Remarks on the Rellich inequality

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Abstract

We study the Rellich inequalities in the framework of equalities. We present equalities which imply the Rellich inequalities by dropping remainders. This provides a simple and direct understanding of the Rellich inequalities as well as the nonexistence of nontrivial extremisers.

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1 Introduction and the main results

In this paper, we prove some equalities which yield the Rellich inequality by dropping remainder terms in $L^2(\mathbb{R}^n)$ setting for $n \geq 5$. Moreover, a characterization is given on H^2-functions which make vanishing remainders on the basis of simple partial differential equations. Our presentation based on equalities presumably gives a clear picture of how the Rellich inequality follows with sharp remainders and implies the nonexistence of nontrivial extremisers.

The Rellich inequality that we study in this paper is the following :

$$\left\| \frac{f}{|x|^2} \right\|_{L^2(\mathbb{R}^n)} \leq \frac{4}{n(n-4)} \|\Delta f\|_{L^2(\mathbb{R}^n)} \tag{1.1}$$

for all $f \in H^2(\mathbb{R}^n)$ with $n \geq 5$, where $H^s = H^s(\mathbb{R}^n)$ is the standard Sobolev space of order $s \in \mathbb{R}$ defined as $(1-\Delta)^{-s/2} L^2(\mathbb{R}^n)$ and $\Delta = \sum_{j=1}^n \partial_j^2$ is the Laplacian in $\mathbb{R}^n$. The inequality (1.1) is basic in the self-adjointness problem of the Schrödinger operators with singular potentials such as $V(x) = \lambda |x|^{-2}$ with $\lambda > -\frac{n(n-4)}{4}$ (See [2, 3, 7, 8, 10, 14, 15, 27, 29, 30] and references therein for related subjects). Moreover, there is a large literature on (1.1) in connection with Hardy type inequalities [1, 4, 5, 6, 9, 11, 12, 13, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28, 31].

In an earlier work [24], we studied the Hardy inequality in L^2 setting by means of the equalities

$$\left(\frac{n-2}{2}\right)^2 \left\|\frac{f}{|x|}\right\|^2_{L^2(\mathbb{R}^n)} = \|\partial_r f\|^2_{L^2(\mathbb{R}^n)} - \||x|^{-\frac{n}{2}+1}\partial_r(|x|^{\frac{n}{2}-1}f)\|^2_{L^2(\mathbb{R}^n)}$$
$$= \|\partial_r f\|^2_{L^2(\mathbb{R}^n)} - \left\|\partial_r f + \frac{n-2}{2|x|}f\right\|^2_{L^2(\mathbb{R}^n)} \tag{1.2}$$

for all $f \in H^1(\mathbb{R}^n)$ with $n \geq 3$, where $\partial_r = \frac{x}{|x|}\cdot\nabla = \sum_{j=1}^n \frac{x_j}{|x|}\partial_j$ denotes the radial derivative in $\mathbb{R}^n$.

The purpose of this paper is to present the corresponding equalities on the Rellich inequality (1.1) and characterize the equality case in terms of vanishing conditions of remainders. To be more specific, we prove the following theorem.

Theorem 1.1. *Let $n \geq 5$. Then the following equalities*

$$\begin{aligned}
&\left(\frac{n(n-4)}{4}\right)^2 \left\|\frac{f}{|x|^2}\right\|^2_{L^2(\mathbb{R}^n)} \\
&= \left\|\partial_r^2 f + \frac{n-1}{|x|}\partial_r f\right\|^2_{L^2(\mathbb{R}^n)} - \left\|\partial_r^2 f + \frac{n-1}{|x|}\partial_r f + \frac{n(n-4)}{4|x|^2}f\right\|^2_{L^2(\mathbb{R}^n)} \\
&\quad - \frac{n(n-4)}{2}\left\|\frac{1}{|x|}\partial_r f + \frac{n-4}{2|x|^2}f\right\|^2_{L^2(\mathbb{R}^n)} \\
&= \left\||x|^{-n+1}\partial_r(|x|^{n-1}\partial_r f)\right\|^2_{L^2(\mathbb{R}^n)} - \left\||x|^{-\frac{n}{2}+1}\partial_r\left(|x|^{-1}\partial_r(|x|^{\frac{n}{2}}f)\right)\right\|^2_{L^2(\mathbb{R}^n)} \\
&\quad - \frac{n(n-4)}{2}\left\||x|^{-\frac{n}{2}+1}\partial_r(|x|^{\frac{n}{2}-2}f)\right\|^2_{L^2(\mathbb{R}^n)} \\
&= \left\||x|^{-n+1}\partial_r(|x|^{n-1}\partial_r f)\right\|^2_{L^2(\mathbb{R}^n)} - \left\||x|^{-\frac{n}{2}-1}\partial_r\left(|x|^{3}\partial_r(|x|^{\frac{n-4}{2}}f)\right)\right\|^2_{L^2(\mathbb{R}^n)} \\
&\quad - \frac{n(n-4)}{2}\left\||x|^{-\frac{n}{2}+1}\partial_r(|x|^{\frac{n}{2}-2}f)\right\|^2_{L^2(\mathbb{R}^n)}
\end{aligned} \tag{1.3}$$

