Optimal embeddings of critical Sobolev-Lorentz-Zygmund spaces

OPTIMAL EMBEDDINGS OF CRITICAL SOBOLEV-LORENTZ-ZYGMUND SPACES

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Abstract

In this paper, we establish the embedding on the critical Sobolev-Lorentz-Zygmund space $H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$ into the generalized Morrey space $\mathcal{M}_{\Phi,r}(\mathbb{R}^n)$ with an optimal Young function Φ. Furthermore, as an application of this embedding, we obtain the almost Lipschitz continuity for functions in $H^{\frac{n}{p}+1}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$. O'Neil's inequality and its reverse play an essential role for the proof of main theorems.

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1 Introduction and main theorems

In this paper, we consider the optimal embedding on the critical Sobolev-Lorentz-Zygmund space $H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$ into the generalized Morrey space $\mathcal{M}_{\Phi,r}(\mathbb{R}^n)$, where $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \leq \infty$, $1 \leq r < \infty$ and $\lambda_1, \cdots, \lambda_m$ are non-negative numbers with $m \in \mathbb{N}$, and Φ is a Young function. One of main purposes is to investigate the optimal Young function Φ with which the embedding $H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_{\Phi,r}(\mathbb{R}^n)$ holds. The Sobolev-Lorentz-Zygmund space $H^s_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$, $s \in \mathbb{R}$, is defined as a Bessel potential space $H^s_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n) := (1-\Delta)^{-\frac{s}{2}} L_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$ in terms of the Lorentz-Zygmund space $L_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$. The space $H^s_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$ extends the Sobolev-Lorentz space and the Sobolev space since $L_{p,q,0,\cdots,0}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$ and $L_{p,p}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, where $L_p(\mathbb{R}^n)$ and $L_{p,q}(\mathbb{R}^n)$ denote the Lebesgue space and the Lorentz space, respectively. We give definitions of those function spaces and related properties in Section 2.

We concern the optimal vanishing and growth orders of the local integrals $\int_E |u(x)|^r dx$ as $|E| \to 0$ or $|E| \to \infty$ for functions in $H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$. In Suzuki-Wadade [21], the authors gave the optimal growth order of the local integrals for functions in $H^{\frac{n}{p}}_{p,q}(\mathbb{R}^n)$ stated as follows:

Theorem A [21]. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \le \infty$ and $1 \le r < \infty$. Then there exists a positive constant C such that the inequality*

$$\left(\int_E |u(x)|^r dx \right)^{\frac{1}{r}} \le C\,|E|^{\frac{1}{r}-\frac{1}{p}} \|u\|_{H^{\frac{n}{p}}_{p,q}} \tag{1.1}$$

holds for all $u \in H^{\frac{n}{p}}_{p,q}(\mathbb{R}^n)$ and all measurable sets E if and only if $p > r$ or $p = r \ge q$, where C is independent of E.

In Theorem A, the necessity for the condition $p > r$ or $p = r \ge q$ comes from the part $|E| \to \infty$ in (1.1). In fact, the vanishing order $|E|^{\frac{1}{r}-\frac{1}{p}}$ as $|E| \to 0$ turns out not to be optimal, and in [21], the authors also proved the following:

Theorem B [21]. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \le \infty$ and $1 \le r < \infty$. Then there exist positive constants δ and C such that the inequality*

$$\left(\int_E |u(x)|^r dx \right)^{\frac{1}{r}} \le C\,|E|^{\frac{1}{r}} \log(\frac{1}{|E|})^{\frac{1}{q'}}\, \|u\|_{H^{\frac{n}{p}}_{p,q}}$$

holds for all $u \in H^{\frac{n}{p}}_{p,q}(\mathbb{R}^n)$ and all measurable sets E satisfying $|E| < \delta$, where C and δ are independent of E, and $q' := \frac{q}{q-1}$.

Theorem B [21] was originally obtained by Brézis-Wainer [3] when $p = q$ which corresponds to the critical Sobolev space $H^{\frac{n}{p}}_p(\mathbb{R}^n)$. Ozawa [16] gave an alternative proof of Theorem B [21] when $p = q$. We also refer to Sawano-Wadade [19], where the authors proved similar embeddings on the critical Sobolev-Morrey space.

Our first goal in this paper is to extend both of Theorem A and Theorem B for functions in $H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$. Concerning an extension of Theorem A, we can show that the inequality (1.1) with $\|u\|_{H^{\frac{n}{p}}_{p,q}}$ replaced by $\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}}$ holds if and only if $p > r$ or $p = r \ge q$ without any modification for the proof of Theorem A in [21]. Therefore, we omit its proof in this paper. However, the vanishing order as $|E| \to 0$ depends on the exponents $\lambda_1, \cdots, \lambda_m$ when we consider an extension of Theorem B to the statement in terms of $H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$. In order to state main theorems, we introduce multiple-logarithmic functions by $\ell_l(t) := \underbrace{\ell_1 \circ \cdots \circ \ell_1}_{l}(t)$ for $t \ge c_l$ with $\ell_1(t) := \log t$, and the constants $c_l > 0$ are determined by $\ell_l(c_l) = 1$. Our first result now reads:

Theorem 1.1. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \le \infty$, $1 \le r < \infty$ and let $\lambda_1, \cdots, \lambda_m$ be non-negative numbers with $m \in \mathbb{N}$. Assume one of the conditions (A)-(C):*

$$\begin{cases} \text{(A) There exists } 0 \le j \le m-1 \text{ such that } \lambda_1 = \cdots = \lambda_j = \frac{1}{q'} \text{ and } \lambda_{j+1} > \frac{1}{q'}\,; \\ \text{(B) There exists } 0 \le j \le m-1 \text{ such that } \lambda_1 = \cdots = \lambda_j = \frac{1}{q'} \text{ and } \lambda_{j+1} < \frac{1}{q'}\,; \\ \text{(C) } \lambda_1 = \cdots = \lambda_m = \frac{1}{q'}, \end{cases}$$

where the conditions (A) and (B) are understood as $\lambda_1 > \frac{1}{q'}$ *and* $\lambda_1 < \frac{1}{q'}$ *when* $j = 0$*, respectively. Then there exist positive constants* C *and* δ *such that the inequalities*

$$\left(\int_E |u(x)|^r dx\right)^{\frac{1}{r}} \le \begin{cases} C\,|E|^{\frac{1}{r}}\,\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}} & \text{if (A) is fulfilled;} \\ C\,|E|^{\frac{1}{r}}\,\ell_{j+1}(\frac{1}{|E|})^{\frac{1}{q'}-\lambda_{j+1}} \displaystyle\prod_{l=j+2}^{m} \ell_l(\frac{1}{|E|})^{-\lambda_l}\,\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}} & \text{if (B) is fulfilled;} \\ C\,|E|^{\frac{1}{r}}\,\ell_{m+1}(\frac{1}{|E|})^{\frac{1}{q'}}\,\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}} & \text{if (C) is fulfilled,} \end{cases} \tag{1.2}$$

hold for all $u \in H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$ *and for all measurable sets* E *satisfying* $|E| < \delta$*, where the constants* C *and* δ *are independent of* E*, and in the middle inequality in* (1.2)*, the right-hand side of* (1.2) *is understood as* $C\,|E|^{\frac{1}{r}}\,\ell_m(\frac{1}{|E|})^{\frac{1}{q'}-\lambda_m}\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}}$ *when* $j = m-1$.

