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Dynamical semigroup for unbounded repeated perturbation of an open system

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We consider a dynamical semigroup for unbounded Kossakowski-Lindblad-Davies generator corresponding to evolution of an open system for a tuned repeated harmonic perturbation. For this evolution, we prove the existence of uniquely determined minimal trace-preserving strongly continuous dynamical semigroups on the space of states. The corresponding dual W^*-dynamical system is shown to be unital quasi-free and completely positive automorphisms of the canonical commutation relation-algebra. © *2016 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4941940]

I. REPEATED PERTURBATION AND SEMIGROUPS

A quantum Hamiltonian dynamical system with repeated harmonic interaction was proposed in Refs. 20 and 22, as a model of physical phenomenon known as the "one-atom maser," see Refs. 6 and 17. The model consists of the isolated one mode cavity, which is pumping by an infinite chain of atoms. In the present paper, we consider the corresponding open system, which is a model of a leaky cavity. This model can be described mathematically through the Kossakowski-Lindblad-Davies (KLD) *dissipative* extension of the Hamiltonian dynamics.[21,3]

Since repeated perturbation of Hamiltonian dynamics is piecewise *constant*, its analysis reduces to study of Quantum Dynamical Semigroups (QDSs) and their generators on the space of states. A similar reduction is also valid for repeated perturbation of open quantum dynamical systems.

The theory of QDS is quite satisfactory for bounded generators and for their bounded KLD extensions.[8] An extension of this theory to the case of *unbounded* dissipative generators was initiated in Refs. 9 and 10 and developed in Refs. 13, 12, 11, and 19 for completely positive maps of Canonical Commutation Relation (CCR)-algebras. The progress in construction of the *minimal* dynamical semigroups for unbounded dissipative generators is essentially due to ideas that come back to Kato.[14] They were extended on quantum dynamics in Ref. 9. The Kato regularisation method inspired the study of uniqueness and trace-preserving (or Markovian) property of the *minimal* QDS for KLD generators, e.g., Refs. 7 and 4 Lecture 3.

This method initiated analysis of singular (with relative bound equal one) perturbations of positive substochastic semigroups on the normal states[16] as well as in abstract space of states.[5]

In the present paper, we consider the problem of construction of unbounded generators for a concrete quantum dynamical system. The model is a dissipative KLD extension of the Hamiltonian dynamical system[21] for an open system interacting with a boson reservoir. We show that generator of the corresponding QDS can be constructed following the Kato regularisation[14] of perturbations with relative bound equal one.

Our main results are the following.

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In Section II, we construct generator of the minimal QDS corresponding to the standard KLD extension of the Hamiltonian dynamical system.[21] We prove (Theorem 2.7) that it generates strongly continuous, contraction, positive, and *trace-preserving* semigroup, i.e., it is a Markov Dynamical Semigroup (MDS) on the space of trace-class operators.

In Section III, we establish explicit formulae for the action of the *dual* MDS on the Weyl CCR-algebra (Theorem 3.1). This allows to prove that the dual MDS is *completely positive* (Theorem 3.6). Finally, we prove that the MDS maps the space of *quasi-free* states into itself, see Theorem 3.9.

Note that evolution driven by repeated perturbation has the form of iterated composition of quantum dynamical semigroups.[6] Although motivated by repeated perturbation, the main results of the present paper concern first of all the *individual* semigroups (Section II) rather than their compositions. In Section III, we consider completely positive quasi-free maps by semigroups constructed in Section II and one application to compositions. For more results related to iteration of the semigroups compositions we address the readers to Ref. 21.

In the rest of this section, we briefly review the Hamiltonian dynamics and recall the standard KLD extension for the open system with a boson reservoir. Then we give a formal definition of generators corresponding to individual semigroups that define evolution of our model of the open boson system for repeated harmonic interaction.[21]

Let a and a^* be the annihilation and the creation operators defined in the Fock space $\mathscr{F}$ generated by a cyclic vector Ω. That is, the Hilbert space $\mathscr{F}$ contains the algebraic span $\mathscr{F}_{\text{fin}}$ of vectors $\{(a^*)^m\Omega\}_{m\geq 0}$ as a dense subset and a, a^* satisfy the CCRs,

$$[a, a^*] = \mathbb{1}, \quad [a, a] = 0, \quad [a^*, a^*] = 0 \quad \text{on} \quad \mathscr{F}_{\text{fin}}.$$

We denote by $\{\mathscr{H}_k\}_{k=0}^N$ the copies of $\mathscr{F}$ for an arbitrary but finite $N \in \mathbb{N}$ and by $\mathscr{H}^{(N)}$ the Hilbert space tensor product of these copies,

$$\mathscr{H}^{(N)} = \bigotimes_{k=0}^{N} \mathscr{H}_k \tag{1.1}$$

and by $\Omega_F := \Omega^{\otimes(N+1)}$, its cyclic vector.

In this space, we define the annihilation and the creation operators

$$b_k := \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes a \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}\,, \;\; b_k^* := \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes a^* \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \tag{1.2}$$

for $k = 0, 1, 2, \ldots, N$, where the operator a, or a^*, is the $(k+1)$th factor. On algebraic tensor product $\mathscr{H}_{\text{fin}}^{(N)} := \mathscr{F}_{\text{fin}}^{\otimes(N+1)}$, these unbounded operators satisfy the CCR,

$$[b_k, b_{k'}^*] = \delta_{k,k'}\mathbb{1}, \quad [b_k, b_{k'}] = [b_k^*, b_{k'}^*] = 0 \;\; (k, k' = 0, 1, 2, \ldots, N). \tag{1.3}$$

We consider the Hamiltonian of the system with time-dependent *repeated harmonic* perturbation,[20,22]

$$H_N(t) = E\, b_0^* b_0 + \epsilon \sum_{k=1}^{N} b_k^* b_k + \eta \sum_{k=1}^{N} \chi_{[(k-1)\tau, k\tau)}(t)\,(b_0^* b_k + \, b_k^* b_0), \tag{1.4}$$

for $t \in [0, N\tau)$, where $\tau, E, \epsilon, \eta > 0$ and $\chi_{[x,y)}(\cdot)$ is the characteristic function of semi-open intervals $[x, y) \subset \mathbb{R}$. Here (1.4) denotes the self-adjoint operator on the dense domain

$$\mathcal{D}_0 := \bigcap_{k=0}^{N} \text{dom}(b_k^* b_k) \;\subset \mathscr{H}^{(N)}. \tag{1.5}$$

Model (1.4) describes the system $\mathcal{S} + C_N$, where subsystem $\mathcal{S}$ corresponding to the kinetic term $E\, b_0^* b_0$ of the Hamiltonian is repeatedly interacting with a long time-equidistant *chain* $C_N = \mathcal{S}_1 + \mathcal{S}_2 + \cdots + \mathcal{S}_N$ of subsystems corresponding to the kinetic terms $\epsilon \sum_{k=1}^{N} b_k^* b_k$. The Hilbert space $\mathscr{H}_0$ corresponds to the subsystem $\mathcal{S}$ and the Hilbert space $\mathscr{H}_k$ to the subsystem $\mathcal{S}_k$ ($k = 1, \ldots, N$). This visualisation is motivated by a number of physical models, see Refs. 6 and 17.

