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Summary: We consider a singular integral along a submanifold of finite type. We prove a certain L^p estimate for the singular integral, which is useful in applying an extrapolation method that shows L^p boundedness of the singular integral under a sharp condition of the kernel.

1 Introduction

Let $B(0, 1) = \{x \in \mathbb{R}^n : |x| < 1\}$ and let $\Phi : B(0, 1) \rightarrow \mathbb{R}^d$ be a smooth function. We assume that Φ is of finite type at the origin, that is, for any $\xi \in S^{d-1}$ (the unit sphere in $\mathbb{R}^d$) there exists a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $|\alpha| \geq 1$ and $\partial_x^\alpha \langle \Phi(x), \xi \rangle|_{x=0} \neq 0$, where $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$.

Let a function Ω in $L^1(S^{n-1})$ satisfy

$$\int_{S^{n-1}} \Omega(\theta) d\sigma(\theta) = 0, \quad (1.1)$$

where $d\sigma$ denotes the Lebesgue surface measure on the unit sphere S^{n-1} in $\mathbb{R}^n$. Throughout this note we assume $n \geq 2$. Let Δ_s , $s \geq 1$, denote the collection of functions h on $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$ satisfying

$$\|h\|_{\Delta_s} = \sup_{j \in \mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} |h(t)|^s dt/t \right)^{1/s} < \infty,$$

where $\mathbb{Z}$ denotes the set of integers. We define

$$\omega(h, t) = \sup_{|s| < tR/2} \int_R^{2R} |h(r-s) - h(r)| dr/r, \quad t \in (0, 1],$$

where the supremum is taken over all s and R such that $|s| < tR/2$ (see [6, 12]). For $\eta > 0$, let Λ^η denote the family of functions h satisfying

$$\|h\|_{\Lambda^\eta} = \sup_{t \in (0, 1]} t^{-\eta} \omega(h, t) < \infty.$$

Define a space $\Lambda_s^\eta = \Delta_s \cap \Lambda^\eta$ and set $\|h\|_{\Lambda_s^\eta} = \|h\|_{\Delta_s} + \|h\|_{\Lambda^\eta}$ for $h \in \Lambda_s^\eta$.

We consider a singular Radon transform of the form:

$$\begin{aligned} T(f)(x) &= \text{p.v.} \int_{B(0,1)} f(x - \Phi(y)) K(y) dy \\ &= \lim_{\epsilon \rightarrow 0} \int_{1 > |y| > \epsilon} f(x - \Phi(y)) K(y) dy \end{aligned} \quad (1.2)$$

for an appropriate function f on $\mathbb{R}^d$, where $K(y) = h(|y|)\Omega(y')|y|^{-n}$, $y' = |y|^{-1}y$, $h \in \Delta_1$. See Stein [13], Fan, Guo, and Pan [4], Al-Salman and Pan [1] and also [2, 5, 14] for this singular integral and related topics.

In the previous works, the operator T was studied under the condition that h is a constant function. In this note, we consider the operator T under a more general condition on h . We shall prove the following:

Theorem 1.1 *Let $q \in (1, 2]$, $\Omega \in L^q(S^{n-1})$ and $h \in \Lambda_1^\eta$ for some $\eta > 0$. Suppose that Ω satisfies the condition (1.1). Let T be defined as in (1.2). Then we have*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq C_p(q-1)^{-1} \|h\|_{\Lambda_1^\eta} \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$, where the constant C_p is independent of q , h and Ω .

Let $L \log L(S^{n-1})$ denote the Zygmund class of the functions F on S^{n-1} satisfying

$$\int_{S^{n-1}} |F(\theta)| \log(2 + |F(\theta)|) d\sigma(\theta) < \infty.$$

Then, as an application of Theorem 1.1 and extrapolation, we have the following theorem.

Theorem 1.2 *Let $h \in \Lambda_1^\eta$ for some $\eta > 0$. Suppose that Ω is in $L \log L(S^{n-1})$ and satisfies the condition (1.1). Let T be as in (1.2). Then we have*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$.

The extrapolation argument that proves Theorem 1.2 from Theorem 1.1 can be found in [8, 9, 10, 11] (see also [15, Chap. XII, pp. 119–120]). If the function h is assumed to be a constant function in Theorem 1.2, we have a result of Al-Salman and Pan shown in [1] (see [1, Theorem 1.1]); so we can give a different proof of the result by applying Theorem 1.1 and extrapolation. Relevant results can be found in [8, 9, 10, 11].

In Section 2, we shall prove Theorem 1.1. Consider a singular integral of the form

$$S(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - P(y)) h(|y|) \Omega(y') |y|^{-n} dy,$$

where $P(y)$ is a polynomial mapping from $\mathbb{R}^n$ to $\mathbb{R}^d$ satisfying $P(-y) = -P(y)$ ($P \neq 0$), $h \in \Delta_s$ for $s \in (1, 2]$ and Ω is a function in $L^q(S^{n-1})$, $q \in (1, 2]$, satisfying (1.1). Then, it has been proved that

$$\|S(f)\|_{L^p(\mathbb{R}^d)} \leq C_p(q-1)^{-1}(s-1)^{-1} \|\Omega\|_{L^q(S^{n-1})} \|h\|_{\Delta_s} \|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$, where the constant C_p is independent of q, s, Ω, h and the polynomial components of P if they are of fixed degree (see [8, Theorem 1]). Outline of our proof of Theorem 1.1 is similar to that of the proof for [8, Theorem 1]. We apply methods of [4] to obtain some basic estimates. We need to assume that $h \in \Lambda_1^\eta$ for some $\eta > 0$ to prove certain Fourier transform estimates. As in [8] (see also [9, 10]), a key idea of the proof of Theorem 1.1 is to apply a Littlewood–Paley decomposition adapted to an appropriate lacunary sequence depending on q for which $\Omega \in L^q(S^{n-1})$.