hold for all $f \in H^2(\mathbb{R}^n)$. Moreover, there does not exist $f \in H^2(\mathbb{R}^n)$ satisfying

$$\left(\frac{n(n-4)}{4}\right)^2 \left\|\frac{f}{|x|^2}\right\|^2_{L^2(\mathbb{R}^n)} = \left\|\partial_r^2 f + \frac{n-1}{|x|}\partial_r f\right\|^2_{L^2(\mathbb{R}^n)}$$
$$= \left\||x|^{-n+1}\partial_r(|x|^{n-1}\partial_r f)\right\|^2_{L^2(\mathbb{R}^n)} \tag{1.4}$$

as well as

$$\left(\frac{n(n-4)}{4}\right)^2 \left\|\frac{f}{|x|^2}\right\|^2_{L^2(\mathbb{R}^n)} = \|\Delta f\|^2_{L^2(\mathbb{R}^n)} \tag{1.5}$$

except $f = 0$.

Equalities (1.3) imply (1.1) by the following theorem and its corollary. For j with $1 \leq j \leq n$, we denote by L_j a spherical derivative defined by

$$L_j = \partial_j - \frac{x_j}{|x|}\partial_r = \partial_j - \sum_{k=1}^{n} \frac{x_j x_k}{|x|^2}\partial_k.$$

Theorem 1.2. *Let $n \geq 5$. Then the following equalities*

$$\begin{aligned}
\|\Delta f\|_{L^2(\mathbb{R}^n)}^2 &= \left\|\partial_r^2 f + \frac{n-1}{|x|}\partial_r f\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|\sum_{j=1}^{n} L_j^2 f\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + \frac{n(n-4)}{2}\sum_{j=1}^{n}\left\|\frac{1}{|x|}L_j f\right\|_{L^2(\mathbb{R}^n)}^2 + 2\sum_{j=1}^{n}\left\|\partial_r L_j f + \frac{n-2}{2|x|}L_j f\right\|_{L^2(\mathbb{R}^n)}^2 \\
&= \||x|^{-n+1}\partial_r(|x|^{n-1}\partial_r f)\|_{L^2(\mathbb{R}^n)}^2 + \left\|\sum_{j=1}^{n} L_j^2 f\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + \frac{n(n-4)}{2}\sum_{j=1}^{n}\left\|\frac{1}{|x|}L_j f\right\|_{L^2(\mathbb{R}^n)}^2 + 2\sum_{j=1}^{n}\||x|^{-\frac{n}{2}+1}\partial_r(|x|^{\frac{n}{2}-1}L_j f)\|_{L^2(\mathbb{R}^n)}^2 \qquad (1.6)
\end{aligned}$$

hold for all $f \in H^2(\mathbb{R}^n)$.

Corollary 1.3. *Let $n \geq 5$. Then the inequality*

$$\|\Delta f\|_{L^2(\mathbb{R}^n)}^2 \geq \left\|\partial_r^2 f + \frac{n-1}{|x|}\partial_r f\right\|_{L^2(\mathbb{R}^n)}^2 \qquad (1.7)$$

holds for all $f \in H^2(\mathbb{R}^n)$. In (1.7), equality holds if and only if f is radial.

We prove Theorems 1.1 and 1.2 in Sections 2 and 3, respectively. For simplicity, we prove the theorems for $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C})$ since the proofs are completed by a density argument. The main idea of the proofs is given by the following lemma.