For $t \in [(n-1)\tau, n\tau)$, only subsystem $\mathcal{S}_n$ interacts with $\mathcal{S}$ and the system $\mathcal{S} + C_N$ is *autonomous* on this time-interval with the self-adjoint Hamiltonian

$$H_n = E\, b_0^* b_0 + \epsilon \sum_{k=1}^{N} b_k^* b_k + \eta\, (b_0^* b_n + b_n^* b_0) \tag{1.6}$$

on domain $\mathcal{D}_0$. To keep operator (1.6) lower semi-bounded, we assume that parameters E, ϵ, η satisfy the condition

(H1)
$$\eta^2 \leqslant E\, \epsilon. \tag{1.7}$$

We denote by $\mathfrak{C}_1(\mathscr{H}^{(N)})$ the Banach space of trace-class operators on $\mathscr{H}^{(N)}$ with trace norm $\|\cdot\|_1$. Its dual space is isometrically isomorphic to the space of bounded operators on $\mathscr{H}^{(N)}$: $\mathfrak{C}_1^*(\mathscr{H}^{(N)}) \simeq \mathcal{L}(\mathscr{H}^{(N)})$. We consider the dual pair corresponding to the bilinear functional

$$\langle \phi\, | A \rangle_{\mathscr{H}^{(N)}} = \mathrm{Tr}_{\mathscr{H}^{(N)}}(\phi\, A), \quad \text{for } (\phi, A) \in \mathfrak{C}_1(\mathscr{H}^{(N)}) \times \mathcal{L}(\mathscr{H}^{(N)}). \tag{1.8}$$

Positive operators in $\mathfrak{C}_1(\mathscr{H}^{(N)})$ with *unit* trace are called *density matrices*. For each density matrix ρ, we consider the normal state $\omega_\rho(\cdot)$ on $\mathcal{L}(\mathscr{H}^{(N)})$ defined by

$$\omega_\rho(\ \cdot\ ) = \langle \rho\, |\ \cdot\ \rangle_{\mathscr{H}^{(N)}}. \tag{1.9}$$

To describe evolution of the open system corresponding to (1.4), we consider the KLD dissipative extension of the Hamiltonian dynamics to non-Hamiltonian *master equation*: $\partial_t \rho(t) = L_\sigma(t)(\rho(t))$, with the time-dependent generator

$$L_\sigma(t)(\rho) := -i\, [H_N(t), \rho] + Q(\rho) - \frac{1}{2}(Q^*(\mathbb{1})\rho + \rho\, Q^*(\mathbb{1})), \tag{1.10}$$

for $t \in [0, N\tau)$.[4,1] The operator Q acts on ρ as

$$Q(\rho) = \sigma_-\, b_0\, \rho\, b_0^* + \sigma_+\, b_0^*\, \rho\, b_0. \tag{1.11}$$

Its dual operator Q^* is defined by the relation $\langle Q(\rho)\, | A \rangle_{\mathscr{H}^{(N)}} = \langle \rho\, | Q^*(A) \rangle_{\mathscr{H}^{(N)}}$,

$$Q^*(A) = \sigma_-\, b_0^*\, A\, b_0 + \sigma_+\, b_0\, A\, b_0^*. \tag{1.12}$$

Since the Hamiltonian part of the dynamics is *piecewise* autonomous, the *formal* generator (1.10) for $t \in [(k-1)\tau, k\tau)$, $k = 1, 2, \ldots, N$, is

$$L_{\sigma,k}(\rho) := -i[H_k, \rho] + Q(\rho) - \frac{1}{2}(Q^*(\mathbb{1})\rho + \rho Q^*(\mathbb{1})). \tag{1.13}$$

Note that the form of generators (1.10), (1.13) corresponds to repeated perturbation of the open system $\mathcal{S} + \mathcal{R}$, i.e., we study $(\mathcal{S} + \mathcal{R}) + C_N$ for external boson reservoir $\mathcal{R}$. Then a formal solution $\rho(t)$ of the Cauchy problem for the master equation corresponding to initial condition $\rho(0) = \rho$ is defined by the evolution map $\{T^\sigma_{t,0}\}_{t \geq 0}$. It is a composition of QDS with generators (1.13),

$$\rho(t) = T^\sigma_{t,0}(\rho) := (T^\sigma_{n,\nu(t)}\, T^\sigma_{n-1} \ldots T^\sigma_2\, T^\sigma_1)(\rho) \tag{1.14}$$

for $t = (n-1)\tau + \nu(t)$ and $n \leqslant N$, where $T^\sigma_{k,s} = e^{sL_{\sigma,k}}$, $T^\sigma_k = T^\sigma_{k,\tau}$ $(k = 1, 2, \ldots, n)$. Consequently, the analysis of evolution for repeated perturbation reduces to the study of QDS on the intervals $[(k-1)\tau, k\tau)$, $k = 1, \ldots, N$.

It is known that for the *standard* KLD generator of form (1.13) with bounded H_k, Q, and Q^*, the corresponding QDS $\{T^\sigma_{k,s}\}_{s \geq 0}$ on $\mathfrak{C}_1(\mathscr{H}^{(N)})$ is norm-continuous, completely positive and trace-preserving, see, e.g., Ref. 8.

Our first aim is to give a rigorous definition of unbounded generator, which has standard KLD-form (1.13) for (unbounded) operators (1.6), (1.11), (1.12), and then to construct QDS for solution (1.14). After that we check for QDS the properties quoted above for semigroup with a bounded generator.

Our next hypothesis imposed on the parameters $\sigma_\pm$ (1.11), (1.12) the conditions,

(H2)
$$0 \leqslant \sigma_+ < \sigma_-. \tag{1.15}$$

Together with **(H1)**, they play an important role in construction of semigroups $\{T^{\sigma}_{k,s}\}_{s\geq 0}$ with the trace-preserving property, cf. Theorem 2.7. Under these hypotheses, complete positivity of the dual semigroups $\{T^{\sigma\,*}_{k,s}\}_{s\geq 0}$ is established in Section III B.

II. MINIMAL DYNAMICAL SEMIGROUP

A. Unbounded generators

First, we define operators related to Hamiltonian (1.6) in Hilbert space (1.1), which we denote now on by $\mathscr{H} \equiv \mathscr{H}^{(N)}$,

$$K_0 = \frac{\sigma_+}{2} b_0 b_0^* + \frac{\sigma_-}{2} b_0^* b_0 + i\left((E-\epsilon) b_0^* b_0 + \epsilon\, \hat{n}\right),\;\; \hat{n} = \sum_{j=0}^{N} b_j^* b_j, \tag{2.1}$$

$$K_n = K_0 + i\,\eta(b_0^* b_n + b_n^* b_0) = \frac{1}{2}\, Q^*(\mathbb{1}) + i\, H_n,\; n = 1,2,\ldots,N. \tag{2.2}$$

Here $E, \epsilon, \eta > 0$ and $\sigma_\pm$ satisfy **(H1)** and **(H2)**, respectively. Domains of these operators are identical to $\mathcal{D}_0$ (1.5), which is dense in $\mathscr{H}$.

Lemma 2.1. For $n = 1,2,\ldots,N$, the operator K_n is m-accretive.

For the proof, see the Appendix.

It is known that for any *m-accretive* A in a Hilbert space, the operator $(-A)$ is the generator of a one-parameter *Strongly Continuous Contraction Semigroup* (SCCS) $\{e^{-tA}\}_{t\geqslant 0}$ on the Hilbert space, in general, e.g., Refs. 15 and 24. Then Lemma 2.1 implies

Corollary 2.2. The operator $(-K_n)$ is generator of a SCCS $\{e^{-t\,K_n}\}_{t\geqslant 0}$ on $\mathscr{H}$ for $n = 1,2,\ldots,N$.

Next we make precise definition of operators (1.13). Since the operators $\{b_n, b_n^*\}_{n=0}^N$ in $\mathscr{H}$ are unbounded, operators (1.13) in the Banach space $\mathfrak{C}_1(\mathscr{H})$ are also unbounded. Let $\Phi : \mathfrak{C}_1(\mathscr{H}) \to \mathfrak{C}_1(\mathscr{H})$ be the positive injection defined by $\Phi(\rho) = (\mathbb{1} + \hat{n})^{-1} \rho (\mathbb{1} + \hat{n})^{-1}$, and we put $\widetilde{\mathscr{D}} := \Phi(\mathfrak{C}_1(\mathscr{H}))$. Note that $\hat{n}$ is a non-negative self-adjoint operator on domain $\mathcal{D}_0$. In fact,

$$\psi_m = \frac{b_0^{*m_0} b_1^{*m_1} \ldots b_N^{*m_N}}{\sqrt{m_0!\, m_1!\, \ldots\, m_N!}}\, \Omega_F \tag{2.3}$$

is the eigenvector of $\hat{n}$ with eigenvalue $\sum_{k=0}^N m_k$ for $m = (m_0, \ldots, m_N) \in \mathbb{Z}_+^{N+1}$. And the set of vectors (2.3) for $m \in \mathbb{Z}_+$ form a *Complete Ortho-Normal System* (CONS) of $\mathscr{H}$.

