let θ be a 2-form on an almost Hermite manifold (M, g, I) . We have*

$$\sum_{i=1}^{2k} \theta(e_i, Ie_i) \Omega^k = 2k(\theta \wedge \Omega^{k-1}).$$

Using (2), we can show the following.

Lemma 15 *If $A^{\nabla''} = 0$ and $A^{\nabla^\perp} = 0$, then we have*

$$\begin{aligned} 8\pi \sum_{i=1}^{2k} c_1(T^\perp M, \nabla^\perp I^\perp)(e_i, Ie_i) &= 2\text{Tr} \mathcal{R}^\perp + \|\bar{\nabla}^\perp I^\perp\|^2, \\ 8\pi \sum_{i=1}^{2k} c_1(TM, \nabla^I)(e_i, Ie_i) &= 4\kappa^* + \|\bar{\nabla} I\|^2. \end{aligned}$$

By Lemmas 13, 14 and 15, we have the following proposition.

Proposition 3 *Let M be a $2k$ -dimensional compact submanifold in $Q^{2n}(c)$ with twistor lift $\tilde{I}$. If $A^{\nabla''} = 0$ and $A^{\nabla^\perp} = 0$, then we have*

$$\begin{aligned} &16k^2 c \int_M \Omega^k - 4 \int_M \kappa \Omega^k - 16k\pi \int_M c_1(TM, \nabla^I) \wedge \Omega^{k-1} + \int_M \|\bar{\nabla} I\|^2 \Omega^k \\ &+ 16k^2 \int_M \|H\|^2 \Omega^k - 32k\pi \int_M c_1(T^\perp M, \nabla^\perp I^\perp) \wedge \Omega^{k-1} \\ &+ 2 \int_M \|\bar{\nabla}^\perp I^\perp\|^2 \Omega^k \geq 0. \end{aligned}$$

The equality holds if and only if M is twistor holomorphic.

As a submanifold in $\mathbb{R}^{2(p_1+p_2)+4}(=Q^{2(p_1+p_2)+4}(0))$, the Calabi-Eckmann manifold $S^{2p_1+1} \times S^{2p_2+1}$ satisfies the assumptions in Proposition 3 but is not twistor holomorphic. In fact, because the almost complex structure on the Calabi-Eckmann manifold is integrable and $\nabla^\perp I^\perp = 0$, we see that $A^{\nabla''} = 0$ and $A^{\nabla^\perp} = 0$. However, we have $\alpha^{(0,2)}(J_1 N_1, J_2 N_2) \neq 0$, where N_i is the unit outward normal vector field of S^{2p_i+1} in $\mathbb{R}^{2p_i+2} \cong \mathbb{C}^{p_i+1}$ and J_i is the standard complex structure of $\mathbb{C}^{p_i+1}$ ($i = 1, 2$). Note that the Calabi-Eckmann manifold with $p_1 + p_2 \geq 1$ does not admit any Kähler structures. If a submanifold is a Kähler manifold, the following corollary holds.

Corollary 2 *Let M be a compact submanifold of $\dim M = 2k$ in $Q^{2n}(c)$ with twistor lift $\tilde{I}$. If M is a Kähler manifold and $A^{\nabla^\perp} = 0$, then we have*

$$\begin{aligned} & 8k^2 c \int_M \Omega^k - 16k\pi \int_M c_1(TM) \wedge \Omega^{k-1} \\ & + 8k^2 \int_M \|H\|^2 \Omega^k - 16k\pi \int_M c_1(T^\perp M) \wedge \Omega^{k-1} + \int_M \|\bar{\nabla}^\perp I^\perp\|^2 \Omega^k \geq 0, \end{aligned}$$

where $c_1(TM)$ (resp. $c_1(T^\perp M)$) is the first Chern class of TM (resp. $T^\perp M$). The equality holds if and only if M is twistor holomorphic.

Proof Because M is a Kähler manifold, we see that $\kappa^* = \kappa$. Using Lemma 15, we have the conclusion. $\square$

The second (resp. fourth) term of the left hand side of the inequality in Corollary 2 depends only on the cohomology classes $c_1(TM)$ (resp. $c_1(T^\perp M)$) and $[\Omega]$, not on the forms representing them.