In Section 3, we shall give analogs of Theorems 1.1 and 1.2 for a maximal singular integral operator related to T . In what follows we also write $\|f\|_{L^p(\mathbb{R}^d)} = \|f\|_p$ and $\|\Omega\|_{L^q(S^{n-1})} = \|\Omega\|_q$. Throughout this note, the letter C will be used to denote non-negative constants which may be different in different occurrences.

2 Proof of Theorem 1.1

Let M be a positive integer. We write $\Phi(y) = (\Phi_1(y), \dots, \Phi_d(y))$. Let $P_j(y)$ be the Taylor polynomial of $\Phi_j(y)$ at the origin defined by

$$P_j(y) = \sum_{|\alpha| \leq M-1} \frac{1}{\alpha!} (\partial_y^\alpha \Phi_j)(0) y^\alpha,$$

where $\alpha! = \alpha_1! \dots \alpha_n!$ and $y^\alpha = y_1^{\alpha_1} \dots y_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $y = (y_1, \dots, y_n)$. We write $P(y) = (P_1(y), P_2(y), \dots, P_d(y))$ and

$$P(y) = \sum_{j=1}^{\ell} Q_j(y), \quad Q_j(y) = \sum_{|\gamma|=N(j)} a_\gamma y^\gamma \quad (a_\gamma \in \mathbb{R}^d),$$

where $0 = N(1) < N(2) < \dots < N(\ell)$, $Q_j \neq 0$ for $j \geq 2$. Let $\beta_m = \rho^{N(m)}$ and $\alpha_m = \tau(q-1)/(qN(m))$ for $2 \leq m \leq \ell$, where $\tau = 4^{-1} \min(1, \eta)$, $\rho \geq 2$. Also, let $\beta_{\ell+1} = \rho^M$ and $\alpha_{\ell+1} = \epsilon_0(q-1)/q$ for some $\epsilon_0 \in (0, 1/4)$. The positive integer M and the positive number ϵ_0 will be specified in the following (see Lemma 2.4 below).

Let T be as in Theorem 1.1. Let $E_k = \{x \in \mathbb{R}^n : \rho^k \leq |x| < \rho^{k+1}\}$, $k \in \mathbb{Z}$, $\rho \geq 2$. Then $T(f)(x) = \sum_{k=-\infty}^{-1} \sigma_k * f(x)$, where $\{\sigma_k\}_{k=-\infty}^{-1}$ is a sequence of Borel measures on $\mathbb{R}^d$ such that

$$\sigma_k * f(x) = \int_{E_k} f(x - \Phi(y)) K(y) dy. \quad (2.1)$$

Put $P^{(m)}(y) = \sum_{j=1}^m Q_j(y)$ for $m = 1, 2, \dots, \ell$ and $P^{(\ell+1)}(y) = \Phi(y)$. Consider a sequence $\mu^{(m)} = \{\mu_k^{(m)}\}_{k=-\infty}^{-1}$ of positive measures on $\mathbb{R}^d$ such that

$$\mu_k^{(m)} * f(x) = \int_{E_k} f(x - P^{(m)}(y)) |K(y)| dy$$

for $m = 1, 2, \dots, \ell + 1$. Note that $\mu_k^{(1)} = (\int_{E_k} |K(y)| dy) \delta_{P(0)}$, where δ_a is Dirac's delta function on $\mathbb{R}^d$ concentrated at a . Let $\sigma^{(m)} = \{\sigma_k^{(m)}\}_{k=-\infty}^{-1}$ be a sequence of Borel

measures on $\mathbb{R}^d$ such that

$$\sigma_k^{(m)} * f(x) = \int_{E_k} f(x - P^{(m)}(y)) K(y) dy,$$

for $m = 1, 2, \dots, \ell + 1$. We note that $\sigma_k^{(1)} = 0$ by (1.1) and

$$(\sigma_k^{(m)} * f)^\wedge(\xi) = \hat{f}(\xi) \int_{E_k} e^{-2\pi i \langle P^{(m)}(y), \xi \rangle} K(y) dy,$$

where $\hat{f}$ denotes the Fourier transform of f . A similar formula holds for $\mu_k^{(m)}$.

Let $\{\gamma(j, k)\}_{k=1}^{r_j}$ be an enumeration of $\{\gamma\}_{|\gamma|=N(j)}$ for $1 \leq j \leq \ell$. Define a linear mapping L_j from $\mathbb{R}^d$ to $\mathbb{R}^{r_j}$ by

$$L_j(\xi) = (\langle a_{\gamma(j,1)}, \xi \rangle, \langle a_{\gamma(j,2)}, \xi \rangle, \dots, \langle a_{\gamma(j,r_j)}, \xi \rangle),$$

for $1 \leq j \leq \ell$. Let $L_{\ell+1}$ be the identity mapping on $\mathbb{R}^d$. Let $s_j = \text{rank } L_j$. For $j \geq 2$, there exist non-singular linear transformations $R_j : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $H_j : \mathbb{R}^{s_j} \rightarrow \mathbb{R}^{s_j}$ such that

$$|H_j \pi_{s_j}^d R_j(\xi)| \leq |L_j(\xi)| \leq C |H_j \pi_{s_j}^d R_j(\xi)|,$$

where $\pi_{s_j}^d(\xi) = (\xi_1, \dots, \xi_{s_j})$ is the projection and C is a constant depending only on r_j (see [5]).