Note that operators (2.2) are relatively bounded with respect to $(\mathbb{1} + \hat{n})$, i.e., $\|K_n \psi\| \leq \alpha\, \|(\mathbb{1} + \hat{n})\psi\|$, $\psi \in \mathcal{D}_0$ hold for some $\alpha > 0$. Then taking into account that operators $b_0(\mathbb{1} + \hat{n})^{-1}$ and $b_0^*(\mathbb{1} + \hat{n})^{-1}$ are bounded, the *restriction* $L_{\sigma,n}\restriction_{\widetilde{\mathscr{D}}}$ to the set $\widetilde{\mathscr{D}} = \Phi(\mathfrak{C}_1(\mathscr{H}))$ of unbounded in $\mathfrak{C}_1(\mathscr{H})$ operator

$$L_{\sigma,n}(\rho) = -K_n \rho - \rho K_n^* + \sigma_- b_0 \rho b_0^* + \sigma_+ b_0^* \rho b_0, \tag{2.4}$$

(see (1.13)) can be defined for any $n = 1,2\ldots,N$ as

$$\begin{aligned} L_{\sigma,n}(\Phi(\rho)) &= -K_n(\mathbb{1} + \hat{n})^{-1} \rho (\mathbb{1} + \hat{n})^{-1} - (\mathbb{1} + \hat{n})^{-1} \rho (K_n(\mathbb{1} + \hat{n})^{-1})^* \\ &+ \sigma_- b_0(\mathbb{1} + \hat{n})^{-1} \rho (b_0(\mathbb{1} + \hat{n})^{-1})^* + \sigma_+ b_0^*(\mathbb{1} + \hat{n})^{-1} \rho (b_0^*(\mathbb{1} + \hat{n})^{-1})^*. \end{aligned} \tag{2.5}$$

Note that the domain $\widetilde{\mathscr{D}}$ is dense in $\mathfrak{C}_1(\mathscr{H})$ since it contains all finite-rank operators, which are the vectors of $\mathcal{D}_0$.

B. Dynamical semigroup on the space of density matrices

To construct dynamical semigroups (DSs) with the generators, which are extensions of (2.4), we recall some results of the Kato-Davies analysis in Refs. 14 and 9. Since these results are applicable to any $n = 1,2,\ldots N$, we describe them only for $n = 1$ and for the corresponding semigroup.

First, we note that the operator K_1 (2.2) satisfies for all $\varphi,\psi \in \mathcal{D}_0$ (1.5) the (conservation) identity,

$$-(K_1\varphi,\psi) - (\varphi, K_1\psi) + \sigma_-(b_0\varphi, b_0\psi) + \sigma_+(b_0^*\varphi, b_0^*\psi) = 0. \tag{2.6}$$

Let $V := \mathfrak{C}_1^{sa}(\mathscr{H})$ denote the Banach subspace of all *self-adjoint* elements of $\mathfrak{C}_1(\mathscr{H})$. The family of maps

$$S_t(\rho) = e^{-tK_1}\,\rho\,(e^{-tK_1})^* \;\; (t \geqslant 0,\; \rho \in V) \tag{2.7}$$

defines a positive SCCS on V. Let the closed operator Z be generator of S_t and dom(Z) denote its domain. Then the set $\mathscr{D} := \Psi(V) = (\mathbb{1} + K_1)^{-1}V((\mathbb{1} + K_1)^{-1})^* \subset \mathrm{dom}(Z)$ is dense in V and

$$Z(\rho) = -K_1\rho - \rho K_1^* \quad \text{holds for } \rho \in \mathscr{D}. \tag{2.8}$$

Since the SCCS semigroup $\{S_t\}_{t\geq 0}$ commutes with the map Ψ, one gets $S_t(\mathscr{D}) \subseteq \mathscr{D}$. Hence the set $\mathscr{D}$ is a *core* of the generator Z. Note also that $S_t(\mathrm{dom}(Z)) \subseteq \mathrm{dom}(Z)$ and that $\mathscr{D} = \widetilde{\mathscr{D}} \cap V$.

There are two positive Z-bounded operators J_- and J_+ on dom(Z) such that

$$J_-(\rho) = b_0\rho b_0^*, \quad J_+(\rho) = b_0^*\rho b_0 \qquad \text{for} \qquad \rho \in \mathscr{D}. \tag{2.9}$$

Then, the operator $\hat{L} := Z + \sigma_-J_- + \sigma_+J_+$ is defined on the domain dom(Z). Whereas we denote by $L := L_{\sigma,n=1}\restriction_{\widetilde{\mathscr{D}}}$ operator (2.4) for $n = 1$ with domain $\widetilde{\mathscr{D}}$. Here we understand (2.8) and (2.9) as in (2.5). Then the conservation identity

$$\mathrm{Tr}_{\mathscr{H}}(\hat{L}(\rho)) = 0 \quad \text{holds for } \rho \in \mathrm{dom}\,(Z), \tag{2.10}$$

and the operator $J := (\sigma_-J_- + \sigma_+J_+)$ is Z-bounded with the relative bound equals to *one*, which require non-perturbative arguments to construct a DS corresponding to $\hat{L}$.

Proposition 2.3. For any $r \in [0,1)$ the operator $Z + r(\sigma_-J_- + \sigma_+J_+)$ with domain dom(Z) *is the generator of a positive SCCS $\{T_{t,r}\}_{t\geqslant 0}$ on V.*

Proposition 2.4. There exists a positive SCCS $\{T_t\}_{t\geqslant 0}$ on V such that

$$\lim_{r\to 1} T_{t,r}(\rho) = T_t(\rho)\,,\; \rho \in V\,,$$

uniformly on each compact interval of $t \geqslant 0$. The generator M of T_t is a closed extension of the operator $\hat{L} = Z + (\sigma_-J_- + \sigma_+J_+)$ with dom$(\hat{L})$ = dom(Z).

Remark 2.5. Since perturbation J has relative bound 1, the operator $\hat{L}$ may have many closed extensions.[15] The semigroup constructed in Proposition 2.4 is minimal in the following sense: if the SCCS $\{T_t'\}_{t\geqslant 0}$ has the generator M', which is another extension of $\hat{L}$, then $T_t' > T_t$ holds for all $t > 0$. Moreover, in spite of (2.6), or conservation identity (2.10), the minimal DS need not be trace-preserving.

Proposition 2.6. If dom(Z) *is a core of the generator M, then the minimal semigroup $\{T_t\}_{t\geqslant 0}$ is trace-preserving, i.e., a Markovian semigroup.*

Now we come back to contraction of DS for our concrete open system (1.6) and (1.13) described by the master equation with the formal individual generators $L_{\sigma,n}$ (2.4) defined on the set $\widetilde{\mathscr{D}}$, or on the subset $\mathscr{D} = \widetilde{\mathscr{D}} \cap V$.

Theorem 2.7. *For each $n = 1,2\ldots,N$, the closure of the operator $L_{\sigma,n}\restriction_{\mathscr{D}}$ is the generator of a trace-preserving SCCS on V.*

Proof. As we noted above, it is enough to consider only the case $n = 1$, i.e., the operator $L\restriction_{\mathscr{D}} = (Z + \sigma_-J_- + \sigma_+J_+)\restriction_{\mathscr{D}}$.

(a) We start by checking that dom(Z) is a core of M constructed in Proposition 2.4. To this aim, we define on V the SCCS $\{R_s\}_{s\geq 0}$,

$$R_s(\rho) = e^{-s\hat{n}}\rho e^{-s\hat{n}}, \;\; (s \geqslant 0,\; \rho \in V).$$

Note that $R_{s>0}(V) \subset \Psi(V) = \mathscr{D}$. This implies

$$R_s(\mathrm{dom}(Z)) \subseteq \mathrm{dom}(Z). \tag{2.11}$$

Moreover,

$$R_s(S_t(\rho)) = S_t(R_s(\rho)) \qquad \text{for} \;\; \rho \in V, \text{ and } \; t, s \geq 0,$$

which by (2.11) and differentiation at $t = +0$ give

$$R_s(Z(\rho)) = Z(R_s(\rho)) \quad \text{for} \quad \rho \in \mathrm{dom}(Z). \tag{2.12}$$

By explicit calculations and by (2.11), one finds that for Z-bounded operators (2.9),

$$J_+(R_s(\rho)) = e^{2s} R_s(J_+(\rho)) \;\text{ and }\; J_-(R_s(\rho)) = e^{-2s} R_s(J_-(\rho)) \tag{2.13}$$

hold for all $\rho \in \mathrm{dom}\,(Z)$. Then (2.12) and (2.13) yield on $\mathrm{dom}(Z)$ the equation

$$(Z + \sigma_+ J_+ + \sigma_- J_-)R_s = R_s(Z + e^{2s}\sigma_+ J_+ + e^{-2s}\sigma_- J_-). \tag{2.14}$$

Now we introduce the operators $\tilde{K}_0$ and $\tilde{K}_1$, which are defined via replacing parameters $\sigma_\pm, E, \epsilon$ and η in K_0 and K_1 (see (2.1), (2.2)) by $\tilde{\sigma}_\pm = e^{\pm 2s}\sigma_\pm$, $\tilde{E} = r(s)E$, $\tilde{\epsilon} = r(s)\epsilon$, and $\tilde{\eta} = r(s)\eta$. Here

$$r(s) := \frac{e^{2s}\sigma_+ + e^{-2s}\sigma_-}{\sigma_+ + \sigma_-}. \tag{2.15}$$

To keep $r(s) \in (0,1)$, we set $s \in (0, 2^{-1}\log \sigma_-/\sigma_+)$. Note that this is possible due to hypothesis **(H2)**: $0 \leqslant \sigma_+ < \sigma_-$, and that the limit $\lim_{s\downarrow 0} r(s) = 1$.