Lemma 1.4. *Let $\mathcal{H}$ be a vector space with Hermitian scalar product $(\cdot|\cdot)$. Also let $a \in \mathbb{R}$, $c > 0$ and $u, v \in \mathcal{H}$. Then the following equalities are equivalent.*

$$\begin{aligned}
&\|u\|^2 = -c\,\mathrm{Re}(u|v) + a. \\
&\mathrm{Re}(u|u + cv) = a. \\
&\|cv\|^2 = \|u + cv\|^2 + \|u\|^2 - 2a. \\
&\frac{1}{c^2}\|u\|^2 = \|v\|^2 - \left\|v + \frac{1}{c}u\right\|^2 + \frac{2a}{c^2}.
\end{aligned}$$

Proof. The lemma follows from the equality

$$\|cv\|^2 = \|u + cv\|^2 + \|u\|^2 - 2\,\mathrm{Re}(u|u + cv).$$

□

Remark. *The lemma was first formulated in [24] for $a = 0$. In [24], the equalities* (1.2) *were derived from*

$$\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^2} dx = -\frac{2}{n-2} \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x|} \overline{\partial_r f(x)} dx$$

by applying the lemma with $\mathcal{H} = L^2(\mathbb{R}^n)$, $u = \frac{f}{|x|}$, $v = \partial_r f$, and $c = \frac{2}{n-2}$.

2 Proof of Theorem 1.1

We introduce the standard polar coordinates $(r, \omega) = \left(|x|, \frac{x}{|x|}\right) \in (0, \infty) \times \mathbb{S}^{n-1}$ and the Lebesgue measure σ on the unit sphere $\mathbb{S}^{n-1}$. We have by integration by parts

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^4} dx \\
&= \int_0^\infty r^{n-5} \int_{\mathbb{S}^{n-1}} |f(r\omega)|^2 d\sigma(\omega) dr \\
&= -\frac{2}{n-4} \operatorname{Re} \int_0^\infty r^{n-4} \int_{\mathbb{S}^{n-1}} f(r\omega) \overline{\omega \cdot \nabla f(r\omega)} d\sigma(\omega) dr \\
&= \frac{2}{(n-3)(n-4)} \operatorname{Re} \int_0^\infty r^{n-3} \int_{\mathbb{S}^{n-1}} \left(|\omega \cdot \nabla f(r\omega)|^2 + f(r\omega) \overline{(\omega \cdot \nabla)^2 f(r\omega)} \right) d\sigma(\omega) dr \\
&= \frac{2}{(n-3)(n-4)} \left(\left\| \frac{1}{|x|} \partial_r f \right\|_{L^2(\mathbb{R}^n)}^2 + \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x|^2} \overline{\partial_r^2 f(x)} dx \right). \qquad (2.1)
\end{aligned}$$

The first norm on the right hand of the last equality in (2.1) is rewritten as

$$\begin{aligned}
\left\| \frac{1}{|x|} \partial_r f \right\|_{L^2(\mathbb{R}^n)}^2 &= \left\| \partial_r \left(\frac{f}{|x|} \right) + \frac{f}{|x|^2} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&= \left\| \partial_r \left(\frac{f}{|x|} \right) \right\|_{L^2(\mathbb{R}^n)}^2 + 2 \operatorname{Re} \left(\partial_r \left(\frac{f}{|x|} \right) \middle| \frac{f}{|x|^2} \right) + \left\| \frac{f}{|x|^2} \right\|_{L^2(\mathbb{R}^n)}^2 . \qquad (2.2)
\end{aligned}$$

We apply (1.2) with f replaced by $\frac{f}{|x|}$ to obtain

$$\left\| \partial_r \left(\frac{f}{|x|} \right) \right\|_{L^2(\mathbb{R}^n)}^2 = \left(\frac{n-2}{2} \right)^2 \left\| \frac{f}{|x|^2} \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| |x|^{-\frac{n}{2}+1} \partial_r (|x|^{\frac{n}{2}-2} f) \right\|_{L^2(\mathbb{R}^n)}^2 . \qquad (2.3)$$

Integrating by parts, we have

$$\begin{aligned}
2 \operatorname{Re} \left(\partial_r \left(\frac{f}{|x|} \right) \middle| \frac{f}{|x|^2} \right) &= \int_{\mathbb{R}^n} \frac{1}{|x|} \partial_r \left(\frac{|f|^2}{|x|^2} \right) dx = \int_{\mathbb{R}^n} \frac{x}{|x|^2} \cdot \nabla \left(\frac{|f|^2}{|x|^2} \right) dx \\
&= -(n-2) \left\| \frac{f}{|x|^2} \right\|_{L^2(\mathbb{R}^n)}^2 . \qquad (2.4)
\end{aligned}$$