As a special case of $m = 1$ in Theorem 1.1, we obtain the following corollary:

Corollary 1.2. *Let* $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \le \infty$, $1 \le r < \infty$ *and* $\lambda \ge 0$*. Then there exist positive constants* C *and* δ *such that the inequalities*

$$\left(\int_E |u(x)|^r dx\right)^{\frac{1}{r}} \le \begin{cases} C\,|E|^{\frac{1}{r}}\,\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda}} & \text{if } \lambda > \frac{1}{q'}\,; \\ C\,|E|^{\frac{1}{r}}\,\log(\frac{1}{|E|})^{\frac{1}{q'}-\lambda}\,\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda}} & \text{if } \lambda < \frac{1}{q'}\,; \\ C\,|E|^{\frac{1}{r}}\,\log\left(\log(\frac{1}{|E|})\right)^{\frac{1}{q'}}\,\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda}} & \text{if } \lambda = \frac{1}{q'}, \end{cases} \tag{1.3}$$

hold for all $u \in H^{\frac{n}{p}}_{p,q,\lambda}(\mathbb{R}^n)$ *and for all measurable sets* E *satisfying* $|E| < \delta$*, where the constants* C *and* δ *are independent of* E.

Remark that Theorem B is corresponding to the middle inequality in (1.3) with $\lambda = 0$ in Corollary 1.2. Furthermore, Corollary 1.2 tells us that the exponent $\lambda = \frac{1}{q'}$ is a threshold so that the logarithmic vanishing order as $|E| \to 0$ appears for the local integrals of functions in $H^{\frac{n}{p}}_{p,q,\lambda}(\mathbb{R}^n)$.

Theorem 1.1 is regarded as the embedding on $H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$ into the generalized Morrey space. The generalized Morrey spaces have been studied extensively, see for instance Kurata-Nishigaki-Sugano [8], Nakai [11, 12] and Sawano-Sugano-Tanaka [17, 18]. Let Φ be a Young function, that is, $\Phi : [0,\infty) \to [0,\infty)$ is a continuous function satisfying $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. Then for a locally integrable function u on $\mathbb{R}^n$, the norm of the generalized Morrey space

$\mathcal{M}_{\Phi,r}(\mathbb{R}^n)$ is given by

$$\|u\|_{\mathcal{M}_{\Phi,r}} := \sup_{Q\in\mathcal{D}(\mathbb{R}^n)} \Phi(|Q|) \left(\frac{1}{|Q|}\int_Q |u(x)|^r dx\right)^{\frac{1}{r}},$$

where $\mathcal{D}(\mathbb{R}^n)$ denotes the set of dyadic cubes in $\mathbb{R}^n$. The space $\mathcal{M}_{\Phi,r}(\mathbb{R}^n)$ extends the Morrey space and then the Lebesgue space. As an immediate consequnce of Theorem 1.1 and Theorem A with $\|u\|_{H^{\frac{n}{p}}_{p,q}}$ replaced by $\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}}$, we obtain the following embeddings:

Corollary 1.3. *Let $n\in\mathbb{N}$, $1<p<\infty$, $1<q\le\infty$, $1\le r<\infty$ and let $\lambda_1,\cdots,\lambda_m$ be non-negative numbers with $m\in\mathbb{N}$. Define Young functions Φ by*

$$\Phi(t) := \begin{cases} (1+t)^{\frac{1}{p}} & \text{if (A) is fulfilled;} \\ (1+t)^{\frac{1}{p}}\, \ell_{j+1}(c_{j+1}+\frac{1}{t})^{\lambda_{j+1}-\frac{1}{q'}} \displaystyle\prod_{l=j+2}^{m} \ell_l(c_l+\frac{1}{t})^{\lambda_l} & \text{if (B) is fulfilled;} \\ (1+t)^{\frac{1}{p}}\, \ell_{m+1}(c_{m+1}+\frac{1}{t})^{-\frac{1}{q'}} & \text{if (C) is fulfilled.} \end{cases} \tag{1.4}$$

Then the continuous embedding $H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_{\Phi,r}(\mathbb{R}^n)$ holds if and only if $p>r$ or $p=r\ge q$.

As another application of Theorem 1.1, we investigate the Lipschitz type continuity for functions in $H^{\frac{n}{p}+1}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$. It is well-known that $H^{\frac{n}{p}+\alpha}_{p}(\mathbb{R}^n)\hookrightarrow C^\alpha(\mathbb{R}^n)$ for $0<\alpha<1$ but $H^{\frac{n}{p}+1}_{p}(\mathbb{R}^n)\not\hookrightarrow Lip\,(\mathbb{R}^n)$, where $C^\alpha(\mathbb{R}^n)$ and $Lip\,(\mathbb{R}^n)$ denote the Hölder space and the Lipschitz space, respectively. Instead, the functions in $H^{\frac{n}{p}+1}_{p}(\mathbb{R}^n)$ admit the almost Lipschitz continuity, see Bŕezis-Wainger [3]. Based on this fact, we next aim to clarify how the exponents $\lambda_1,\cdots,\lambda_m$ influence the Lipschitz type continuity for functions in $H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$. Our second theorem reads as follows:

Theorem 1.4. *Let $n\in\mathbb{N}$, $1<p<\infty$, $1<q\le\infty$, and let $\lambda_1,\cdots,\lambda_m$ be non-negative numbers with $m\in\mathbb{N}$. Assume one of the conditions (A)-(C) in Theorem 1.1. Then there exist positive constants C and δ such that the inequalities*

$$|u(x)-u(y)| \le \begin{cases} C\,|x-y|\,\|u\|_{H^{\frac{n}{p}+1}_{p,q,\lambda_1,\cdots,\lambda_m}} & \text{if (A) is fulfilled;} \\ C\,|x-y|\,\ell_{j+1}(\frac{1}{|x-y|})^{\frac{1}{q'}-\lambda_{j+1}} \displaystyle\prod_{l=j+2}^{m} \ell_l(\frac{1}{|x-y|})^{-\lambda_l}\,\|u\|_{H^{\frac{n}{p}+1}_{p,q,\lambda_1,\cdots,\lambda_m}} & \text{if (B) is fulfilled;} \\ C\,|x-y|\,\ell_{m+1}(\frac{1}{|x-y|})^{\frac{1}{q'}}\,\|u\|_{H^{\frac{n}{p}+1}_{p,q,\lambda_1,\cdots,\lambda_m}} & \text{if (C) is fulfilled,} \end{cases}$$

hold for all $u\in H^{\frac{n}{p}+1}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$ and for all points x and y satisfying $|x-y|<\delta$, where the constants C and δ are independent of x and y.