By (2.15) and definitions (2.2), (2.8), one gets the identities

$$K_1 = \frac{\tilde{K}_1}{r(s)} - \frac{\sigma_+\sigma_-}{\sigma_+ + \sigma_-}\frac{\sinh 2s}{r(s)}\,\mathbb{1} \;\; , \;\; Z = \frac{\tilde{Z}}{r(s)} + \frac{2\sigma_+\sigma_-}{\sigma_+ + \sigma_-}\frac{\sinh 2s}{r(s)}\,\mathbb{1}. \tag{2.16}$$

which hold on $\mathrm{dom}(Z)$ by the closure. Here operator $\tilde{Z}$ is given by the same expression as (2.8), but with $\tilde{K}_1$ instead of K_1. Therefore, the equality

$$Z + \tilde{\sigma}_+ J_+ + \tilde{\sigma}_- J_- = \frac{1}{r(s)}\Big[\frac{2\sigma_+\sigma_-}{\sigma_+ + \sigma_-}\sinh 2s\;\mathbb{1} + \tilde{Z} + r(s)(\tilde{\sigma}_+ J_+ + \tilde{\sigma}_- J_-)\Big],$$

also holds on $\mathrm{dom}(Z)$. Together with (2.14), this yields on $\mathrm{dom}(Z)$ the relation

$$\begin{aligned}&(\lambda\mathbb{1} - Z - \sigma_+ J_+ - \sigma_- J_-)\, R_s = \\ &\frac{1}{r(s)} R_s\Big[\Big(r(s)\lambda - \frac{2\sigma_+\sigma_-}{\sigma_+ + \sigma_-}\sinh 2s\Big)\mathbb{1} - \tilde{Z} - r(s)(\tilde{\sigma}_+ J_+ + \tilde{\sigma}_- J_-)\Big].\end{aligned} \tag{2.17}$$

Now, for arbitrary $\lambda > 0$, we can choose $s \in (0,1)$ small enough such that

$$r(s)\lambda - \frac{2\sigma_+\sigma_-}{\sigma_+ + \sigma_-}\sinh 2s > 0. \tag{2.18}$$

By Proposition 2.3, the operator $\tilde{Z} + r(s)(\tilde{\sigma}_+ J_+ + \tilde{\sigma}_- J_-)$ is generator of a SCCS. Hence by the Hille-Yosida theorem, its resolvent set includes $\mathbb{C}_+ := \{\, z \in \mathbb{C}\,|\, \mathrm{Re}\, z > 0\,\}$. This together with (2.18) imply that the last factor in the right-hand side of (2.17) is invertible and that the range $\mathrm{ran}((\lambda\mathbb{1} - Z - \sigma_+ J_+ - \sigma_- J_-)\, R_s\,(V))$ of the operator in the left-hand side coincides with the set $R_s\,(V)$. Since for the SCCS $\{R_s\}_{s\geq 0}$ it is dense in V, the range of $(\lambda\mathbb{1} - Z - \sigma_+ J_+ - \sigma_- J_-) = \lambda\mathbb{1} - \hat{L}$ is also dense in V.

Note that by Proposition 2.4, the operator $\hat{L} = Z + \sigma_+ J_+ + \sigma_- J_-$ on the domain $\mathrm{dom}\,(Z)$ is closable since it has the closed extension M. Let $M_0 = \overline{\hat{L}}$ be the closure of $\hat{L}$. Then we have $\lambda\mathbb{1} - M \supseteq \lambda - M_0 \supset \lambda\mathbb{1} - \hat{L}$, which implies for $\lambda > 0$,

$$(\lambda\mathbb{1} - M)^{-1} \supset (\lambda\mathbb{1} - M_0)^{-1} \supset (\lambda\mathbb{1} - \hat{L})^{-1}. \tag{2.19}$$

By the conclusion in the previous paragraph, the domain of the last operator in (2.19) is dense. Hence, by Proposition 2.4, the first operator in (2.19) is a closed bounded extension of the last. Since the second operator $(\lambda\mathbb{1} - M_0)^{-1}$ is the closure of the last one and since it is a restriction of the bounded operator $(\lambda \mathbb{1} - M)^{-1}$, then it is also a bounded operator on V. This yields $M = M_0$ and implies that the minimal semigroup constructed by Kato regularisation (Proposition 2.6) from $\hat{L}$ is trace-preserving.

(b) To finish the proof, one has to show that $\mathscr{D}$ is a core of M. To this end, we recall that since the SCCS semigroup $\{e^{t\,Z}\}_{t\geq 0}$ (2.7) commutes with the map Ψ, the set $\mathscr{D} = \Psi(V)$ is a core of Z and that in (a) we established that dom (Z) is a core of $M = \overline{\hat{L}}$. Then, since the restrictions: $\hat{L}\restriction_{\mathscr{D}} = L\restriction_{\mathscr{D}}$, and the operators $J_\pm$ are Z-bounded, the closure $\overline{(Z + \sigma_- J_- + \sigma_+ J_+)\restriction_{\mathscr{D}}}$ coincides with M. This completes the proof of the theorem for $n = 1$. □

Remark 2.8. The set of density matrices $\{\, \rho \in \mathfrak{C}_1(\mathscr{H}) \mid \rho \geqslant 0, \mathrm{Tr}_{\mathscr{H}}\, \rho = 1 \,\} \subset V$ *is obviously invariant subset of* $\mathfrak{C}_1(\mathscr{H})$ *for the Markov Dynamical Semigroups (MDS)* $\{T^{\sigma}_{n,t}\}_{t\geqslant 0}$, $n = 1, 2, \ldots, N$. *On the other hand, the semigroups* $\{T^{\sigma}_{n,t}\}_{t\geqslant 0}$ *can be extended to the MDS on the Banach space* $\mathfrak{C}_1(\mathscr{H})$ *by linearity.*

III. MARKOV DYNAMICAL SEMIGROUP ON DUAL SPACE

A. Dual dynamics

Equivalent and often more convenient description of the evolution $\rho \mapsto T^{\sigma}_{t,0}(\rho), \rho \in \mathfrak{C}_1(\mathscr{H})$ is the dual evolution $\{T^{\sigma\,*}_{t,0}\}_{t\geqslant 0}$ on the dual space $\mathfrak{C}^*_1(\mathscr{H}) \simeq \mathcal{L}(\mathscr{H})$.

For repeated perturbation, we have to study semigroups $\{T^{\sigma\,*}_{n,t}\}_{t\geqslant 0}$ dual to the SCCS $\{T^{\sigma}_{n,t}\}_{t\geqslant 0}$ constructed in Theorem 2.7,

$$\langle T^{\sigma}_{n,t}(\rho) \mid A\rangle_{\mathscr{H}} = \langle \rho \mid T^{\sigma\,*}_{n,t}(A)\rangle_{\mathscr{H}} \quad \text{for} \quad (\rho, A) \in \mathfrak{C}_1(\mathscr{H}) \times \mathcal{L}(\mathscr{H}),\ n = 1, \ldots, N. \tag{3.1}$$

Since the maps $T^{\sigma}_{n,t}$ are trace-preserving, the dual semigroups are *unital* (unity-preserving) contractions. They are also called the MDSs.

Because the semigroup $\{T^{\sigma}_{n,t}\}_{t\geqslant 0}$ has unbounded generator, the adjoint semigroup $\{T^{\sigma\,*}_{n,t}\}_{t\geqslant 0}$ is not strongly continuous on the dual space $\mathcal{L}(\mathscr{H})$. Duality relation (3.1) and the strong continuity of semigroup $\{T^{\sigma}_{n,t}\}_{t\geqslant 0}$ merely imply the weak*-continuity of $T^{\sigma\,*}_{n,t}$ on $\mathcal{L}(\mathscr{H})$. Therefore, the pair $(\mathcal{L}(\mathscr{H}), T^{\sigma\,*}_{n,t})$ is a W^*-dynamical system.