Let φ be a function in $C^\infty(\mathbb{R})$ satisfying $\varphi(r) = 1$ for $|r| < 1/2$ with support in $\{|r| \leq 1\}$. Define a sequence $\tau^{(m)} = \{\tau_k^{(m)}\}_{k=-\infty}^{-1}$ of Borel measures by

$$\hat{\tau}_k^{(m)}(\xi) = \hat{\sigma}_k^{(m+1)}(\xi) \Phi_{k,m+1}(\xi) - \hat{\sigma}_k^{(m)}(\xi) \Phi_{k,m}(\xi) \quad (2.2)$$

for $m = 1, 2, \dots, \ell$, where

$$\Phi_{k,m}(\xi) = \prod_{j=m+1}^{\ell+1} \varphi\left(\beta_j^k |H_j \pi_{s_j}^d R_j(\xi)|\right)$$

if $1 \leq m \leq \ell$ and $\Phi_{k,\ell+1} = 1$. Then $\sigma_k = \sigma_k^{(\ell+1)} = \sum_{m=1}^{\ell} \tau_k^{(m)}$. We note that

$$\Phi_{k,m+1}(\xi) \varphi\left(\beta_{m+1}^k |H_{m+1} \pi_{s_{m+1}}^d R_{m+1}(\xi)|\right) = \Phi_{k,m}(\xi) \quad (1 \leq m \leq \ell). \quad (2.3)$$

For $1 \leq m \leq \ell$, let $T_\rho^{(m)}(f) = \sum_{k=-\infty}^{-1} \tau_k^{(m)} * f$. Then $T = \sum_{m=1}^{\ell} T_\rho^{(m)}$.

For a sequence $\nu = \{\nu_k\}_{k=-\infty}^{-1}$ of finite Borel measures on $\mathbb{R}^d$, let $\nu^*(f)(x) = \sup_k |\nu_k| * f(x)|$, where $|\nu_k|$ denotes the total variation. We consider the maximal operators $(\mu^{(m)})^*$ ($1 \leq m \leq \ell + 1$). We also write $(\mu^{(\ell+1)})^* = \mu_\rho^*$.

Let $\theta \in (0, 1)$. For $p \in (1, \infty)$ let $p' = p/(p-1)$ and $\delta(p) = |1/p - 1/p'|$. Then we prove the following two propositions.

Proposition 2.1 *Let $p > 1 + \theta$ and $1 \leq j \leq \ell + 1$. Then we have*

$$\left\| (\mu^{(j)})^*(f) \right\|_{L^p(\mathbb{R}^d)} \leq C(\log \rho) \|h\|_{\Lambda_1^\eta} \|\Omega\|_{L^q(S^{n-1})} B^{2/p} \|f\|_{L^p(\mathbb{R}^d)}, \quad (2.4)$$

where $B = \left(1 - \rho^{-\theta\kappa/q'}\right)^{-1}$ for some positive constant κ such that

$$(1 - \beta_m^{-\theta\alpha_m})^{-1} \leq B$$

for all m with $2 \leq m \leq \ell + 1$. The constant C is independent of $q \in (1, 2]$, $h \in \Lambda_1^\eta$, $\Omega \in L^q(S^{n-1})$ and ρ .

Proposition 2.2 *Let $p \in (1 + \theta, (1 + \theta)/\theta)$ and $1 \leq m \leq \ell$. Then*

$$\|T_\rho^{(m)}(f)\|_{L^p(\mathbb{R}^d)} \leq C(\log \rho) \|h\|_{\Lambda_1^\eta} \|\Omega\|_{L^q(S^{n-1})} B^{1+\delta(p)} \|f\|_{L^p(\mathbb{R}^d)},$$

where B is as in Proposition 2.1 and the constant C is independent of $q \in (1, 2]$, $h \in \Lambda_1^\eta$, $\Omega \in L^q(S^{n-1})$ and ρ .

We can easily derive Theorem 1.1 from Proposition 2.2. Proposition 2.1 is used to prove Proposition 2.2. To prove Proposition 2.2 we also need the following.

Lemma 2.3 *Let $q \in (1, 2]$, $\Omega \in L^q(S^{n-1})$, $h \in \Lambda_1^\eta$ and $A = (\log \rho) \|h\|_{\Lambda_1^\eta} \|\Omega\|_q$. Let $\tau_k^{(m)}$ be as in (2.2). Then, for $1 \leq m \leq \ell$ we have*

$$\|\tau_k^{(m)}\| = |\tau_k^{(m)}|(\mathbb{R}^d) \leq c_1 A, \quad (2.5)$$

$$|\hat{\tau}_k^{(m)}(\xi)| \leq c_2 A \left(\beta_{m+1}^k |L_{m+1}(\xi)| \right)^{-\alpha_{m+1}}, \quad (2.6)$$

$$|\hat{\tau}_k^{(m)}(\xi)| \leq c_3 A \left(\beta_{m+1}^{k+1} |L_{m+1}(\xi)| \right)^{\alpha_{m+1}}, \quad (2.7)$$

for all $k \in \mathbb{Z}$ satisfying $k \leq L$ with some constants c_i ($1 \leq i \leq 3$), where L is a negative integer, $L \leq -4$, which will be determined in Lemma 2.4 below.

To prove Lemma 2.3 we need the following two lemmas.

Lemma 2.4 *Let $1 < q \leq 2$, $\Omega \in L^q(S^{n-1})$, $h \in \Lambda_1^\eta$ and let σ_k be as in (2.1). Then, there exist a positive integer M , a positive number $\epsilon_0 \in (0, 1/4)$ and a negative integer L , $L \leq -4$, such that*

$$|\hat{\sigma}_k(\xi)| \leq C(\log \rho) \left(|\xi| \rho^{kM} \right)^{-\epsilon_0/q'} \|h\|_{\Lambda_1^\eta} \|\Omega\|_q$$

for $k \leq L$. The constants M , ϵ_0 , L and C are independent of ρ , q , h and Ω .