The case $m = 1$ in Theorem 1.4 yields the following corollary :

Corollary 1.5. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \le \infty$ and $\lambda \ge 0$. Then there exist positive constants C and δ such that the inequalities*

$$|u(x) - u(y)| \le \begin{cases} C\,|x-y|\,\|u\|_{H^{\frac{n}{p}+1}_{p,q,\lambda}} & \text{if } \lambda > \frac{1}{q'}\,; \\ C\,|x-y|\,\log\left(\frac{1}{|x-y|}\right)^{\frac{1}{q'}-\lambda}\,\|u\|_{H^{\frac{n}{p}+1}_{p,q,\lambda}} & \text{if } \lambda < \frac{1}{q'}\,; \\ C\,|x-y|\,\log\left(\log(\frac{1}{|x-y|})\right)^{\frac{1}{q'}}\,\|u\|_{H^{\frac{n}{p}+1}_{p,q,\lambda}} & \text{if } \lambda = \frac{1}{q'}, \end{cases} \tag{1.5}$$

hold for all $u \in H^{\frac{n}{p}+1}_{p,q,\lambda}(\mathbb{R}^n)$ and for all points x and y satisfying $|x-y| < \delta$, where the constants C and δ are independent of x and y.

In [3], the middle inequality in (1.5) with $p = q$ and $\lambda = 0$ was proved. Moreover, Corollary 1.5 tells us that the exponent $\lambda = \frac{1}{q'}$ is a threshold so that $H^{\frac{n}{p}}_{p,q,\lambda}(\mathbb{R}^n)$ can be embedded into $Lip\,(\mathbb{R}^n)$.

Finally, we consider the optimality for the inequalities (1.2) in Theorem 1.1 with respect to the vanishing orders as $|E| \to 0$, which also implies the optimality for the Young functions (1.4) in Corollary 1.3. As a result, we can observe that the vanishing orders as $|E| \to 0$ are optimal in terms of the multiple logarithmic functions. Our final theorem is stated as follows :

Theorem 1.6. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \le \infty$, $1 < r < \infty$, and let $\lambda_1, \cdots, \lambda_m$ be non-negative numbers with $m \in \mathbb{N}$. Take $k \ge m$ with $k \in \mathbb{N}$ and $\varepsilon > 0$. Assume one of the conditions (A)-(C) in Theorem 1.1.*

(i) If $q < \infty$, then there exist $u \in H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$ and positive constants C and δ such that the inequalities

$$\begin{aligned} &\left(\int_E |u(x)|^r dx\right)^{\frac{1}{r}} \\ &\ge \begin{cases} C\,|E|^{\frac{1}{r}} & \text{if (A) is fulfilled;} \\ C\,|E|^{\frac{1}{r}}\,\ell_{j+1}(\frac{1}{|E|}) \displaystyle\prod_{l=j+1}^{m} \ell_l(\frac{1}{|E|})^{-\lambda_l} \prod_{l=j+1}^{k-1} \ell_l(\frac{1}{|E|})^{-\frac{1}{q}}\,\ell_k(\frac{1}{|E|})^{-\frac{1}{q}-\varepsilon} & \text{if (B) is fulfilled;} \\ C\,|E|^{\frac{1}{r}}\,\ell_{m+1}(\frac{1}{|E|}) \displaystyle\prod_{l=m+1}^{k} \ell_l(\frac{1}{|E|})^{-\frac{1}{q}}\,\ell_{k+1}(\frac{1}{|E|})^{-\frac{1}{q}-\varepsilon} & \text{if (C) is fulfilled,} \end{cases} \end{aligned} \tag{1.6}$$

hold for all measurable sets E satisfying $|E| < \delta$, where u, C and δ are independent of E.

(ii) If $q = \infty$, then there exist $u \in H^{\frac{n}{p}}_{p,\infty,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$ and positive constants C and δ such that

the inequalities

$$\left(\int_E |u(x)|^r dx\right)^{\frac{1}{r}} \geq \begin{cases} C\,|E|^{\frac{1}{r}} \quad \textit{if (A) is fulfilled;} \\ C\,|E|^{\frac{1}{r}}\,\ell_{j+1}(\frac{1}{|E|}) \displaystyle\prod_{l=j+1}^{m} \ell_l(\frac{1}{|E|})^{-\lambda_l} \quad \textit{if (B) is fulfilled;} \\ C\,|E|^{\frac{1}{r}}\,\ell_{m+1}(\frac{1}{|E|}) \quad \textit{if (C) is fulfilled,} \end{cases}$$

hold for all measurable sets E satisfying $|E| < \delta$, where u, C and δ are independent of E.

Theorem 1.6 implies that the vanishing orders as $|E| \to 0$ for the inequalities (1.2) in Theorem 1.1 are best-possible when $q = \infty$ and they are also sharp even when $q < \infty$ in terms of the multiple logarithmic functions. It is worth noting that the last two inequalities in (1.6) become sharper as $k \in \mathbb{N}$ is getting larger.

The inequalities characterizing critical function spaces in terms of Sobolev's embedding such as Sobolev-Lorentz spaces, Sobolev-Morrey spaces, Besov spaces, Triebel-Lizorkin spaces and functions of bounded mean oscillation called BMO have been extensively studied, see for instance Brézis-Wainger [3], Chen-Zhu [4], Edmunds-Triebel [5], Machihara-Ozawa-Wadade [9], Nagayasu-Wadade [10], Ogawa [14], Ogawa-Ozawa [15], Ozawa [16], Sawano-Wadade [19], Wadade [22, 23, 24] and so on. In those papers, the authors established critical embeddings by proving Trudinger-Moser type inequalities, Gagliardo-Nirenberg type inequalities, Brézis-Gallouët-Wainger type inequalities and the logarithmic Hardy inequalities. Our main subject in this paper is concerned with the optimal embedding from the critical Sobolev-Lorentz-Zygmund space into the generalized Morrey space, which is regarded as one of the characterization for the critical Sobolev-Lorentz-Zygmund space. However, as far as we know, this kind of embeddings discussed in this paper is little known compared to the embeddings related to the corresponding Trudinger-Moser type inequalities and so on. We will discuss the relations between those critical embeddings in the forthcoming paper.

This paper is organized as follows. Section 2 is devoted to give the definition of the Sobolev-Lorentz-Zygmund space and to collect the elementary properties concerning the rearrangement of funtions. We shall prove main theorems in Section 3.

2 Preliminaries

In this section, we first recall the definition of the Lorentz-Zygmund space. To this end, we define the rearrangement of measurable functions. For a measurable function f on $\mathbb{R}^n$ with $n \in \mathbb{N}$, let $f_* : [0, \infty) \to [0, \infty]$ be the distribution function of f defined by

$$f_*(\lambda) := |\{x \in \mathbb{R}^n\,;\, |f(x)| > \lambda\}| \quad \text{for} \;\; \lambda \geq 0,$$

where $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$, and then the rearrangement $f^* : [0, \infty) \to [0, \infty]$ of f is defined by

$$f^*(t) := \inf\{\lambda > 0\,;\, f_*(\lambda) \leq t\} \quad \text{for} \;\; t \geq 0.$$

Moreover, $f^{**} : (0,\infty) \to [0,\infty]$ denotes the average function of f^* defined by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(\tau)d\tau \quad \text{for } t > 0.$$