Let $\mathscr{A}(\mathscr{H})$ denote the Weyl CCR-algebra on $\mathscr{H}$. This unital algebra is generated as operator-norm closure of the linear span $\mathscr{A}_{\mathrm{fin}}(\mathscr{H})$ of the Weyl operators

$$W(\zeta) = \exp[i(\langle \zeta, b\rangle + \langle b, \zeta\rangle)/\sqrt{2}], \tag{3.2}$$

where the sesquilinear form notations

$$\langle \zeta, b\rangle := \sum_{j=0}^{N} \bar{\zeta}_j b_j, \qquad \langle b, \zeta\rangle := \sum_{j=0}^{N} \zeta_j b^*_j \tag{3.3}$$

are used. We comment that CCR (1.3) has the Weyl form:

$$W(\zeta_1)W(\zeta_2) = e^{-i\,\mathrm{Im}\langle \zeta_1, \zeta_2\rangle/2}\, W(\zeta_1 + \zeta_2) \qquad \text{for} \qquad \zeta_1, \zeta_2 \in \mathbb{C}^{N+1}. \tag{3.4}$$

and the algebra $\mathscr{A}(\mathscr{H})$ is dense subset of $\mathcal{L}(\mathscr{H})$ in the weak as well as in the strong operator topologies. (see, e.g., Ref. 2, Lectures 4 and 5).

In the rest of this section, we give the explicit form for the action of $\{T^{\sigma\,*}_{n,t}\}_{t\geqslant 0}$ for $1 \leqslant n \leqslant N$ on the Weyl operators. To this aim, we introduce $(N+1) \times (N+1)$ Hermitian matrices J_n, X_n, and Y_n by

$$(J_n)_{jk} = \begin{cases} 1 & (j = k = 0 \text{ or } j = k = n) \\ 0 & \text{otherwise} \end{cases}, \tag{3.5}$$

$$(X_n)_{jk} = \begin{cases} (E-\epsilon)/2 & (j,k) = (0,0) \\ -(E-\epsilon)/2 & (j,k) = (n,n) \\ \eta & (j,k) = (0,n) \ , \\ \eta & (j,k) = (n,0) \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

and

$$Y_n = \epsilon I + \frac{E-\epsilon}{2} J_n + X_n \qquad \text{for} \qquad n = 1, \dots, N, \tag{3.7}$$

where I is the $(N+1)\times(N+1)$ identity matrix. By P_0 we denote the $(N+1)\times(N+1)$ matrix: $(P_0)_{jk} = \delta_{j0}\delta_{k0}\,(j,k = 1,2,\dots,N)$. Then Hamiltonian (1.6) takes the form

$$H_n = \sum_{j,k=0}^{N} (Y_n)_{jk} b_j^* b_k. \tag{3.8}$$

Theorem 3.1. *For $n = 1,2,\dots,N$, the action of $\{T_{n,t}^{\sigma\,*}\}_{t\geqslant 0}$ on the Weyl operator has the form*

$$T_{n,t}^{\sigma\,*}(W(\zeta)) = \Gamma_{n,t}^{\sigma}(\zeta) W(U_n^{\sigma}(t)\zeta)\ , \ \ \zeta \in \mathbb{C}^{N+1}\ , \tag{3.9}$$

$$\Gamma_{n,t}^{\sigma}(\zeta) = \exp\Big[-\frac{1}{4}\, \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} (\langle \zeta, \zeta\rangle - \langle U_n^{\sigma}(t)\zeta, U_n^{\sigma}(t)\zeta\rangle)\Big], \tag{3.10}$$

and

$$U_n^{\sigma}(t) = \exp\Big[\, it\Big(Y_n + i\,\frac{\sigma_- - \sigma_+}{2} P_0\Big)\Big]. \tag{3.11}$$

Remark 3.2. The main effect of non-zero $\sigma_{\mp}$, in comparison to the case $\sigma_{\mp} = 0$,[20] *may be summarised as an imaginary shift of the energy parameter,*

$$E \to E_{\sigma} := E + i\,\frac{\sigma_- - \sigma_+}{2}, \quad 0 \le \sigma_+ < \sigma_-.$$

*Note that by **(H2)** $\mathrm{Im}(E_{\sigma}) > 0$. Thereby the semigroup $\{U_n^{\sigma}(t)\}_{t\geqslant 0}$ is contraction.*

Proof (of Theorem 3.1). Without loss of generality, we only consider $n = 1$. We put

$$\Omega(t) = \Gamma_{1,t}^{\sigma}(\zeta) \quad \text{and} \quad \zeta(t) = U_1^{\sigma}(t)\zeta. \tag{3.12}$$

1° The operator-valued equation

$$\partial_t(\Omega(t)W(\zeta(t))) = \Omega(t)\Big(i[H_1, W(\zeta(t))] + \sigma_-\, b_0^*\, W(\zeta(t))\, b_0 \\ -\frac{\sigma_-}{2}\{b_0^* b_0, W(\zeta(t))\} + \sigma_+\, b_0\, W(\zeta(t))\, b_0^* - \frac{\sigma_+}{2}\{b_0 b_0^*, W(\zeta(t))\}\Big). \tag{3.13}$$

holds on $\mathcal{D}_0$ (1.5). Here the derivative in the left-hand side is valid in the strong-operator convergence sense. Note that equation (3.13) follows straightforwardly from the formulae

$$\partial_t W(\zeta(t)) = \Big(i\frac{\langle \partial_t \zeta(t), b\rangle + \langle b, \partial_t\zeta(t)\rangle}{\sqrt{2}} \\ +\frac{1}{2}\Big[i\frac{\langle \zeta(t), b\rangle + \langle b, \zeta(t)\rangle}{\sqrt{2}}, i\frac{\langle \partial_t \zeta(t), b\rangle + \langle b, \partial_t\zeta(t)\rangle}{\sqrt{2}}\Big]\Big) W(\zeta(t))$$

and

$$[b_k, W(\zeta(t))] = i\,\frac{\zeta(t)_k}{\sqrt{2}} W(\zeta(t))\ , \quad [b_k^*, W(\zeta(t))] = -i\,\frac{\overline{\zeta(t)_k}}{\sqrt{2}} W(\zeta(t)),$$

which have sense on the number-operator domain $\mathcal{D}_0$. See, e.g., Ref. 2, Lecture 5.

2° For any $\rho \in \mathscr{D} = (\mathbb{1} + \hat{n})^{-1} V (\mathbb{1} + \hat{n})^{-1}$, the following equality holds:

$$\partial_t\, \mathrm{Tr}[\rho\, \Omega(t) W(\zeta(t))] = \mathrm{Tr}[(L_{\sigma,1}\rho)\Omega(t)W(\zeta(t))]. \tag{3.14}$$

In fact, let $\rho = (\mathbb{1} + \hat{n})^{-1}\nu(\mathbb{1} + \hat{n})^{-1}$, where $\nu \in V$ is approximated by a family of finite-rank self-adjoint operators $\{\nu_k\}_{k\geqslant 1}$, i.e., $\nu_k \to \nu$, when $k \to \infty$, in the trace-norm topology. Then from 1° with the help of (2.5), we obtain

$$\partial_t \mathrm{Tr}[(\mathbb{1} + \hat{n})^{-1}\nu_k(\mathbb{1} + \hat{n})^{-1}\Omega(t)W(\zeta(t))] = \mathrm{Tr}\big[L_{\sigma,1}\big((\mathbb{1} + \hat{n})^{-1}\nu_k(\mathbb{1} + \hat{n})^{-1}\big)\Omega(t)W(\zeta(t))\big].$$

Note that one also gets that the limit

$$\lim_{k\to\infty} L_{\sigma,1}\big((\mathbb{1} + \hat{n})^{-1}\nu_k(\mathbb{1} + \hat{n})^{-1}\big) = L_{\sigma,1}\big((\mathbb{1} + \hat{n})^{-1}\nu(\mathbb{1} + \hat{n})^{-1}\big)$$

in the trace-norm, since by (2.5), the expression of $L_{\sigma,1}\big((\mathbb{1} + \hat{n})^{-1}(\nu_k - \nu)(\mathbb{1} + \hat{n})^{-1}\big)$ is the sum of the products of $\nu_k - \nu$ and k-independent bounded operators . Then

$$\partial_t \mathrm{Tr}[(\mathbb{1} + \hat{n})^{-1}\nu_k(\mathbb{1} + \hat{n})^{-1}\Omega(t)W(\zeta(t))] \to \mathrm{Tr}\big[L_{\sigma,1}\big((\mathbb{1} + \hat{n})^{-1}\nu(\mathbb{1} + \hat{n})^{-1}\big)\Omega(t)W(\zeta(t))\big]$$

holds uniformly in t. On the other hand, the limit

$$\mathrm{Tr}[(\mathbb{1} + \hat{n})^{-1}\nu_k(\mathbb{1} + \hat{n})^{-1}\Omega(t)W(\zeta(t))] \to \mathrm{Tr}[(\mathbb{1} + \hat{n})^{-1}\nu(\mathbb{1} + \hat{n})^{-1}\Omega(t)W(\zeta(t))]$$

also holds for $k \to \infty$ uniformly in t. Then we obtain the assertion by the standard argument on differentiation under the limit.