Lemma 2.5 *Let $\rho \geq 2$, $k \in \mathbb{Z}$, $1 < q \leq 2$, $h \in \Lambda_1^\eta$ and $\Omega \in L^q(S^{n-1})$. Let P be a real-valued polynomial on $\mathbb{R}^n$ of degree $m \geq 1$. Write*

$$P(x) = \sum_{|\alpha|=m} a_\alpha y^\alpha + Q(y),$$

where $\deg Q \leq m - 1$ if $Q \neq 0$. Then there exists a constant $C > 0$ independent of ρ, k, q, h, Ω and the coefficients of the polynomial P such that

$$\left| \int_{\rho^k \leq |y| < \rho^{k+1}} \exp(iP(x)) h(|x|) \Omega(x') |x|^{-n} dx \right| \leq C(\log \rho) \|h\|_{\Lambda_1^\eta} \|\Omega\|_q \left(\rho^{km} \sum_{|\alpha|=m} |a_\alpha| \right)^{-\tau/(mq')},$$

where $\tau = 4^{-1} \min(1, \eta)$.

We can prove Lemma 2.5 similarly to the proof of Lemma 2.4 of [4]. To prove Lemma 2.4 we need the following two results, which can be found in [4].

Lemma 2.6 Let $\Phi : B(0, 1) \rightarrow \mathbb{R}^d$ be smooth and of finite type at the origin. Define $G_m : B(0, 1) \times S^{d-1} \rightarrow \mathbb{R}$ by

$$G_m(x, \xi) = \sum_{|\alpha|=m} \langle \xi, \partial_x^\alpha \Phi(x) \rangle x^\alpha \frac{m!}{\alpha!}$$

for $m \geq 1$. Then, there exist constants $R, \delta \in (0, 1/4)$ and a mapping ℓ from S^{d-1} to a finite set of positive integers such that

$$C_\Phi := \sup_{\xi \in S^{d-1}} \int_{|x| \leq R} |G_{\ell(\xi)}(x, \xi)|^{-\delta} dx < \infty.$$

Lemma 2.7 Let $\psi, \varphi \in C^\infty(\mathbb{R})$ be real-valued. Let $s \in (0, 1]$ and $a, b \in \mathbb{R}$ with $a < b$. Suppose that φ is compactly supported and that

$$\begin{aligned} |(d/dx)^k \psi(x)| &\leq s \quad \text{for } x \in [a, b], \\ |(d/dx)^{(k+1)} \psi(x)| &\leq 1 \quad \text{for } x \in [a-s, b+s], \end{aligned}$$

where k is a positive integer. Then, there exists a positive constant C depending only on k and φ such that

$$\left| \int_a^b \exp(i\lambda\psi(x)) \varphi(x) dx \right| \leq C |\lambda|^{-\epsilon/k} \int_{a-s}^{b+s} |(d/dx)^k \psi(x)|^{-\epsilon(1+1/k)} dx$$

for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $\epsilon \in (0, 1]$.

Define a function F on an appropriate subinterval of $\mathbb{R}_+$ by $F(t) = \langle \xi, \Phi(tx) \rangle$ for fixed $\xi \in S^{d-1}$ and $x \in B(0, 1)$. Then, we note that $(d/dt)^m F(t) = t^{-m} G_m(tx, \xi)$, where G_m is as in Lemma 2.6.

Proof of Lemma 2.4: Take an integer $\nu \geq 1$ and $a \in [2, 4]$ such that $\rho = a^\nu$. Let Φ, δ, R and $\ell(\xi)$ be as in Lemma 2.6. Put $\ell_0 = \max_{\xi \in S^{d-1}} \ell(\xi)$. Let L be a negative integer such that

$$\left| (d/dr)^\ell \langle \xi', \Phi(\rho^k sr\theta) \rangle \right| < 1/2$$

for $1 \leq \ell \leq \ell_0 + 1$, $s \in [1, \rho]$, $r \in (0, 5)$, $\xi' \in S^{d-1}$ and $\theta \in S^{n-1}$ whenever $k \leq L$ and such that $2^{k+2} < R$ if $k \leq L$. Then, when $\xi \in \mathbb{R}^d \setminus \{0\}$ and $k \leq L$, we write

$$\begin{aligned}\hat{\sigma}_k(\xi) &= \sum_{j=0}^{\nu-1} \int_{\rho^k a^j}^{\rho^k a^{j+1}} \int_{S^{n-1}} \exp(-2\pi i \langle \xi, \Phi(r\theta) \rangle) h(r) \Omega(\theta) d\sigma(\theta) dr/r \\ &= \sum_{j=0}^{\nu-1} \int_1^a \int_{S^{n-1}} \exp(-2\pi i \langle \xi, \Phi(\rho^k a^j r \theta) \rangle) h(\rho^k a^j r) \Omega(\theta) d\sigma(\theta) dr/r.\end{aligned}$$

Let $\phi \in C^\infty(\mathbb{R})$ satisfy $\text{supp}(\phi) \subset (0, 10^{-9})$, $\phi \geq 0$, $\int \phi(s) ds = 1$. Define $h_j(r) = \int_{s < r/2} h(\rho^k a^j(r-s)) \phi_u(s) ds$, $r > 0$, where $\phi_u(s) = u^{-1} \phi(u^{-1}s)$, $u > 0$. Then, if $u < 1$,

$$\int_1^a |h(\rho^k a^j r) - h_j(r)| dr/r \leq C\omega(h, u). \quad (2.8)$$

We take $u = (|\xi| \rho^{kM})^{-\zeta/q'}$ for a suitable M with $M \geq \ell_0$ and $\zeta > 0$, which will be specified below. We assume $|\xi| \rho^{kM} \geq 1$ for the moment. Define

$$s_k(\xi) = \sum_{j=0}^{\nu-1} \int_1^a \int_{S^{n-1}} \exp(-2\pi i \langle \xi, \Phi(\rho^k a^j r \theta) \rangle) h_j(r) \Omega(\theta) d\sigma(\theta) dr/r.$$

Then, by (2.8)

$$\begin{aligned}|\hat{\sigma}_k(\xi) - s_k(\xi)| &\leq C(\log \rho) \|\Omega\|_1 \omega(h, u) \\ &\leq C(\log \rho) \|\Omega\|_1 \|h\|_{\Lambda^\eta} (|\xi| \rho^{kM})^{-\eta\zeta/q'},\end{aligned} \quad (2.9)$$

where we have used the fact that $\nu \approx \log \rho$.