In what follows, we assume $f^*(t) < +\infty$ for all $t > 0$. Then f^* is right-continuous and non-increasing on $(0,\infty)$, and hence, f^{**} is continuous and non-increasing on $(0,\infty)$ with $f^*(t) \le f^{**}(t)$ for all $t > 0$. We now introduce the Lorentz-Zygmund space by using the rearrangement. Let $1 \le p, q \le \infty$, and let $\lambda_1, \cdots, \lambda_m$ be non-negative numbers with $m \in \mathbb{N}$. Then the Lorentz-Zygmund space $L_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$ is a function space equipped with the norm given by

$$\|f\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}} := \left(\int_0^\infty \left(t^{\frac{1}{p}} \prod_{l=1}^m \ell_l(c_l + \frac{1}{t})^{\lambda_l} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}},$$

where $\ell_l(t) := \underbrace{\ell_1 \circ \cdots \circ \ell_1}_{l}(t)$ for $t \ge c_l$ with $\ell_1(t) := \log t$, and the constants $c_l > 0$ are determined by $\ell_l(c_l) = 1$. When $q = \infty$, the norm $\|f\|_{L_{p,\infty,\lambda_1,\cdots,\lambda_m}}$ can be defined by the usual modification. Remark that the Lorentz-Zygmund space generalizes the Lorentz space $L_{p,q}(\mathbb{R}^n)$ since $\|f\|_{L_{p,q,0,\cdots,0}} = \|f\|_{L_{p,q}}$.

We can take f^* replaced by f^{**} in $\|f\|_{L_{p,q}}$ as another equivalent norm on $L_{p,q}(\mathbb{R}^n)$ if $p \ne 1$. Indeed, the following Hardy inequality guarantees its equivalence,

$$\left(\int_0^\infty \left(\frac{t^{\frac{1}{p}}}{t}\int_0^t f(s)ds\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \le p' \left(\int_0^\infty \left(t^{\frac{1}{p}} f(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \tag{2.1}$$

for non-negative measurable functions f, where $p' := \frac{p}{p-1}$. For the proof of (2.1), see O'Neil [13, Lemma 2.3] and references therein. Furthermore, since f^* and f^{**} are non-increasing functions in $(0,\infty)$, we get the following decay estimates. For any $t > 0$, we have

$$f^*(t) \le \left(\frac{q}{p}\right)^{\frac{1}{q}} t^{-\frac{1}{p}} \|f\|_{L_{p,q}}, \tag{2.2}$$

and then if $p > 1$, together with (2.1), we obtain for any $t > 0$,

$$f^{**}(t) \le p' \left(\frac{q}{p}\right)^{\frac{1}{q}} t^{-\frac{1}{p}} \|f\|_{L_{p,q}}.$$

Next, we recall the pointwise rearrangement inequality for the convolution of functions proved by O'Neil [13, Theorem 1.7]. In fact, for measurable functions f and g on $\mathbb{R}^n$, we have

$$(f * g)^{**}(t) \le t\, f^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)ds \quad \text{for } t > 0. \tag{2.3}$$

Moreover, we make use of the reverse O'Neil inequality established in Kozono-Sato-Wadade [7, Lemma 2.2]. In fact, there exists a positive constant C such that the inequality

$$(f * g)^{**}(t) \ge C\left(t\, f^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)ds\right) \tag{2.4}$$

holds for all $t>0$ and for all measurable functions f and g on $\mathbb{R}^n$ which are both non-negative, radially symmetric and non-increasing in the radial direction $|x|$.

In this paper, we frequently use the Bessel potential $G_s * f := (1-\Delta)^{-\frac{s}{2}} f$ and the Riesz potential $I_s * f := (-\Delta)^{-\frac{s}{2}} f$ for $0<s<n$. More precisely, the kernel functions I_s and G_s are defined respectively by

$$\begin{cases} I_s(x) := \dfrac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} |x|^{-(n-s)} ; \\ G_s(x) := \dfrac{1}{(4\pi)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)} \displaystyle\int_0^\infty e^{-\pi \frac{|x|^2}{t} - \frac{t}{4\pi}} t^{-\frac{n-s}{2}} \frac{dt}{t} \end{cases}$$

for $x \in \mathbb{R}^n \setminus \{0\}$, where Γ denotes the Gamma function. Based on the Lorentz-Zygmund space, we define the Sobolev-Lorentz-Zygmund space $H^s_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$ by $H^s_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n) := (I-\Delta)^{-\frac{s}{2}} L_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n) = G_s * L_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$ equipped with the norm $\|u\|_{H^s_{p,q,\lambda_1,\cdots,\lambda_m}} := \|(I-\Delta)^{\frac{s}{2}} u\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}}$. The space $H^s_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$ extends the Sobolev-Lorentz space $H^s_{p,q}(\mathbb{R}^n)$ and then the Sobolev space $H^s_p(\mathbb{R}^n)$ since we have $L_{p,q,0,\cdots,0}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$ and $L_{p,p}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. We now collect the elementary properties of I_s and G_s in the following lemma.

Lemma 2.1. *Let $n \in \mathbb{N}$ and $0<s<n$.*

(i) I_s and G_s are non-negative, radially symmetric and non-increasing in the radial direction, so that $I_s^*(t) = I_s(x)$ and $G_s^*(t) = G_s(x)$ if $|x| = \left(\frac{t}{\omega_n}\right)^{\frac{1}{n}} > 0$, where $\omega_n := \frac{2\pi^{\frac{\pi}{2}}}{n\Gamma(\frac{n}{2})}$ denotes the volume of the unit ball in $\mathbb{R}^n$.

(ii) $G_s(x) \le I_s(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$, which implies $G_s^*(t) \le I_s^*(t)$, $G_s^{**}(t) \le I_s^{**}(t)$ for all $t>0$, and $\displaystyle\lim_{|x|\downarrow 0} \frac{G_s(x)}{I_s(x)} = \lim_{t\downarrow 0} \frac{G_s^*(t)}{I_s^*(t)} = 1$.

(iii) $\|G_s\|_{L_1(\mathbb{R}^n)} = 1$ and there exists a positive constant C such that the following inequalities hold,

$$G_s(x) \le \begin{cases} C\,|x|^{-(n-s)} & \text{for } x \in \mathbb{R}^n \setminus \{0\} ; \\ C\,e^{-|x|} & \text{for } x \in \mathbb{R}^n \quad \text{with } |x| \ge 1. \end{cases}$$

Since the facts in Lemma 2.1 are well-known, we omit the detailed proof here, see Stein [20] for instance. Furthermore, we refer to Almgren-Lieb [1], Bennett-Sharpley [2] and Kokilashvili-Krbec [6] for further information about the rearrangement theory.

3 Proof of main theorems

In this section, we shall prove main theorems.

Proof of Theorem 1.1. First, letting $(1-\Delta)^{\frac{n}{2p}} u = f$, we have $u = G_{\frac{n}{p}} * f$, where $G_{\frac{n}{p}}$ denotes the Bessel kernel. Thus the inequality (1.2) can be written equivalently as

$$\left(\int_E |G_{\frac{n}{p}} * f(x)|^r dx \right)^{\frac{1}{r}}$$

$$\leq \begin{cases} C\,|E|^{\frac{1}{r}}\,\|f\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}} & \text{if (A) is fulfilled}\,; \\ C\,|E|^{\frac{1}{r}}\,\ell_{j+1}(\frac{1}{|E|})^{\frac{1}{q'}-\lambda_{j+1}}\displaystyle\prod_{l=j+2}^{m}\ell_l(\frac{1}{|E|})^{-\lambda_l}\,\|f\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}} & \text{if (B) is fulfilled}\,; \\ C\,|E|^{\frac{1}{r}}\,\ell_{m+1}(\frac{1}{|E|})^{\frac{1}{q'}}\,\|f\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}} & \text{if (C) is fulfilled,} \end{cases}$$

for all $f \in L_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$ and all measurable sets E having small measure.