3° Equality (3.14) also holds for $\rho \in \mathrm{dom}\, \overline{L_{\sigma,1}\restriction_{\mathscr{D}}}$. Here $\overline{L_{\sigma,1}\restriction_{\mathscr{D}}}$ denotes the closure of the restriction $L_{\sigma,1}\restriction_{\mathscr{D}}$ (cf. Theorem 2.7).

In fact, for any $\rho \in \mathrm{dom}\, \overline{L_{\sigma,1}\restriction_{\mathscr{D}}}$, there exists a sequence $\{\rho_k\}_{k\geqslant 1} \subset \mathscr{D}$ such that

$$\rho_k \to \rho, \quad L_{\sigma,1}\restriction_{\mathscr{D}} \rho_k \to \overline{L_{\sigma,1}\restriction_{\mathscr{D}}}\, \rho\,,$$

as $k \to \infty$, in the trace-norm topology. Then we obtain the assertion by differentiation under the limit as in 2°.

4° For each $\rho \in \mathrm{dom}\, \overline{L_{\sigma,1}\restriction_{\mathscr{D}}}$, $\zeta \in \mathbb{C}^{N+1}$ and $t \geqslant 0$, the following equality holds:

$$\mathrm{Tr}\,[T^{\sigma}_{1,t}(\rho)W(\zeta)] = \mathrm{Tr}\,[\rho\,\Omega(t)W(\zeta(t))]. \tag{3.15}$$

To this aim, we define the function

$$f(s,t) := \mathrm{Tr}\,[T^{\sigma}_{1,s}(\rho)\Omega(t)W(\zeta(t))] \quad \text{for} \quad s,t \geqslant 0.$$

Then Theorem 2.7 and the Hille-Yosida theorem yield $T^{\sigma}_{1,s}(\rho) \in \mathrm{dom}\, \overline{L_{\sigma,1}\restriction_{\mathscr{D}}}$ and $\partial_s f(s,t)$ $= \mathrm{Tr}\,[\overline{L_{\sigma,1}\restriction_{\mathscr{D}}}(T^{\sigma}_{1,s}(\rho))\Omega(t)W(\zeta(t))]$, which is equal to $\partial_t f(s,t)$ by 3°. Then we obtain $\partial_s f(t-s,s) =$ 0 and assertion (3.15) follows from $f(t,0) = f(0,t)$.

5° Since $T^{\sigma}_{1,t}$ is bounded and $\mathrm{dom}\, \overline{L_{\sigma,1}\restriction_{\mathscr{D}}}$ is dense in V, (3.15) holds for any $\rho \in V$. Note that any $\rho \in \mathfrak{C}_1(\mathscr{H})$ can be presented as a linear combination of elements from V. The theorem then follows by Remark 2.8 and by duality (3.1). □

From (1.14), the dual evolution map for the repeated perturbation is given by

$$T^{\sigma\,*}_{t,0} = T^{\sigma\,*}_{1} \cdots T^{\sigma\,*}_{n-1} T^{\sigma\,*}_{n,\nu(t)}, \tag{3.16}$$

where $t = (n-1)\tau + \nu(t)$, $n \leqslant N$.

Corollary 3.3. The composition of dual evolutions (3.16) on the Weyl operator is

$$T^{\sigma\,*}_{N\tau,0}(W(\zeta)) = \exp\Big[-\frac{1}{4}\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\big(\langle\zeta,\zeta\rangle - \langle U^{\sigma}_1\cdots U^{\sigma}_N\,\zeta, U^{\sigma}_1\cdots U^{\sigma}_N\,\zeta\rangle\big)\Big] W(U^{\sigma}_1\cdots U^{\sigma}_N\,\zeta)\,, \tag{3.17}$$

where we denote $U^{\sigma}_n := U^{\sigma}_n(\tau)$.

To illustrate the above statements by an example, we consider the evolution of the initial state given by product of the Gibbs states,

$$\rho = \rho_0 \otimes \bigotimes_{k=1}^{N} \rho_k\,,\ \rho_0 = e^{-\beta_0 a^*a}/Z(\beta_0)\,,\ \rho_j = e^{-\beta a^*a}/Z(\beta)\,(j = 1,2\ldots,N), \tag{3.18}$$

which has the characteristic function (see Ref. 20),

$$\omega_\rho(W(\zeta)) = \langle \rho \,|\, W(\zeta)\rangle = \exp\Big[-\frac{|\zeta_0|^2}{4}\Big(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big) - \frac{\langle\zeta,\zeta\rangle}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big]. \tag{3.19}$$

From Corollary 3.3, we obtain the following proposition about time evolution of the Gibbs state for the open system $(\mathcal{S}+\mathcal{R})+C$.

Proposition 3.4. Let ρ be initial density matrix (3.18). Then

$$\omega_{T^{\sigma}_{N\tau,0}\rho}(W(\zeta)) = \langle \rho \,|\, T^{\sigma\,*}_{N\tau,0}(W(\zeta))\rangle = \exp\Big[-\frac{1}{4}\langle\zeta, X^{\sigma}(N\tau)\zeta\rangle\Big],$$

where $X^{\sigma}(N\tau)$ is the $(N+1)\times(N+1)$ matrix given by

$$X^{\sigma}(N\tau) = U^{\sigma\,*}_N \cdots U^{\sigma\,*}_1\Big[\Big(-\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} + \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big)I + \Big(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big)P_0\Big]$$
$$\times U^{\sigma}_1 \cdots U^{\sigma}_N + \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} I. \tag{3.20}$$

B. Completely positive quasi-free maps and states

Let $\mathfrak{A}(\mathfrak{S},\sigma)$ be the (abstract) Weyl CCR-algebra for a linear space $\mathfrak{S}$ and a symplectic form σ on $\mathfrak{S}$. Recall that a bounded linear unital map $\mathfrak{T} : \mathfrak{A}(\mathfrak{S},\sigma) \to \mathfrak{A}(\mathfrak{S},\sigma)$ is *quasi-free* if there exists a linear map $U : \mathfrak{S} \to \mathfrak{S}$, and a map $\Gamma : \mathfrak{S} \to \mathbb{C}$ such that

$$\mathfrak{T}(W(\zeta)) = \Gamma(\zeta) W(U\zeta) \quad \text{hold for all} \quad \zeta \in \mathfrak{S}. \tag{3.21}$$

We also recall that for two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, a map $\mathfrak{T} : \mathfrak{A} \to \mathfrak{B}$ is *completely positive* (CP) if

$$\sum_{k,k'=1}^{K} y_k^* \, \mathfrak{T}(x_k^* x_{k'}) \, y_{k'} \geqslant 0 \tag{3.22}$$

holds for all $\{x_k\}_{k=1}^K \subset \mathfrak{A}$ and $\{y_k\}_{k=1}^K \subset \mathfrak{B}$ for any $K \geqslant 1$. See [Ref. 18, Ch. 8], [Refs. 1 and 4]. Using (3.21), one can define the map $\mathfrak{T}$ for a given U and Γ on the algebraic span of Weyl operators which is dense in $\mathfrak{A}(\mathfrak{S},\sigma)$. For the problem of extension of $\mathfrak{T}$ to a CP map on $\mathfrak{A}(\mathfrak{S},\sigma)$, we refer the following result in Ref. 11.