By Lemma 2.7

$$\begin{aligned}&\left| \int_1^w \exp(-2\pi i \langle \xi, \Phi(\rho^k a^j t \theta) \rangle) dt \right| \\ &\leq C |\xi|^{-\epsilon/\ell(\xi')} \int_{1/2}^{a+1/2} \left| G_{\ell(\xi')}(\rho^k a^j r \theta, \xi') \right|^{-\epsilon(1+1/\ell(\xi'))} dr\end{aligned}$$

for $w \in [1, a]$, where $\xi' = \xi/|\xi|$. Also, $|h_j(a)| \leq Cu^{-1} \|h\|_{\Delta_1}$, $\int_1^a |h_j(r)| dr/r \leq C \|h\|_{\Delta_1}$, $\int_1^a |h'_j(r)| dr/r \leq Cu^{-1} \|h\|_{\Delta_1}$. Therefore, applying integration by parts, we see that

$$\begin{aligned}&\left| \int_1^a \exp(-2\pi i \langle \xi, \Phi(\rho^k a^j r \theta) \rangle) h_j(r) dr/r \right| \\ &\leq Cu^{-1} \|h\|_{\Delta_1} |\xi|^{-\epsilon/\ell(\xi')} \int_{1/2}^{a+1/2} \left| G_{\ell(\xi')}(\rho^k a^j r \theta, \xi') \right|^{-\epsilon(1+1/\ell(\xi'))} dr/r.\end{aligned}$$

Note that

$$\begin{aligned} & \int_{S^{n-1}} \left(\int_{1/2}^{a+1/2} \left| G_{\ell(\xi')}(\rho^k a^j r \theta, \xi') \right|^{-\epsilon(1+1/\ell(\xi'))} dr/r \right) |\Omega(\theta)| d\sigma(\theta) \\ & \leq C(\rho^k a^j)^{-n} \int_{|x| \leq 2\rho^k a^{j+1}} |G_{\ell(\xi')}(x, \xi')|^{-2\epsilon} |\Omega(x')| dx =: I, \end{aligned}$$

where $\epsilon \in (0, 1]$. Since $2\rho^k a^{j+1} < R$, by Hölder's inequality we have

$$I \leq C(\rho^k a^j)^{-n} (\rho^k a^j)^{n/q} \|\Omega\|_q \left(\int_{|x| \leq R} |G_{\ell(\xi')}(x, \xi')|^{-2\epsilon q'} dx \right)^{1/q'}.$$

Therefore

$$\begin{aligned} & \sum_{j=0}^{v-1} (\rho^k a^j)^{-n} \int_{|x| \leq 2\rho^k a^{j+1}} |G_{\ell(\xi')}(x, \xi')|^{-2\epsilon} |\Omega(x')| dx \\ & \leq C \|\Omega\|_q \rho^{-kn/q'} \left(\sum_{j=0}^{v-1} a^{-jn/q'} \right) \left(\int_{|x| \leq R} |G_{\ell(\xi')}(x, \xi')|^{-2\epsilon q'} dx \right)^{1/q'} \\ & \leq C(\log \rho) \|\Omega\|_q \rho^{-kn/q'} \left(\int_{|x| \leq R} |G_{\ell(\xi')}(x, \xi')|^{-2\epsilon q'} dx \right)^{1/q'}, \end{aligned}$$

since $v \approx \log \rho$. Using these estimates, we have

$$\begin{aligned} & \left| \sum_{j=0}^{v-1} \int_1^a \int_{S^{n-1}} \exp \left(-2\pi i \langle \xi, \Phi(\rho^k a^j r \theta) \rangle \right) h_j(r) \Omega(\theta) d\sigma(\theta) dr/r \right| \\ & \leq C(\log \rho) u^{-1} \|h\|_{\Delta_1} |\xi|^{-\epsilon/\ell(\xi')} \|\Omega\|_q \rho^{-kn/q'} \left(\int_{|x| \leq R} |G_{\ell(\xi')}(x, \xi')|^{-2\epsilon q'} dx \right)^{1/q'}, \end{aligned}$$

where C is independent of ϵ, ρ, q, h and Ω . If we put $\epsilon = \delta/(2q')$, then by Lemma 2.6 we have

$$|s_k(\xi)| \leq CC_{\Phi}^{1/q'} (\log \rho) \|h\|_{\Delta_1} \|\Omega\|_q (|\xi| \rho^{kM})^{\zeta/q'} (|\xi| \rho^{2kn\ell(\xi')/\delta})^{-\delta/(2q'\ell(\xi'))}.$$

Therefore, if M is a positive integer such that $M - 1 < 2n\ell_0/\delta \leq M$ and $\zeta < \delta/(2\ell_0)$,

$$|s_k(\xi)| \leq CC_{\Phi}^{1/q'} (\log \rho) \|h\|_{\Delta_1} \|\Omega\|_q (|\xi| \rho^{kM})^{-(\delta/(2\ell_0) - \zeta)/q'}. \quad (2.10)$$

Combining (2.9) and (2.10), we can see that

$$|\hat{\sigma}_k(\xi)| \leq C(\log \rho) \|h\|_{\Lambda_1^\eta} \|\Omega\|_q (|\xi| \rho^{kM})^{-\epsilon_0/q'},$$

where $\epsilon_0 = \min(\eta\zeta, \delta/(2\ell_0) - \zeta)$. If $|\xi| \rho^{kM} \leq 1$, the conclusion of Lemma 2.4 follows from the estimate $|\hat{\sigma}_k(\xi)| \leq C(\log \rho) \|h\|_{\Delta_1} \|\Omega\|_1$ (see (2.14) below with $m = \ell + 1$). This completes the proof of Lemma 2.4. $\square$

Proof of Lemma 2.5: Let

$$I(x) = \int_1^\rho \exp \left(i \left[(\rho^k t)^m \sum_{|\alpha|=m} a_\alpha x^\alpha + Q(\rho^k t x) \right] \right) h(\rho^k t) dt/t.$$