By O'Neil's inequality (2.3), we obtain

$$\left(\int_E |G_{\frac{n}{p}} * f(x)|^r dx\right)^{\frac{1}{r}} = \left(\int_0^{|E|} (G_{\frac{n}{p}} * f)^*(t)^r dt\right)^{\frac{1}{r}} \leq \left(\int_0^{|E|} \left(t\,G^{**}_{\frac{n}{p}}(t)f^{**}(t)\right)^r dt\right)^{\frac{1}{r}}$$
$$+\left(\int_0^{|E|} \left(\int_t^{\infty} G^*_{\frac{n}{p}}(s)f^*(s)ds\right)^r dt\right)^{\frac{1}{r}} \leq \left(\int_0^{|E|} \left(t\,G^{**}_{\frac{n}{p}}(t)f^{**}(t)\right)^r dt\right)^{\frac{1}{r}}$$
$$+\left(\int_0^{|E|} \left(\int_t^{|E|} G^*_{\frac{n}{p}}(s)f^*(s)ds\right)^r dt\right)^{\frac{1}{r}} + \left(\int_0^{|E|} \left(\int_{|E|}^{\infty} G^*_{\frac{n}{p}}(s)f^*(s)ds\right)^r dt\right)^{\frac{1}{r}} =: I_1 + I_2 + I_3.$$

We first estimate I_1. For small $t > 0$, by the decay estimate (2.2) and Lemma 2.1, we see

$$t\,G^{**}_{\frac{n}{p}}(t)f^{**}(t) = \frac{1}{t}\int_0^t G^*_{\frac{n}{p}}(s)ds\int_0^t f^*(s)ds \leq \frac{C}{t}\int_0^t s^{-\frac{1}{p'}}ds\int_0^t s^{-\frac{1}{p}}ds\,\|f\|_{L_{p,q}}$$
$$= C\,\|f\|_{L_{p,q}} \leq C\,\|f\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}},$$

and then $I_1 \leq C\,|E|^{\frac{1}{r}}\|f\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}}$.

Next, we proceed to the estimate of I_2. By using (2.2) and Lemma 2.1, we have

$$I_2 \leq C\left(\int_0^{|E|}\left(\int_t^{|E|} s^{-\frac{1}{p'}-\frac{1}{p}}ds\right)^r dt\right)^{\frac{1}{r}}\|f\|_{L_{p,q}} \leq C\left(\int_0^{|E|}\left(\log\frac{|E|}{t}\right)^r dt\right)^{\frac{1}{r}}\|f\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}}$$
$$= C\left(\int_0^1\left(\log\frac{1}{s}\right)^r ds\right)^{\frac{1}{r}}|E|^{\frac{1}{r}}\|f\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}} = C\,|E|^{\frac{1}{r}}\|f\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}}.$$

Finally, we estimate I_3. For small $\delta > 0$, we have

$$I_3 = |E|^{\frac{1}{r}}\int_{|E|}^{\infty} G^*_{\frac{n}{p}}(s)f^*(s)ds = |E|^{\frac{1}{r}}\int_{|E|}^{\delta} G^*_{\frac{n}{p}}(s)f^*(s)ds + |E|^{\frac{1}{r}}\int_{\delta}^{\infty} G^*_{\frac{n}{p}}(s)f^*(s)ds =: I_{31} + I_{32}.$$

We can estimate I_{32} as follows. By using (2.2) and Lemma 2.1 again, we see for any $\alpha > \frac{1}{p'}$,

$$I_{32} \leq C\,|E|^{\frac{1}{r}}\int_{\delta}^{\infty} s^{-\alpha-\frac{1}{p}}ds\,\|f\|_{L_{p,q}} \leq C\,|E|^{\frac{1}{r}}\|f\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}}.$$

Furthermore, by Lemma 2.1 and Hölder's inequality, I_{31} is estimated as

$$
\begin{aligned}
I_{31} &\leq C\,|E|^{\frac{1}{r}} \int_{|E|}^{\delta} s^{-\frac{1}{p'}-\frac{1}{p}} \prod_{l=1}^{m} \ell_l(\frac{1}{s})^{-\lambda_l} s^{\frac{1}{p}} \prod_{l=1}^{m} \ell_l(\frac{1}{s})^{\lambda_l} f^*(s)ds \\
&\leq C\,|E|^{\frac{1}{r}} \left(\int_{|E|}^{\delta} \prod_{l=1}^{m} \ell_l(\frac{1}{s})^{-\lambda_l q'} \frac{ds}{s} \right)^{\frac{1}{q'}} \|f\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}}. \qquad (3.1)
\end{aligned}
$$

By applying L'Hopital's rule, we can investigate the growth orders as $|E| \to 0$ of the integral in the right-hand side of (3.1) under the conditions (A)-(C). As results, we obtain

$$
\left(\int_{|E|}^{\delta} \prod_{l=1}^{m} \ell_l(\frac{1}{s})^{-\lambda_l q'} \frac{ds}{s} \right)^{\frac{1}{q'}} \leq \begin{cases} C \quad \text{if (A) is fulfilled}\,; \\ C\,\ell_{j+1}(\frac{1}{|E|})^{\frac{1}{q'}-\lambda_{j+1}} \displaystyle\prod_{l=j+2}^{m} \ell_l(\frac{1}{|E|})^{-\lambda_l} \quad \text{if (B) is fulfilled}\,; \\ C\,\ell_{m+1}(\frac{1}{|E|})^{\frac{1}{q'}} \quad \text{if (C) is fulfilled,} \end{cases}
$$

and hence,

$$
I_{31} \leq \begin{cases} C\,|E|^{\frac{1}{r}}\,\|f\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}} \quad \text{if (A) is fulfilled}\,; \\ C\,|E|^{\frac{1}{r}}\,\ell_{j+1}(\frac{1}{|E|})^{\frac{1}{q'}-\lambda_{j+1}} \displaystyle\prod_{l=j+2}^{m} \ell_l(\frac{1}{|E|})^{-\lambda_l}\,\|f\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}} \quad \text{if (B) is fulfilled}\,; \\ C\,|E|^{\frac{1}{r}}\,\ell_{m+1}(\frac{1}{|E|})^{\frac{1}{q'}}\,\|f\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}} \quad \text{if (C) is fulfilled.} \end{cases}
$$

Thus summing up all estimates above, we obtain desired conclusions. □

Next, we give a proof of Corollary 1.3. It is an immediate consequence of Theorem 1.1 and Theorem A with $\|u\|_{H^{\frac{n}{p}}_{p,q}}$ replaced by $\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}}$.