Proposition 3.5. For a given linear map $U : \mathfrak{S} \to \mathfrak{S}$, let σ_U be another symplectic form defined by

$$\sigma_U(\alpha,\beta) = \sigma(\alpha,\beta) - \sigma(U\alpha, U\beta) \qquad \text{for} \quad \alpha,\beta \in \mathfrak{S}. \tag{3.23}$$

Then the necessary and sufficient condition of that map (3.21) can be extended to a completely positive map on $\mathfrak{A}(\mathfrak{S},\sigma)$ is the existence of a state ω on $\mathfrak{A}(\mathfrak{S},\sigma_U)$ such that $\Gamma(\zeta) = \omega(W_U(\zeta))$ for Weyl operators $W_U \in \mathfrak{A}(\mathfrak{S},\sigma_U)$.

Theorem 3.6. *As a map on $\mathscr{A}(\mathscr{H})$, the dual MDS $\{T^{\sigma\,*}_{n,t}\}_{t\geq 0}$ given by duality (3.1) is quasi-free and completely positive for $n = 1,2,\ldots N$.*

Proof. It is obvious from (3.9) and its contraction property that the dual MDS maps $\mathscr{A}(\mathscr{H})$ into itself and that it is quasi-free.

For a fixed n and t, we put $U = U^{\sigma}_n(t)$. By setting $\mathfrak{S} = \mathbb{C}^{N+1}$ and $\sigma(\,\cdot\,,\,\cdot\,) = \mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle$, then $\mathfrak{A}(\mathfrak{S},\sigma) = \mathscr{A}(\mathscr{H})$ holds and the action of $T^{\sigma\,*}_{n,t}$ has form (3.21). Since U is a contraction (Remark 3.2), there is a linear map $C : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ such that

$$\langle C\alpha, C\beta\rangle = \langle\alpha,\beta\rangle - \langle U\alpha, U\beta\rangle, \quad \alpha,\beta \in \mathbb{C}^{N+1}. \tag{3.24}$$

Then we can consider the CCR-algebra $\mathfrak{A}(\mathbb{C}^{N+1},\sigma_U)$ with symplectic form

$$\sigma_U(\alpha,\beta) = \sigma(\alpha,\beta) - \sigma(U\alpha,U\beta) = \mathrm{Im}\langle C\alpha, C\beta\rangle, \tag{3.25}$$

as the C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ generated by the Weyl system $\{W(C\zeta) \mid \zeta \in \mathbb{C}^{N+1}\}$. Here $W_U(\zeta) = W(C\zeta)$ satisfies CCR with symplectic form σ_U, where $W(\zeta)$ is given by (3.2).

Let ρ be product density matrix (3.18) with $\beta_0 = \beta = \log \sigma_-/\sigma_+ > 0$ (cf. **H.2**). Then we have the corresponding normal state ω_ρ on $\mathcal{L}(\mathcal{H})$. Let ω be the restriction of ω_ρ to $\mathfrak{A}(\mathbb{C}^{N+1}, \sigma_U)$. From (3.19), one gets

$$\omega(W_U(\zeta)) = \langle \rho \,|\, W(C\zeta) \rangle = \exp\Big[- \frac{\|C\zeta\|^2}{4} \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} \Big] .$$

Comparing with $\Gamma^{\sigma}_{n,t}(\zeta)$ in (3.10) and (3.24), we see that the ω plays the role in Proposition 3.5. Then, there exists a CP map on $\mathscr{A}(\mathcal{H})$ whose action on the Weyl operators coincides with that of $T^{\sigma\,*}_{n,t}$. From the continuity, the coincidence of these maps on $\mathscr{A}(\mathcal{H})$ follows. Thus, the complete positivity of $T^{\sigma\,*}_{n,t}$ has been proved. □

Since a composition of quasi-free CP maps is clearly quasi-free and CP, (3.16) imply

Corollary 3.7. The dual evolutions $T^{\sigma\,*}_{t,0}$ is the completely positive quasi-free map on the Weyl CCR-algebra $\mathscr{A}(\mathcal{H})$ for $t \in [0, N\tau)$.

Corollary 3.8. The dual evolutions $T^{\sigma\,*}_{t,0}$ and $T^{\sigma\,*}_{n,t}$ $(n = 1, \ldots, N)$ are the completely positive maps on $\mathcal{L}(\mathcal{H})$ for $t \in [0, N\tau)$.

Proof. For $\mathfrak{T} = T^{\sigma;*}_{n,t}$, arbitrarily fixed unit vector $\varphi \in \mathcal{H}$ and $K \in \mathbb{N}$, put

$$\Phi(\{A_k\}_{k=1}^K, \{B_k\}_{k=1}^K) = (\varphi, \sum_{k,k'=1}^{K} B_k^* \,\mathfrak{T}(A_k^* A_{k'})\, B_{k'}\varphi), \tag{3.26}$$

where $\{A_k\}_{k=1}^K, \{B_k\}_{k=1}^K \subset \mathcal{L}(\mathcal{H})$. Since the CCR algebra $\mathscr{A}(\mathcal{H})$ is a dense subset of $\mathcal{L}(\mathcal{H})$ in the strong operator topology, we may take $\{A_{k,j}\}_{j\in\mathbb{N}}, \{B_{k,j}\}_{j\in\mathbb{N}} \subset \mathscr{A}(\mathcal{H})$ such that

$$\mathrm{s} - \lim_{j\to\infty} A_{k,j} = A_k \qquad \text{and} \qquad \mathrm{s} - \lim_{j\to\infty} B_{k,j} = B_k$$

for every $k = 1, \ldots, K$. Recalling that $\mathfrak{T} = T^{\sigma\,*}_{n,t}$ is CP on $\mathscr{A}(\mathcal{H})$, we have

$$0 \leqslant \Phi(\{A_{k,j}\}_{k=1}^K, \{B_{k,j}\}_{k=1}^K) = \sum_{k,k'=1}^{K} \langle T^{\sigma}_{n,t}(B_{k,j} P_\varphi B^*_{k',j}), A^*_{k,j} A_{k',j} \rangle ,$$

where P_φ is the projection operator on $\mathcal{H}$ onto its one dimensional subspace spanned by φ. Note that

$$T^{\sigma}_{n,t}(B_{k,j} P_\varphi B^*_{k',j}) \to T^{\sigma}_{n,t}(B_k P_\varphi B^*_{k'}) \qquad \text{in} \quad \mathfrak{C}_1(\mathcal{H})$$

as $j \to \infty$ since $B_{k,j} P_\varphi B^*_{k',j} \to B_k P_\varphi B^*_{k'}$ in $\mathfrak{C}_1(\mathcal{H})$ and $T^{\sigma}_{n,t}$ is bounded on $\mathfrak{C}_1(\mathcal{H})$. Note also that $A^*_{k,j} A_{k',j}$ converges to $A^*_k A_{k'}$ weakly. By the principle of uniform boundedness, $\{A^*_{k,j} A_{k',j}\}_{j\in\mathbb{N}}$ is a bounded set. Together with weak continuity of normal states $\langle \rho | \cdot \rangle : \mathcal{L}(\mathcal{H}) \to \mathbb{C}$, this yields that

$$\Phi(\{A_k\}_{k=1}^K, \{B_k\}_{k=1}^K) = \lim_{j\to\infty} \Phi(\{A_{k,j}\}_{k=1}^K, \{B_{k,j}\}_{k=1}^K) \geqslant 0.$$

Thereby we have proved the complete positivity of $\mathfrak{T} = T^{\sigma\,*}_{n,t}$ on $\mathcal{L}(\mathcal{H})$. Proofs for the other maps are almost verbatim. □

As we have seen, $\mathfrak{T}\big(\mathscr{A}(\mathcal{H})\big) \subset \mathscr{A}(\mathcal{H})$ for $\mathfrak{T} = T^{\sigma\,*}_{t,0}, T^{\sigma\,*}_{n,t}$ holds. Moreover, $\mathfrak{T}$ is positive unital map. Therefore for any state ω on $\mathscr{A}(\mathcal{H})$, $\omega \circ \mathfrak{T}$ is also a state on $\mathscr{A}(\mathcal{H})$.

Recall that a state ω on $\mathscr{A}(\mathcal{H})$ is said to be quasi-free if there exist a linear form L and a non-negative sesquilinear form q on $\mathbb{C}^{N+1}$ such that

$$\omega(W(\zeta)) = \exp[L(\zeta) - q(\zeta, \zeta)]$$

holds for every $\zeta \in \mathbb{C}^{N+1}$.[23] By (3.9) and (3.17), it is obvious that $\omega \circ \mathfrak{T}$ is quasi-free if ω is. Let us summarize them in the following assertion.