By (2.3) and (2.4), we rewrite (2.2) as

$$\left\|\frac{1}{|x|}\partial_r f\right\|^2_{L^2(\mathbb{R}^n)} = \left(\frac{n-4}{2}\right)^2 \left\|\frac{f}{|x|^2}\right\|^2_{L^2(\mathbb{R}^n)} + \left\||x|^{-\frac{n}{2}+1}\partial_r(|x|^{\frac{n}{2}-2}f)\right\|^2_{L^2(\mathbb{R}^n)}. \tag{2.5}$$

The second integral on the right hand side of the last equality in (2.1) is rewritten as

$$\begin{aligned} &\operatorname{Re}\int_{\mathbb{R}^n} \frac{f(x)}{|x|^2}\overline{\partial_r^2 f(x)}dx \\ &= \operatorname{Re}\int_{\mathbb{R}^n} \frac{f(x)}{|x|^2}\overline{\left(\partial_r^2 f(x) + \frac{n-1}{|x|}\partial_r f(x)\right)}dx - (n-1)\operatorname{Re}\int_{\mathbb{R}^n}\frac{f(x)}{|x|^3}\overline{\partial_r f(x)}dx, \end{aligned} \tag{2.6}$$

where the last integral is given by

$$\operatorname{Re}\int_{\mathbb{R}^n}\frac{f}{|x|^3}\overline{\partial_r f}dx = \frac{1}{2}\int_{\mathbb{R}^n}\frac{1}{|x|^3}\partial_r(|f|^2)dx = \frac{1}{2}\int_{\mathbb{R}^n}\frac{x}{|x|^4}\cdot\nabla(|f|^2)dx = -\frac{n-4}{2}\left\|\frac{f}{|x|^2}\right\|^2_{L^2(\mathbb{R}^n)}. \tag{2.7}$$

It follows from (2.1), (2.5), (2.6) and (2.7) that

$$\begin{aligned} &\frac{n(n-4)}{4}\left\|\frac{f}{|x|^2}\right\|^2_{L^2(\mathbb{R}^n)} \\ &= -\operatorname{Re}\int_{\mathbb{R}^n}\frac{f(x)}{|x|^2}\overline{\left(\partial_r^2 f(x) + \frac{n-1}{|x|}\partial_r f(x)\right)}dx - \left\||x|^{-\frac{n}{2}+1}\partial_r(|x|^{\frac{n}{2}-2}f)\right\|^2_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{2.8}$$

Then (1.3) follows from (2.8) by applying Lemma 1.4 with $\mathcal{H} = L^2(\mathbb{R}^n)$, $u = \frac{f}{|x|^2}$, $v = \partial_r^2 f + \frac{n-1}{|x|}\partial_r f$, $c = \frac{4}{n(n-4)}$ and $a = -c\||x|^{-\frac{n}{2}+1}\partial_r(|x|^{\frac{n}{2}-2}f)\|^2_{L^2(\mathbb{R}^n)}$.

We now assume that $f \in H^2(\mathbb{R}^n)$ satisfies (1.4). Then by (1.3), it follows $\partial_r(|x|^{\frac{n}{2}-2}f) = 0$, which is equivalent to the existence of $\varphi : \mathbb{S}^{n-1} \to \mathbb{C}$ such that $|x|^{\frac{n}{2}-2}f(x) = \varphi(\frac{x}{|x|})$ almost everywhere. In that case $\frac{f}{|x|^2} \in L^2(\mathbb{R}^n)$ if and only if $\frac{1}{|x|^n}|\varphi(\frac{x}{|x|})|^2 \in L^1(\mathbb{R}^n)$, where the last condition if and only if $\varphi \equiv 0$, which in turn implies $f \equiv 0$. In the case where $f \in H^2(\mathbb{R}^n)$ satisfies (1.5), where the problem is reduced to the case (1.4) just we have argued if we can prove (1.7). Therefore, the proof of the last part of the theorem will be completed after the completion of the proof of Corollary 1.3.