Proof of Corollary 1.3. First, assume $p > r$ or $p = r \geq q$. Then by applying Theorem 1.1 and Theorem A with $\|u\|_{H^{\frac{n}{p}}_{p,q}}$ replaced by $\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}}$, we see for any measurable set E,

$$
\begin{aligned}
&\left(\int_E |u(x)|^r dx \right)^{\frac{1}{r}} \\
&\leq \begin{cases} C\,|E|^{\frac{1}{r}}(1+|E|)^{-\frac{1}{p}}\,\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}} \quad \text{if (A) is fulfilled}\,; \\ C\,|E|^{\frac{1}{r}}(1+|E|)^{-\frac{1}{p}}\,\ell_{j+1}(c_{j+1}+\frac{1}{|E|})^{\frac{1}{q'}-\lambda_{j+1}} \displaystyle\prod_{l=j+2}^{m} \ell_l(c_l+\frac{1}{|E|})^{-\lambda_l}\,\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}} \quad \text{if (B) is fulfilled}\,; \\ C\,|E|^{\frac{1}{r}}(1+|E|)^{-\frac{1}{p}}\,\ell_{m+1}(c_{m+1}+\frac{1}{|E|})^{\frac{1}{q'}}\,\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}} \quad \text{if (C) is fulfilled,} \end{cases}
\end{aligned}
\qquad (3.2)
$$

which imply the continuous embeddings $H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_{\Phi,r}(\mathbb{R}^n)$ with Young functions (1.4). Conversely, since the conditions $p > r$ or $p = r \geq q$ are necessary for Theorem A with $\|u\|_{H^{\frac{n}{p}}_{p,q}}$ replaced by $\|u\|_{H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}}$, they are necessary also for the inequalities (3.2). Thus we finish the proof of Corollary 1.3. $\square$

We proceed to the proof of Theorem 1.4, which will be proved by utilizing Theorem 1.1.

Proof of Theorem 1.4. We consider only the case of the condition (C) since other cases can be treated in a quite same way. Let x and y be distinct points in $\mathbb{R}^n$, and let Q be a closed cube in $\mathbb{R}^n$ with its side $\rho = |x-y|$ containing x and y. For any $z \in Q$, we have

$$u(z) - u(x) = \int_0^1 \nabla u\,(tz + (1-t)x) \cdot (z-x)\,dt,$$

and then

$$|u(z) - u(x)| \leq \sqrt{n}\,\rho \int_0^1 |\nabla u\,(tz + (1-t)x)|\,dt. \tag{3.3}$$

Defining $u_Q := \frac{1}{|Q|}\int_Q u(z)dz$ and integrating (3.3) with respect to z over Q, we obtain

$$\begin{aligned}|u_Q - u(x)| &\leq \frac{1}{|Q|}\int_Q |u(z)-u(x)|dz \leq \sqrt{n}\,\rho^{1-n}\int_0^1\int_Q |\nabla u\,(tz+(1-t)x)|\,dzdt \\ &= \sqrt{n}\,\rho^{1-n}\int_0^1 t^{-n}\int_{tQ+(1-t)x}|\nabla u(\zeta)|\,d\zeta dt. \end{aligned}\tag{3.4}$$

Here, applying Theorem 1.1 with $r = 1$, we have for any small $|Q|$,

$$\begin{aligned}\int_{tQ+(1-t)x}|\nabla u(\zeta)|\,d\zeta &\leq C\,|tQ|\,\ell_{m+1}(\frac{1}{|tQ|})^{\frac{1}{q'}}\,\|\nabla u\|_{H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}} \\ &\leq C\,t^n\rho^n\ell_{m+1}(\frac{1}{t^n\rho^n})^{\frac{1}{q'}}\,\|u\|_{H^{\frac{n}{p}+1}_{p,q,\lambda_1,\cdots,\lambda_m}}. \end{aligned}\tag{3.5}$$

Thus combining (3.4) with (3.5) yields for any small $|Q|$,

$$|u_Q - u(x)| \leq C\,\rho\int_0^1 \ell_{m+1}(\frac{1}{t^n\rho^n})^{\frac{1}{q'}}dt\,\|u\|_{H^{\frac{n}{p}+1}_{p,q,\lambda_1,\cdots,\lambda_m}} \leq C\,\rho\,\ell_{m+1}(\frac{1}{\rho})^{\frac{1}{q'}}\,\|u\|_{H^{\frac{n}{p}+1}_{p,q,\lambda_1,\cdots,\lambda_m}}. \tag{3.6}$$

Interchanging roles of x and y, we obtain (3.6) with x replaced by y, and then we have a desired conclusion. $\square$

In the end, we shall show Theorem 1.6. The reverse O'Neil inequality (2.4) is an essential tool to estimate the local integrals from below.

Proof of Theorem 1.6. First, we consider the case $q<\infty$. Assume the condition (A). In this case, we define $f_0(x):=|x|^{\alpha n}\chi_{\{x\in\mathbb{R}^n\,;\,|x|<\delta\}}(x)$, where α is any number satisfying $-\frac{1}{p}<\alpha<0$, and $\delta>0$ will be chosen small enough later, and then we have

$$f_0^*(t)=\tilde{f}_0\left(\left(\frac{t}{\omega_n}\right)^{\frac{1}{n}}\right)\simeq g_0(t):=t^{\alpha}\chi_{(0,\delta)}(t)$$

for all $t>0$ with some small $\delta>0$, where $\tilde{f}_0(|x|)=f_0(x)$, and $\omega_n:=\frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}$ is the volume of the unit ball in $\mathbb{R}^n$. That is, there exist positive constants C and $\tilde{C}$ such that

$$\tilde{C}\,g_0(t)\le f_0^*(t)\le C\,g_0(t) \tag{3.7}$$

hold for all $t>0$ with some small $\delta>0$. Then by the definition of the Lorentz-Zygmund norm and the latter estimate in (3.7) and, $\frac{1}{p}+\alpha>0$, we obtain

$$\begin{aligned}
\|f_0\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}} &\le C\left(\int_0^{\infty}\left(t^{\frac{1}{p}}\prod_{l=1}^{m}\ell_l(c_l+\frac{1}{t})^{\lambda_l}g_0(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\\
&\le C\left(\int_0^{\delta}\left(t^{\frac{1}{p}+\alpha}\prod_{l=1}^{m}\ell_l(\frac{1}{t})^{\lambda_l}\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\le C\left(\int_0^{\delta}t^{\frac{q}{2}(\frac{1}{p}+\alpha)-1}dt\right)^{\frac{1}{q}}<+\infty,
\end{aligned}$$

which implies $f_0\in L_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$, or equivalently, $u_0:=G_{\frac{n}{p}}*f_0\in H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$. On the other hand, for any measurable set E satisfying $|E|<\frac{\delta}{2}$, by the former estimate in (3.7), the Hardy inequality (2.1), the reverse O'Neil inequality (2.4) and Lemma 2.1, we see

$$\begin{aligned}
\int_E|G_{\frac{n}{p}}*f_0(x)|^r dx &= \int_0^{|E|}(G_{\frac{n}{p}}*f_0)^*(t)^r dt\ge C\int_0^{|E|}(G_{\frac{n}{p}}*f_0)^{**}(t)^r dt\\
&\ge C\int_0^{|E|}\left(t\,G^{**}_{\frac{n}{p}}(t)f_0^{**}(t)+\int_t^{\infty}G^*_{\frac{n}{p}}(s)f_0^*(s)ds\right)^r dt\\
&\ge C\int_0^{|E|}\left(\int_t^{\delta}G^*_{\frac{n}{p}}(s)f_0^*(s)ds\right)^r dt\ge C\int_0^{|E|}\left(\int_t^{\delta}G^*_{\frac{n}{p}}(s)g_0(s)ds\right)^r dt\\
&\ge C\int_0^{|E|}\left(\int_t^{\delta}s^{-\frac{1}{p'}}g_0(s)ds\right)^r dt\ge C\int_0^{|E|}\left(\int_{\frac{\delta}{2}}^{\delta}s^{\alpha-\frac{1}{p'}}ds\right)^r dt=C\,|E|,
\end{aligned} \tag{3.8}$$

which is a desired inequality.