5 The conformal areas for twistor holomorphic surfaces.

We begin by recalling the conformal area for a surface in $S^m(=Q^m(1))$ which is introduced in [9]. Although the notion of the conformal volume is defined more generally, we only explain it for our setting. Let M be an oriented surface and $f : M \rightarrow S^m$ a conformal immersion. The group of conformal diffeomorphisms of S^m is denoted by G . We say that the conformal immersions f_1 and f_2 are equivalent if there exists $\rho \in G$ such that $f_1 = \rho \circ f_2$. The equivalent class of a conformal immersion f is denoted by $[f]$. The conformal area $CA(m, [f])$ of $[f]$ is defined by

$$CA(m, [f]) := \sup_{\rho \in G} \int_M dv_{(\rho \circ f)^* g_0},$$

where g_0 is the standard metric of S^m . By definition, it is easy to see that the conformal area depends on the equivalent class. Using Corollary 2, we have the following theorem.

Theorem 1 *Let M be a compact twistor holomorphic surface in S^{2n} immersed by f . We have*

$$CA(2n, [f]) + \frac{1}{8} \int_M \|\bar{\nabla}^\perp I^\perp\|^2 dv_g \leq 2\pi \int_M c_1(TM) + 2\pi \int_M c_1(T^\perp M). \quad (9)$$

If there exists $\rho \in G$ such that $\rho \circ f$ is minimal, then the equality holds.

Proof By Corollary 2, we have

$$A(f) + \int_M \|H_f\|^2 dv_g + \frac{1}{8} \int_M \|\bar{\nabla}^\perp I^\perp\|^2 dv_g = 2\pi \int_M c_1(TM) + 2\pi \int_M c_1(T^\perp M),$$

where $A(f)$ is the area of $(M, g(= f^*g_0))$ and H_f is the mean curvature vector field of f . For any $h \in G$, it holds that

$$\begin{aligned} A(h \circ f) + \int_M \|H_{h \circ f}\|^2 dv_{(h \circ f)^*g_0} + \frac{1}{8} \int_M \|\bar{\nabla}^\perp I^\perp\|^2 dv_{(h \circ f)^*g_0} \\ = 2\pi \int_M c_1(TM) + 2\pi \int_M c_1(T^\perp M), \end{aligned}$$

because the property that M is twistor holomorphic is invariant under conformal transformations. By Lemma 8, we see that

$$A(h \circ f) + \frac{1}{8} \int_M \|\bar{\nabla}^\perp I^\perp\|^2 dv_g \leq 2\pi \int_M c_1(TM) + 2\pi \int_M c_1(T^\perp M)$$

for any $h \in G$. Hence, the desired inequality holds. Next we assume that there exists $\rho \in G$ such that $\rho \circ f$ is minimal. Then we have

$$A(\rho \circ f) + \frac{1}{8} \int_M \|\bar{\nabla}^\perp I^\perp\|^2 dv_g = 2\pi \int_M c_1(TM) + 2\pi \int_M c_1(T^\perp M).$$

By the definition of the conformal area, we see that the equality holds. $\square$

Note that each term of the left hand side of (9) is a conformal invariant. Consider the Clifford torus in S^3 minimally immersed by f . Let $\iota : S^3 \rightarrow S^4$ be the totally geodesic embedding. Then $\iota \circ f$ is *not* twistor holomorphic. On the other hand, we have

$$2\pi \int_M c_1(TM) + 2\pi \int_M c_1(T^\perp M) = 0.$$

Therefore the assumption that a surface is twistor holomorphic is needed to obtain (9).

The conformal area satisfies $CA(m, [f]) \geq 4\pi k \geq 4\pi$, where k is the maximum multiplicity of f (see [9]). Note that the maximum multiplicity of f is defined by

$$\max_{p \in S^{2n}} \#\{x \in M \mid f(x) = p \in \tilde{M}\}.$$

Corollary 3 *Let M be a compact twistor holomorphic surface in $Q^{2n}(c)$ immersed by f with genus q and maximum multiplicity k . Then we have*

$$2(k + q - 1) \leq \int_M c_1(T^\perp M).$$

In particular, we have

$$2q \leq \int_M c_1(T^\perp M). \quad (10)$$

If the equality holds in (10), then f is an embedding.

Proof Because each term in the inequality is invariant under conformal transformations, we may assume that f is a twistor holomorphic immersion into S^{2n} . The conformal area satisfies $CA(2n, [f]) \geq 4\pi k$. From Theorem 1, it follows that

$$4\pi k \leq CA(2n, [f]) \leq 4\pi(1 - q) + 2\pi \int_M c_1(T^\perp M).$$

If the equality

$$2q = \int_M c_1(T^\perp M)$$

holds, then $k = 1$, so f is an embedding. $\square$

Remark 3 It is easy to see that

$$4\pi \leq CA(2n, [f]) \leq 4\pi + 2\pi \int_M c_1(T^\perp M)$$

for any twistor holomorphic immersion $f : M \rightarrow S^{2n}$.