3 Proof of Theorem 1.2

We start with the equality

$$\Delta f = \partial_r^2 f + \frac{n-1}{|x|}\partial_r f + \sum_{j=1}^n L_j^2 f,$$

which is verified by a direct calculation. Then we expand the scalar product as

$$\|\Delta f\|_{L^2(\mathbb{R}^n)}^2 = \left\|\partial_r^2 f + \frac{n-1}{|x|}\partial_r f\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|\sum_{j=1}^n L_j^2 f\right\|_{L^2(\mathbb{R}^n)}^2 + 2\,\mathrm{Re}\left(\partial_r^2 f + \frac{n-1}{|x|}\partial_r f \,\middle|\, \sum_{j=1}^n L_j^2 f\right). \tag{3.1}$$

From now on we consider the last scalar product. For simplicity, let

$$g = \partial_r^2 f + \frac{n-1}{|x|}\partial_r f = |x|^{-n+1}\partial_r(|x|^{n-1}\partial_r f) \text{ and } h_j = L_j f.$$

By integration by parts,

$$(g|L_j h_j) = -(L_j g|h_j) + (n-1)\left(g\,\middle|\,\frac{x_j}{|x|^2}h_j\right).$$

This gives

$$\left(g\,\middle|\,\sum_{j=1}^n L_j^2 f\right) = -\sum_{j=1}^n (L_j g|h_j) \tag{3.2}$$

since $\sum_{j=1}^n x_j L_j = 0$. We also notice that $L_j\partial_r = \left(\partial_r + \frac{1}{|x|}\right)L_j$ and that $L_j(|x|^\lambda u) = |x|^\lambda L_j u$ for any $\lambda \in \mathbb{R}$ to obtain

$$\begin{aligned} L_j g &= L_j(|x|^{-n+1}\partial_r(|x|^{n-1}\partial_r f)) = |x|^{-n+1}L_j\partial_r(|x|^{n-1}\partial_r f) \\ &= |x|^{-n+1}\left(\partial_r + \frac{1}{|x|}\right)L_j(|x|^{n-1}\partial_r f) = |x|^{-n+1}\left(\partial_r + \frac{1}{|x|}\right)|x|^{n-1}L_j\partial_r f \\ &= |x|^{-n+1}\left(\partial_r + \frac{1}{|x|}\right)|x|^{n-1}\left(\partial_r + \frac{1}{|x|}\right)h_j = \partial_r^2 h_j + \frac{n+1}{|x|}\partial_r h_j + \frac{n-1}{|x|^2}h_j. \end{aligned} \tag{3.3}$$

By (3.3) and (1.2), the real part of the left hand side of (3.2) is calculated as

$$\begin{aligned} -\mathrm{Re}\sum_{j=1}^n (L_j g|h_j) &= -\sum_{j=1}^n \mathrm{Re}\left((\partial_r^2 h_j|h_j) + (n+1)\,\mathrm{Re}\left(\frac{1}{|x|}\partial_r h_j\,\middle|\,h_j\right) + (n-1)\left\|\frac{1}{|x|}h_j\right\|_{L^2(\mathbb{R}^n)}^2\right) \\ &= -\sum_{j=1}^n \left(\frac{(n-1)(n-2)}{2}\left\|\frac{1}{|x|}h_j\right\|_{L^2(\mathbb{R}^n)}^2 - \|\partial_r h_j\|_{L^2(\mathbb{R}^n)}^2\right. \\ &\qquad \left. -\frac{(n+1)(n-2)}{2}\left\|\frac{1}{|x|}h_j\right\|_{L^2(\mathbb{R}^n)}^2 + (n-1)\left\|\frac{1}{|x|}h_j\right\|_{L^2(\mathbb{R}^n)}^2\right) \\ &= \sum_{j=1}^n \left(\|\partial_r h_j\|_{L^2(\mathbb{R}^n)}^2 - \left\|\frac{1}{|x|}h_j\right\|_{L^2(\mathbb{R}^n)}^2\right) \\ &= \sum_{j=1}^n \left(\frac{n(n-4)}{4}\left\|\frac{1}{|x|}h_j\right\|_{L^2(\mathbb{R}^n)}^2 + \||x|^{-\frac{n}{2}+1}\partial_r(|x|^{\frac{n}{2}-1}h_j)\|_{L^2(\mathbb{R}^n)}^2\right). \end{aligned} \tag{3.4}$$

By (3.1), (3.2) and (3.4), we obtain (1.6).

Proof of Corollary 1.3. The inequality (1.7) follows immediately from (1.6). In (1.7), equality holds only if $\sum_{j=1}^{n} L_j^2 f = 0$, which is equivalent to the fact that f is radial since

$$\frac{1}{|x|^2} \sum_{1 \le j < k \le n} (x_j \partial_k - x_k \partial_j)^2 f = \sum_{j=1}^{n} L_j^2 f.$$

Conversely, if f is radial, then $L_j f = 0$ for all j and (1.7) is realized as an equality. $\square$

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