Note that

$$\int_{\rho^k \leq |y| < \rho^{k+1}} \exp(iP(x)) h(|x|) \Omega(x') |x|^{-n} dx = \int_{S^{n-1}} \Omega(\theta) I(\theta) d\sigma(\theta).$$

Let $a \in [2, 4]$ and $\nu \geq 1$ be as in the proof of Lemma 2.4. Decompose $I(x) = \sum_{j=0}^{\nu-1} I_j(x)$, where

$$I_j(x) = \int_1^a \exp \left(i \left[(\rho^k a^j t)^m \sum_{|\alpha|=m} a_\alpha x^\alpha + Q(\rho^k a^j t x) \right] \right) h(\rho^k a^j t) dt/t.$$

Let $h_j(t) = \int_{s < t/2} h(\rho^k a^j(t-s)) \phi_u(s) ds$ be as in the proof of Lemma 2.4 and

$$\tilde{I}_j(x) = \int_1^a \exp \left(i \left[(\rho^k a^j t)^m \sum_{|\alpha|=m} a_\alpha x^\alpha + Q(\rho^k a^j t x) \right] \right) h_j(t) dt/t.$$

Then by (2.8) $|I_j(x) - \tilde{I}_j(x)| \leq C\omega(h, u)$, $0 < u < 1$. So,

$$\begin{aligned} & \left| \int_{S^{n-1}} \Omega(\theta) I_j(\theta) d\sigma(\theta) - \int_{S^{n-1}} \Omega(\theta) \tilde{I}_j(\theta) d\sigma(\theta) \right| \\ & \leq \int_{S^{n-1}} |\Omega(\theta)| |I_j(\theta) - \tilde{I}_j(\theta)| d\sigma(\theta) \\ & \leq C\omega(h, u) \|\Omega\|_1 \leq C \|h\|_{\Lambda^\eta} \|\Omega\|_1 u^\eta \end{aligned} \quad (2.11)$$

for $0 \leq j \leq \nu - 1$. Also, since $|I(x)| \leq C(\log \rho) \|h\|_{\Delta_1}$,

$$\left| \int_{S^{n-1}} \Omega(\theta) I(\theta) d\sigma(\theta) \right| \leq C(\log \rho) \|h\|_{\Delta_1} \|\Omega\|_1. \quad (2.12)$$

Now, we assume that $b := \rho^{km} \sum_{|\alpha|=m} |a_\alpha| \geq 1$ and put $u = (a^{jm} b)^{-1/(4mq')}$. Then, as in the proof of Lemma 2.4, an integration by parts argument implies that

$$|\tilde{I}_j(x)| \leq Cu^{-1} \|h\|_{\Delta_1} \left| (\rho^k a^j)^m \sum_{|\alpha|=m} a_\alpha x^\alpha \right|^{-1/m}, \quad (2.13)$$

since

$$\begin{aligned} & \left| \int_1^w \exp \left(i \left[(\rho^k a^j t)^m \sum_{|\alpha|=m} a_\alpha x^\alpha + Q(\rho^k a^j t x) \right] \right) dt \right| \\ & \leq C \left| (\rho^k a^j)^m \sum_{|\alpha|=m} a_\alpha x^\alpha \right|^{-1/m} \end{aligned}$$

for $w \in [1, a]$, which follows from van der Corput's lemma. We also have $|\tilde{I}_j(x)| \leq C\|h\|_{\Delta_1}$. Combining this with (2.13), we have

$$|\tilde{I}_j(x)| \leq Cu^{-1}\|h\|_{\Delta_1} \min \left(1, \left| (\rho^k a^j)^m \sum_{|\alpha|=m} a_\alpha x^\alpha \right|^{-1/(2mq')} \right)$$

and hence by Hölder's inequality and [7, Corollary 1]

$$\begin{aligned} \left| \int_{S^{n-1}} \Omega(\theta) \tilde{I}_j(\theta) d\sigma(\theta) \right| &\leq \int_{S^{n-1}} |\Omega(\theta) \tilde{I}_j(\theta)| d\sigma(\theta) \leq \|\Omega\|_q \|\tilde{I}_j\|_{q'} \\ &\leq Cu^{-1}\|h\|_{\Delta_1} \|\Omega\|_q \left(\int_{S^{n-1}} \left| (\rho^k a^j)^m \sum_{|\alpha|=m} a_\alpha \theta^\alpha \right|^{-1/(2m)} d\sigma(\theta) \right)^{1/q'} \\ &\leq C\|h\|_{\Delta_1} \|\Omega\|_q \left((\rho^k a^j)^m \sum_{|\alpha|=m} |a_\alpha| \right)^{-1/(4mq')}. \end{aligned}$$

By this estimate and (2.11) we see that

$$\begin{aligned} \left| \int_{S^{n-1}} \Omega(\theta) I_j(\theta) d\sigma(\theta) \right| &\leq C \left(\|h\|_{\Lambda^\eta} \|\Omega\|_1 + \|h\|_{\Delta_1} \|\Omega\|_q \right) \left((\rho^k a^j)^m \sum_{|\alpha|=m} |a_\alpha| \right)^{-\tau/(mq')}, \end{aligned}$$

where $\tau = 4^{-1} \min(1, \eta)$. Thus

$$\begin{aligned} \left| \int_{S^{n-1}} \Omega(\theta) I(\theta) d\sigma(\theta) \right| &\leq \sum_{j=0}^{v-1} \left| \int_{S^{n-1}} \Omega(\theta) I_j(\theta) d\sigma(\theta) \right| \\ &\leq C(\log \rho) \left(\|h\|_{\Lambda^\eta} \|\Omega\|_1 + \|h\|_{\Delta_1} \|\Omega\|_q \right) \left(\rho^{km} \sum_{|\alpha|=m} |a_\alpha| \right)^{-\tau/(mq')}, \end{aligned}$$

if $\rho^{km} \sum_{|\alpha|=m} |a_\alpha| \geq 1$. Along with (2.12), this implies the conclusion of Lemma 2.5. $\square$

Proof of Lemma 2.3: We easily see that

$$\|\sigma_k^{(m)}\| \leq C\|\Omega\|_1 \int_{\rho^k}^{\rho^{k+1}} |h(r)| dr/r \leq C(\log \rho) \|h\|_{\Delta_1} \|\Omega\|_1 \quad (2.14)$$

for $1 \leq m \leq \ell + 1$. By (2.14) and (2.2) we have

$$\|\tau_k^{(m)}\| \leq C(\log \rho) \|h\|_{\Delta_1} \|\Omega\|_1 \quad (2.15)$$

for $1 \leq m \leq \ell$. By (2.15) and Hölder's inequality we have (2.5).