Next, assume the condition (B). In this case, we define functions $f_{\varepsilon,k}(x)$ by

$$f_{\varepsilon,k}(x):=\prod_{l=1}^{j}\ell_l(\frac{1}{|x|})^{-1}\prod_{l=j+1}^{m}\ell_l(\frac{1}{|x|})^{-\lambda_l}\prod_{l=j+1}^{k-1}\ell_l(\frac{1}{|x|})^{-\frac{1}{q}}\,\ell_k(\frac{1}{|x|})^{-\frac{1}{q}-\varepsilon}|x|^{-\frac{n}{p}}\chi_{\{x\in\mathbb{R}^n\,;\,|x|<\delta\}}(x),$$

where $\delta > 0$ will be taken small enough later. It is easy to see that $f_{\varepsilon,k}$ are non-negative, radially symmetric and non-increasing with respect to the radial direction $|x|$. Thus we have

$$
\begin{aligned}
f^*_{\varepsilon,k}(t) &= \tilde{f}_{\varepsilon,k}\left(\left(\frac{t}{\omega_n}\right)^{\frac{1}{n}}\right) \\
&\simeq g_{\varepsilon,k}(t) := \prod_{l=1}^{j} \ell_l(\frac{1}{t})^{-1} \prod_{l=j+1}^{m} \ell_l(\frac{1}{t})^{-\lambda_l} \prod_{l=j+1}^{k-1} \ell_l(\frac{1}{t})^{-\frac{1}{q}} \ell_k(\frac{1}{t})^{-\frac{1}{q}-\varepsilon} t^{-\frac{1}{p}} \chi_{(0,\delta)}(t) \qquad (3.9)
\end{aligned}
$$

for all $t > 0$ with some small $\delta > 0$. Then by (3.9), we have

$$
\begin{aligned}
\|f_{\varepsilon,k}\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}} &\leq C\left(\int_0^\infty \left(t^{\frac{1}{p}} \prod_{l=1}^{m} \ell_l(c_l + \frac{1}{t})^{\lambda_l} g_{\varepsilon,k}(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\leq C\left(\int_0^\delta \prod_{l=1}^{k-1} \ell_l(\frac{1}{t})^{-1} \ell_k(\frac{1}{t})^{-1-q\varepsilon} \frac{dt}{t}\right)^{\frac{1}{q}} < +\infty,
\end{aligned}
$$

which implies $f_{\varepsilon,k} \in L_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$, or equivalently, $u_{\varepsilon,k} := G_{\frac{n}{p}} * f_{\varepsilon,k} \in H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$. By using L'Hopital's rule, we see that there exists a small positive constant $\tilde{\delta} < \delta$ such that the inequalities

$$
\int_t^\delta s^{-\frac{1}{p'}} g_{\varepsilon,k}(s)ds \simeq \ell_{j+1}(\frac{1}{t}) \prod_{l=j+1}^{m} \ell_l(\frac{1}{t})^{-\lambda_l} \prod_{l=j+1}^{k-1} \ell_l(\frac{1}{t})^{-\frac{1}{q}} \ell_k(\frac{1}{t})^{-\frac{1}{q}-\varepsilon} \qquad (3.10)
$$

hold for all $0 < t < \tilde{\delta}$. Thus by carrying out the same estimates in (3.8) and using (3.9) and (3.10), for any measurable set E with $|E| < \tilde{\delta}$, we have

$$
\begin{aligned}
&\int_E |G_{\frac{n}{p}} * f_{\varepsilon,k}(x)|^r dx \geq C\int_0^{|E|}\left(\int_t^\delta s^{-\frac{1}{p'}} g_{\varepsilon,k}(s)ds\right)^r dt \\
&\geq C\int_0^{|E|}\left(\ell_{j+1}(\frac{1}{t}) \prod_{l=j+1}^{m} \ell_l(\frac{1}{t})^{-\lambda_l} \prod_{l=j+1}^{k-1} \ell_l(\frac{1}{t})^{-\frac{1}{q}} \ell_k(\frac{1}{t})^{-\frac{1}{q}-\varepsilon}\right)^r dt \\
&\geq C\,|E|\left(\ell_{j+1}(\frac{1}{|E|}) \prod_{l=j+1}^{m} \ell_l(\frac{1}{|E|})^{-\lambda_l} \prod_{l=j+1}^{k-1} \ell_l(\frac{1}{|E|})^{-\frac{1}{q}} \ell_k(\frac{1}{|E|})^{-\frac{1}{q}-\varepsilon}\right)^r,
\end{aligned}
$$

where the last inequality can be derived by noticing that the function

$$
\ell_{j+1}(\frac{1}{t}) \prod_{l=j+1}^{m} \ell_l(\frac{1}{t})^{-\lambda_l} \prod_{l=j+1}^{k-1} \ell_l(\frac{1}{t})^{-\frac{1}{q}} \ell_k(\frac{1}{t})^{-\frac{1}{q}-\varepsilon}
$$

is decreasing for small $t > 0$.

Next, assume the condition (C). In this case, we define functions $f_{\varepsilon,k}(x)$ by

$$f_{\varepsilon,k}(x) := \prod_{l=1}^{m} \ell_l(\frac{1}{|x|})^{-1} \prod_{l=m+1}^{k} \ell_l(\frac{1}{|x|})^{-\frac{1}{q}} \ell_{k+1}(\frac{1}{|x|})^{-\frac{1}{q}-\varepsilon} |x|^{-\frac{n}{p}} \chi_{\{x\in\mathbb{R}^n\,;\,|x|<\delta\}}(x),$$

and we have

$$f^*_{\varepsilon,k}(t) = \tilde{f}_{\varepsilon,k}\left(\left(\frac{t}{\omega_n}\right)^{\frac{1}{n}}\right) \simeq g_{\varepsilon,k}(t) := \prod_{l=1}^{m} \ell_l(\frac{1}{t})^{-1} \prod_{l=m+1}^{k} \ell_l(\frac{1}{t})^{-\frac{1}{q}} \ell_{k+1}(\frac{1}{t})^{-\frac{1}{q}-\varepsilon} t^{-\frac{1}{p}} \chi_{(0,\delta)}(t) \tag{3.11}$$