Lemma 16 *Let M be a compact twistor holomorphic surface in $Q^{2n}(c)$ immersed by f with $\int_M c_1(T^\perp M) = 0$. Then M is totally umbilic.*

Proof If M is a compact twistor holomorphic surface with $\int_M c_1(T^\perp M) = 0$, then $q = 0$. Using Theorem 1, we see

$$\begin{aligned} 4\pi &\leq CA(2n, [f]) \leq CA(2n, [f]) + \frac{1}{8} \int_M \|\bar{\nabla}^\perp I^\perp\|^2 dv_g \\ &\leq 2\pi \int_M c_1(TM) + 2\pi \int_M c_1(T^\perp M) = 4\pi, \end{aligned}$$

and hence, $\bar{\nabla}^\perp I^\perp = 0$. By Lemmas 13 and 15, we have

$$\int_M (c - K + \|H\|^2) dv_g = 0,$$

where K is the Gaussian curvature of M . On the other hand, by the Gauss equation, we have $2c = 2K + \|\alpha\|^2 - 4\|H\|^2$, so $2(c - K + \|H\|^2) = \|\alpha\|^2 - 2\|H\|^2$. Therefore we obtain

$$0 = \int_M (c - K + \|H\|^2) dv_g = \int_M (\|\alpha\|^2 - 2\|H\|^2) dv_g = \int_M \|\gamma\|^2 dv_g,$$

where γ is defined by $\gamma(X, Y) = \alpha(X, Y) - g(X, Y)H$. Hence we see that $\gamma = 0$, that is, M is totally umbilic. $\square$

In Corollary 3 in [4], it is shown that the Euler number of any compact twistor holomorphic surface in $Q^4(c)$ must be positive except the case the surface is totally umbilic (in [4], the complex structure on the normal bundle is the opposite of that used here). By Corollary 3 and Lemma 16, we can summarize the following corollaries as a generalization of this result.

Corollary 4 *Let M be a compact twistor holomorphic surface in $Q^{2n}(c)$ immersed by f . Then we have*

$$0 \leq \int_M c_1(T^\perp M).$$

The equality holds if and only if M is totally umbilic.

In the case of a twistor holomorphic surface in $Q^4(c)$, we can improve Corollary 3 in [4].

Corollary 5 *Let M be a compact twistor holomorphic surface in $Q^4(c)$ immersed by f . Then we have*

$$2q \leq \int_M c_1(T^\perp M).$$

The equality holds if and only if f is totally umbilic.

Proof It is sufficient to show that f is umbilic if the equality holds. From the assumption, we see that f is an embedding. We may assume that f is an embedding into $\mathbb{R}^4 = Q^4(0)$. On the other hand, it holds that $2I_f = \int_M c_1(T^\perp M)$, where I_f is the self-intersection number of f . Because f is an embedding, we have $\int_M c_1(T^\perp M) = 0$. Then f is totally umbilic. $\square$

Remark 4 Let L be a holomorphic line bundle over a compact Riemann surface with genus q . It is well-known that if

$$\int_M c_1(L) > 2q,$$

then L is very ample. Hence, the normal bundle of a twistor holomorphic immersion f into $Q^4(c)$ that is not totally umbilic is very ample with respect to the Koszul-Malgrange holomorphic structure described by $\nabla^\perp$ by Corollary 5. For example, the normal bundle of the Veronese immersion into S^4 , which is a superminimal immersion, satisfies these conditions. Examples of superminimal surfaces of higher genus can be found in [4].

Finally, we can also obtain the following corollary, which is essentially proved in [2].

Corollary 6 *Let M be a compact superminimal surface in S^{2n} immersed by f . Then the area of M is an integer multiple of 2π .*

Proof Because M is superminimal, we have $\bar{\nabla}^\perp I^\perp = 0$ by Proposition 1, and the equality in (9) holds. For a minimal surface, the conformal area coincides with the usual area (see [9]). Thus we have

$$(\text{the area of } M) = CA(2n, [f]) = 2\pi \int_M c_1(TM) + 2\pi \int_M c_1(T^\perp M).$$

□

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