Let $k \leq L$, where L is as in Lemma 2.4. By Lemmas 2.4 and 2.5 we have $|\hat{\sigma}_k^{(m)}(\xi)| \leq CA (\beta_m^k |L_m(\xi)|)^{-\alpha_m}$ for $m = 2, \dots, \ell + 1$. Also, we note that $|\Phi_{k,m}(\xi)|$ is bounded by $C (\beta_{m+1}^k |L_{m+1}(\xi)|)^{-N}$ for all $N > 0$, when $1 \leq m \leq \ell$. Using these estimates and (2.14) in the definition of $\tau_k^{(m)}$ in (2.2), we have (2.6).

To prove (2.7), we note that

$$\left| \hat{\sigma}_k^{(m+1)}(\xi) - \hat{\sigma}_k^{(m)}(\xi) \right| \leq C(\log \rho) \|h\|_{\Delta_1} \|\Omega\|_1 \beta_{m+1}^{k+1} |L_{m+1}(\xi)|. \quad (2.16)$$

Also, by (2.3) we see that

$$|\Phi_{k,m+1}(\xi) - \Phi_{k,m}(\xi)| \leq C \beta_{m+1}^k |L_{m+1}(\xi)|. \quad (2.17)$$

The estimates (2.14), (2.16) and (2.17) imply

$$|\hat{\tau}_k^{(m)}(\xi)| \leq C(\log \rho) \|h\|_{\Delta_1} \|\Omega\|_1 \beta_{m+1}^{k+1} |L_{m+1}(\xi)|, \quad (2.18)$$

since

$$|\hat{\tau}_k^{(m)}(\xi)| \leq \left| \left(\hat{\sigma}_k^{(m+1)}(\xi) - \hat{\sigma}_k^{(m)}(\xi) \right) \Phi_{k,m+1}(\xi) \right| + \left| \left(\Phi_{k,m+1}(\xi) - \Phi_{k,m}(\xi) \right) \hat{\sigma}_k^{(m)}(\xi) \right|.$$

By (2.15) we also have $|\hat{\tau}_k^{(m)}(\xi)| \leq C(\log \rho) \|h\|_{\Delta_1} \|\Omega\|_1$. This estimate and (2.18) imply (2.7). This completes the proof of Lemma 2.3. $\square$

Proof of Proposition 2.2: Let $\tilde{T}_\rho^{(m)}(f) = \sum_{k \leq L} \tau_k^{(m)} * f$ for $1 \leq m \leq \ell$, where L is as in Lemma 2.3. Then, to prove Proposition 2.2 it suffices to show a version of Proposition 2.2 for $\tilde{T}_\rho^{(m)}$ with bounds similar to those for $T_\rho^{(m)}$, since $\|T_\rho^{(m)}(f) - \tilde{T}_\rho^{(m)}(f)\|_p \leq CA \|f\|_p$ for $1 \leq p \leq \infty$, where A is as in Lemma 2.3. Let $\{\psi_k\}_{k=-\infty}^\infty$ be a sequence of non-negative functions in $C^\infty(\mathbb{R})$ such that each ψ_k is supported in $[\beta_{m+1}^{-k-1}, \beta_{m+1}^{-k+1}]$, $\sum_k \psi_k(t)^2 = 1$ for $t > 0$ and

$$|(d/dt)^j \psi_k(t)| \leq c_j |t|^{-j}, \quad j = 1, 2, \dots,$$

where the constants c_j are independent of β_{m+1} . This is possible since $\beta_{m+1} \geq 2$. Let

$$\left(S_k^{(m+1)}(f) \right)^\wedge(\xi) = \psi_k \left(|H_{m+1} \pi_{s_{m+1}}^d R_{m+1}(\xi)| \right) \hat{f}(\xi).$$

We also write $S_k^{(m+1)} = S_k$. Put

$$D_j^{(m)}(f) = \sum_{k=-\infty}^L S_{j+k} \left(\tau_k^{(m)} * S_{j+k}(f) \right).$$

Then $\tilde{T}_\rho^{(m)} = \sum_j D_j^{(m)}$. Plancherel's theorem and the estimates (2.5)–(2.7) imply that

$$\begin{aligned} \|D_j^{(m)}(f)\|_2^2 &\leq \sum_{k \leq L} C \int_{\Delta(j+k)} |\hat{\tau}_k^{(m)}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \\ &\leq CA^2 \min \left(1, \beta_{m+1}^{-2\alpha_{m+1}(|j|-2)} \right) \sum_{k \leq L} \int_{\Delta(j+k)} |\hat{f}(\xi)|^2 d\xi \\ &\leq CA^2 \min \left(1, \beta_{m+1}^{-2\alpha_{m+1}(|j|-2)} \right) \|f\|_2^2, \end{aligned}$$

where $\Delta(k) = \{\beta_{m+1}^{-k-1} \leq |H_{m+1}\pi_{s_{m+1}}^d R_{m+1}(\xi)| \leq \beta_{m+1}^{-k+1}\}$. Thus we have

$$\|D_j^{(m)}(f)\|_2 \leq CA \min\left(1, \beta_{m+1}^{-\alpha_{m+1}(|j|-2)}\right) \|f\|_2. \quad (2.19)$$

By (2.19) we have

$$\|\tilde{T}_\rho^{(m)}(f)\|_2 \leq \sum_j \|D_j^{(m)}(f)\|_2 \leq CAB \|f\|_2, \quad (2.20)$$

since $B \geq \left(1 - \beta_{m+1}^{-\alpha_{m+1}}\right)^{-1}$, where B is as in Proposition 2.1.