for all $t > 0$ with some small $\delta > 0$. Then by (3.11), we obtain

$$\begin{aligned}
\|f_{\varepsilon,k}\|_{L_{p,q,\lambda_1,\cdots,\lambda_m}} &\le C\left(\int_0^\infty \left(t^{\frac{1}{p}} \prod_{l=1}^{m} \ell_l(c_l + \frac{1}{t})^{\lambda_l} g_{\varepsilon,k}(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\le C\left(\int_0^\delta \prod_{l=1}^{k} \ell_l(\frac{1}{t})^{-1} \ell_{k+1}(\frac{1}{t})^{-1-q\varepsilon} \frac{dt}{t}\right)^{\frac{1}{q}} < +\infty,
\end{aligned}$$

which implies $f_{\varepsilon,k} \in L_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$, or equivalently, $u_{\varepsilon,k} := G_{\frac{n}{p}} * f_{\varepsilon,k} \in H^{\frac{n}{p}}_{p,q,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$. Here, we see that there exists a small positive constant $\tilde{\delta} < \delta$ such that the inequalities

$$\int_t^\delta s^{-\frac{1}{p'}} g_{\varepsilon,k}(s)ds \simeq \ell_{m+1}(\frac{1}{t}) \prod_{l=m+1}^{k} \ell_l(\frac{1}{t})^{-\frac{1}{q}} \ell_{k+1}(\frac{1}{t})^{-\frac{1}{q}-\varepsilon} \tag{3.12}$$

hold for all $0 < t < \tilde{\delta}$. Thus by carrying out the same estimates in (3.8) and using (3.11) and (3.12), for any measurable set E with $|E| < \tilde{\delta}$, we have

$$\begin{aligned}
\int_E |G_{\frac{n}{p}} * f_{\varepsilon,k}(x)|^r dx &\ge C\int_0^{|E|} \left(\int_t^\delta s^{-\frac{1}{p'}} g_{\varepsilon,k}(s)ds\right)^r dt \\
&\ge C\int_0^{|E|} \left(\ell_{m+1}(\frac{1}{t}) \prod_{l=m+1}^{k} \ell_l(\frac{1}{t})^{-\frac{1}{q}} \ell_{k+1}(\frac{1}{t})^{-\frac{1}{q}-\varepsilon}\right)^r dt \\
&\ge C\,|E| \left(\ell_{m+1}(\frac{1}{|E|}) \prod_{l=m+1}^{k} \ell_l(\frac{1}{|E|})^{-\frac{1}{q}} \ell_{k+1}(\frac{1}{|E|})^{-\frac{1}{q}-\varepsilon}\right)^r,
\end{aligned}$$

where the last inequality can be derived by noticing that the function

$$\ell_{m+1}(\frac{1}{t}) \prod_{l=m+1}^{k} \ell_l(\frac{1}{t})^{-\frac{1}{q}} \ell_{k+1}(\frac{1}{t})^{-\frac{1}{q}-\varepsilon}$$

is decreasing for small $t > 0$.

We proceed to the case $q = \infty$. First, assume the condition (A). However, this case can be treated in a same way as the case $q < \infty$ with the condition (A). Therefore, we omit it.

Next, assume the condition (B). In this case, we define a function $f_0(x)$ by

$$f_0(x) := \prod_{l=1}^{j} \ell_l(\frac{1}{|x|})^{-1} \prod_{l=j+1}^{m} \ell_l(\frac{1}{|x|})^{-\lambda_l} |x|^{-\frac{n}{p}} \chi_{\{x\in\mathbb{R}^n\,;\,|x|<\delta\}}(x),$$

where $\delta > 0$ will be taken small enough later, and we have

$$f_0^*(t) = \tilde{f}_0\left(\left(\frac{t}{\omega_n}\right)^{\frac{1}{n}}\right) \simeq g_0(t) := \prod_{l=1}^{j} \ell_l(\frac{1}{t})^{-1} \prod_{l=j+1}^{m} \ell_l(\frac{1}{t})^{-\lambda_l}\, t^{-\frac{1}{p}} \chi_{(0,\delta)}(t) \tag{3.13}$$

for all $t > 0$ with some small $\delta > 0$. Then by (3.13), we obtain $f_0 \in L_{p,\infty,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$, or equivalently, $u_0 := G_{\frac{n}{p}} * f_0 \in H^{\frac{n}{p}}_{p,\infty,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$. Here, we see that there exists a small positive constant $\tilde{\delta} < \delta$ such that the inequalities

$$\int_t^\delta s^{-\frac{1}{p'}} g_0(s)ds \simeq \ell_{j+1}(\frac{1}{t}) \prod_{l=j+1}^{m} \ell_l(\frac{1}{t})^{-\lambda_l}$$

hold for all $0 < t < \tilde{\delta}$. Thus by carrying out the same estimates in (3.8), for any measurable set E with $|E| < \tilde{\delta}$, we have

$$\begin{aligned}
&\int_E |G_{\frac{n}{p}} * f_0(x)|^r dx \geq C \int_0^{|E|} \left(\int_t^\delta s^{-\frac{1}{p'}} g_0(s)ds\right)^r dt \\
&\geq C \int_0^{|E|} \left(\ell_{j+1}(\frac{1}{t}) \prod_{l=j+1}^{m} \ell_l(\frac{1}{t})^{-\lambda_l}\right)^r dt \geq C\,|E| \left(\ell_{j+1}(\frac{1}{|E|}) \prod_{l=j+1}^{m} \ell_l(\frac{1}{|E|})^{-\lambda_l}\right)^r.
\end{aligned}$$

Finally, assume the condition (C). In this case, we define a function $f_0(x)$ by

$$f_0(x) := \prod_{l=1}^{m} \ell_l(\frac{1}{|x|})^{-1} |x|^{-\frac{n}{p}} \chi_{\{x\in\mathbb{R}^n\,;\,|x|<\delta\}}(x),$$

where $\delta > 0$ will be taken small enough later, and we have

$$f_0^*(t) = \tilde{f}_0\left(\left(\frac{t}{\omega_n}\right)^{\frac{1}{n}}\right) \simeq g_0(t) := \prod_{l=1}^{m} \ell_l(\frac{1}{t})^{-1} t^{-\frac{1}{p}} \chi_{(0,\delta)}(t) \tag{3.14}$$

for all $t > 0$ with some small $\delta > 0$. Then by (3.14), we obtain $f_0 \in L_{p,\infty,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$, or equivalently, $u_0 := G_{\frac{n}{p}} * f_0 \in H^{\frac{n}{p}}_{p,\infty,\lambda_1,\cdots,\lambda_m}(\mathbb{R}^n)$. Here, we see that there exists a small positive constant $\tilde{\delta} < \delta$ such that the inequalities $\int_t^\delta s^{-\frac{1}{p'}} g_0(s)ds \simeq \ell_{m+1}(\frac{1}{t})$ hold for all $0 < t < \tilde{\delta}$.

Therefore, by carrying out the same estimates in (3.8), for any measurable set E with $|E| < \tilde{\delta}$, we have

$$\int_E |G_{\frac{n}{p}} * f_0(x)|^r dx \geq C \int_0^{|E|} \left(\int_t^{\delta} s^{-\frac{1}{p'}} g_0(s) ds \right)^r dt \geq C \int_0^{|E|} \ell_{m+1}(\frac{1}{t})^r dt \geq C|E|\, \ell_{m+1}(\frac{1}{|E|})^r.$$

Thus we complete the proof of Theorem 1.6. □

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