Theorem 3.9. *The operators $T^{\sigma\,*}_{t,0}, T^{\sigma\,*}_{n,t}$ $(n = 1, \ldots, N)$ map the set of quasi-free state on $\mathscr{A}(\mathcal{H})$ into itself.*

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APPENDIX: PROOF OF LEMMA 2.1

Proof. The operator K_0 with its domain $\mathcal{D}_0$ (1.5) is closed with discrete spectrum $\mathfrak{S}(K_0) \subset \mathbb{C}_+ = \{z \in \mathbb{C} | \operatorname{Re} z > 0\}$. In fact, for $m \in \mathbb{Z}^{N+1}$, $\big(2^{-1}(\sigma_+ + \sigma_-) + iE\big)m_0 + 2^{-1}\sigma_+ + i\epsilon \sum_{j=1}^{N} m_j$ is its eigenvalue and ψ_m in (2.3) is the corresponding eigenvectors. It is enough to consider the case $n = 1$ only and to prove the following three claims:[24]

(i) the operator K_1 is closed;

(ii) the *numerical range* of K_1 is contained in $\overline{\mathbb{C}}_+$;

(iii) there exists $z \in \mathbb{C}$ such that $\operatorname{Re} z > 0$ and z belongs to the *resolvent set* $\rho(-K_1)$ of the operator $-K_1$.

For (i), we have to show that there exist constants $c \in (0,1)$ and $C > 0$ such that

$$\|\eta(b_0^* b_1 + b_1^* b_0)\varphi\| \leqslant c\|K_0\varphi\| + C\|\varphi\|, \tag{A1}$$

for every $\varphi \in \mathcal{D}_0$. Estimate (A1) follows directly from the CCR number-operator bounds for bosons. By conditions $\eta^2 \leqslant E\epsilon$, **(H.1)**, and $0 < \sigma_+ + \sigma_-$, **(H.2)**, we obtain

$$c = \sqrt{2}\eta\left(E + \sqrt{E^2 + (\sigma_+ + \sigma_-)^2/4}\right)^{-1/2} \epsilon^{-1/2} < 1.$$

To show (ii), let $\varphi \in \mathcal{D}_0$, $\|\varphi\| = 1$. Then one immediately gets

$$(\varphi, K_1\varphi) = \frac{\sigma_+}{2}\|b_0^*\varphi\|^2 + \frac{\sigma_-}{2}\|b_0\varphi\|^2 + iE\|b_0\varphi\|^2 + i\epsilon\sum_{j=1}^{N}\|b_j\varphi\|^2 + 2i\eta \operatorname{Re}(b_0\varphi, b_1\varphi) \subset \overline{\mathbb{C}}_+.$$

For (iii), we note that by virtue of $\mathfrak{S}(K_0) \subset \mathbb{C}_+$, $z \in \rho(-K_0)$ and $\|(z\mathbb{1} + K_0)^{-1}\| \leqslant 1/\operatorname{Re} z$ hold, if $\operatorname{Re} z > 0$. Moreover, we get estimate $\|K_0(z\mathbb{1} + K_0)^{-1}\| \leqslant 1$, if $\operatorname{Re} z \geqslant 0$ and $\operatorname{Im} z \geqslant 0$ hold. Hence, by (A1), we obtain for this value of z that

$$\|\eta(b_0^* b_1 + b_1^* b_0)(z\mathbb{1} + K_0)^{-1}\| \leqslant c\|K_0(z\mathbb{1} + K_0)^{-1}\| + C\|(z\mathbb{1} + K_0)^{-1}\| \leqslant c + \frac{C}{\operatorname{Re} z}. \tag{A2}$$

Then, if $\operatorname{Re} z$ is large enough, the right-hand side of (A2) is less than one. For these values of z, the resolvent identity for K_1 and K_0 yields the boundedness of

$$(z\mathbb{1} + K_1)^{-1} = (z\mathbb{1} + K_0)^{-1}\big(\mathbb{1} + i\eta(b_0^* b_1 + b_1^* b_0)(z\mathbb{1} + K_0)^{-1}\big)^{-1},$$

which proves assertion (iii) and the lemma. □

[1] Alicki, R. and Fannes, M., *Quantum Dynamical Systems* (Oxford University Press, 2001).
[2] *Open Quantum Systems I, The Hamiltonian Approach*, edited by S. Attal, A. Joye, and C.-A. Pillet, Lecture Notes in Mathematics Vol. 1880 (Springer-Verlag, Berlin-Heidelberg, 2006).
[3] *Open Quantum Systems II, The Markovian Approach*, edited by S. Attal, A. Joye, and C.-A. Pillet, Lecture Notes in Mathematics Vol. 1881 (Springer-Verlag, Berlin-Heidelberg, 2006).
[4] *Open Quantum Systems III, Recent Developments*, edited by S. Attal, A. Joye, and C.-A. Pillet, Lecture Notes in Mathematics Vol. 1882 (Springer-Verlag, Berlin-Heidelberg, 2006).
[5] Arlotti, L., Lods, B., and Mokhtar-Kharroubi, M., "On perturbed substochastic semigroups in abstract state spaces," Z. Anal. Anwend. **30**, 457–495 (2011).
[6] Bruneau, L., Joye, A., and Merkli, M., "Repeated interactions in open quantum systems," J. Math. Phys. **55**, 075204 (2014).
[7] Chebotarev, A. M. and Fagnola, F., "Sufficient conditions for conservativity of minimal quantum dynamical semigroups," J. Funct. Anal. **153**, 382–404 (1998).
[8] Davies, E. B., *Quantum Theory of Open Systems* (Academic Press, London, 1976).
[9] Davies, E. B., "Quantum dynamical semigroups and the neutron diffusion equation," Rep. Math. Phys. **11**, 169–188 (1977).

[10] Davies, E. B., "Generators of dynamical semigroups," J. Funct. Anal. **34**, 421–432 (1979).
[11] Demoen, B., Vanheuverzwijn, P., and Verbeure, A., "Completely positive maps of the CCR-algebra," Lett. Math. Phys. **2**, 161–166 (1977).
[12] Evans, D. E. and Lewis, J. T., "Some semigroups of completely positive maps on the the CCR algebra," J. Funct. Anal. **26**, 369–377 (1977).
[13] Fannes, M., "Quasi-free states and automorphisms of the CCR-algebra," Commun. Math. Phys. **51**, 55–56 (1976).
[14] Kato, T., "On the semi-groups generated by Kolmogoroff's differential equations," J. Math. Soc. Jpn **6**, 1–15 (1954).
[15] Kato, T., *Perturbation Theory for Linear Operators*, 2nd ed. (Springer-Verlag, Berlin-Heidelberg, 1995).
[16] Mokhtar-Kharroubi, M., "On perturbed positive semigroups on the Banach space of trace class operators," Infinite Dimens. Anal., Quantum Probab. Relat. Top. **11**, 405–425 (2008).
[17] Nachtergaele, B., Vershynina, A., and Zagrebnov, V. A., "Non-equilibrium states of a photon cavity pumped by an atomic beam," Ann. Henri Poincare **15**, 213–262 (2014).
[18] Petz, D., *An Invitation to the Algebra of Canonical Commutation Relations* (KU Leuven University Press, Leuven, 1990).
[19] Pulè, J. V., "Positive maps of the CCR algebra with a finite number of non-zero truncated functions," Ann. Inst. Henri Poincare, Sect. A **33**(4), 395–408 (1980).
[20] Tamura, H. and Zagrebnov, V. A., "A model of non-autonomous dynamics driven by repeated harmonic interaction," Theor. Math. Phys. (to be published); e-print arXiv:1404.2998 [math.FA] (2014).
[21] Tamura, H. and Zagrebnov, V. A., "Dynamical semigroups for unbounded repeated perturbation of open system," e-print arXiv:1506.08591v1 [math.OA] (2015).
[22] Tamura, H. and Zagrebnov, V. A., "Exactly soluble quantum model for repeated harmonic perturbation," J. Stat. Mech. **2015**, P10005.
[23] Verbeure, A. F., *Many-Body Boson Systems* (Springer-Verlag, Berlin, 2011).
[24] Zagrebnov, V. A., *Topics in the Theory of Gibbs Semigroups* (KU Leuven University Press, Leuven, 2003).