Taking Proposition 2.1 for granted for the moment and recalling the definition of $\tau_k^{(m)}$ in (2.2), by change of variables and a well-known theorem for L^p boundedness of maximal functions (see [5, Section 6]) we have

$$\begin{aligned} \|(\tau^{(m)})^*(f)\|_p &\leq C \|(\mu^{(m+1)})^*(|f|)\|_p + C \|(\mu^{(m)})^*(|f|)\|_p \\ &\leq C_p AB^{2/p} \|f\|_p \end{aligned} \quad (2.21)$$

for $p > 1 + \theta$.

By (2.5), (2.21) and the proof of Lemma in [3, p. 544], we have the following.

Lemma 2.8 *Let $u \in (1 + \theta, 2]$, $1/v - 1/2 = 1/(2u)$. Then we have*

$$\left\| \left(\sum_{k \leq L} |\tau_k^{(m)} * g_k|^2 \right)^{1/2} \right\|_v \leq (c_1 C_u)^{1/2} AB^{1/u} \left\| \left(\sum_{k \leq L} |g_k|^2 \right)^{1/2} \right\|_v,$$

where the constants c_1 and C_u are as in (2.5) and (2.21), respectively.

Also, the Littlewood–Paley theory implies that

$$\|D_j^{(m)}(f)\|_p \leq c_p \left\| \left(\sum_{k \leq L} |\tau_k^{(m)} * S_{j+k}(f)|^2 \right)^{1/2} \right\|_p, \quad (2.22)$$

$$\left\| \left(\sum_k |S_k(f)|^2 \right)^{1/2} \right\|_p \leq c_p \|f\|_p, \quad (2.23)$$

where $1 < p < \infty$ and c_p is independent of β_{m+1} and the linear transformations R_{m+1}, H_{m+1} .

Let $1 + \theta < p \leq 4/(3 - \theta)$. Then, there exists $u \in (1 + \theta, 2]$ such that $1/p = 1/2 + (1 - \theta)/(2u)$. Let $1/v - 1/2 = 1/(2u)$. Then, by (2.22), (2.23) and Lemma 2.8 we have

$$\|D_j^{(m)}(f)\|_v \leq CAB^{1/u} \|f\|_v, \quad (2.24)$$

where C is independent of ρ and the linear transformations R_i , H_i , $2 \leq i \leq \ell + 1$. Noting that $1/p = \theta/2 + (1 - \theta)/v$ and interpolating between (2.19) and (2.24), we have

$$\|D_j^{(m)}(f)\|_p \leq CAB^{(1-\theta)/u} \min\left(1, \beta_{m+1}^{-\theta\alpha_{m+1}(|j|-2)}\right) \|f\|_p,$$

which implies that

$$\begin{aligned} \|\tilde{T}_\rho^{(m)}(f)\|_p &\leq \sum_j \|D_j^{(m)}(f)\|_p \leq CAB^{(1-\theta)/u} \left(1 - \beta_{m+1}^{-\theta\alpha_{m+1}}\right)^{-1} \|f\|_p \quad (2.25) \\ &\leq CAB^{2/p} \|f\|_p. \end{aligned}$$

A duality and interpolation argument using (2.20) and (2.25) implies the conclusion of Proposition 2.2 with $T_\rho^{(m)}$ replaced by $\tilde{T}_\rho^{(m)}$, which proves Proposition 2.2. $\square$

We now prove Proposition 2.1 by induction on j . First, the inequality $(\mu^{(1)})^*(f)(x) \leq C(\log \rho)\|h\|_{\Delta_1}\|\Omega\|_1|f(x - P(0))|$ implies the estimate (2.4) for $j = 1$. Next, we prove (2.4) for $j = m$ by assuming (2.4) for $j = m - 1$, $2 \leq m \leq \ell + 1$. Define a sequence $\eta^{(m)} = \{\eta_k^{(m)}\}_{k=-\infty}^{-1}$ of Borel measures on $\mathbb{R}^d$ by

$$\hat{\eta}_k^{(m)}(\xi) = \varphi\left(\beta_m^k |H_m \pi_{s_m}^d R_m(\xi)|\right) \hat{\mu}_k^{(m-1)}(\xi),$$

where $\varphi \in C_0^\infty(\mathbb{R})$ is as in the definition of $\tau_k^{(m)}$ in (2.2). Then, from (2.4) with $j = m - 1$, it follows that

$$\left\|(\eta^{(m)})^*(f)\right\|_p \leq C \left\|(\mu^{(m-1)})^*(f)\right\|_p \leq CAB^{2/p} \|f\|_p \quad (2.26)$$

for $p > 1 + \theta$, where A, B are as above. As in the proof of Lemma 2.3, we have

$$\begin{aligned} \|\eta_k^{(m)}\| + \|\mu_k^{(m)}\| &\leq C\|\mu_k^{(m-1)}\| + \|\mu_k^{(m)}\| \quad (2.27) \\ &\leq C\|\Omega\|_1 \int_{\rho^k}^{\rho^{k+1}} |h(r)| dr/r \\ &\leq C(\log \rho)\|h\|_{\Delta_1}\|\Omega\|_1 \leq CA. \end{aligned}$$

Let $k \leq L$, where L is as above. Since

$$\begin{aligned} &|\hat{\mu}_k^{(m)}(\xi) - \hat{\eta}_k^{(m)}(\xi)| \\ &\leq |\hat{\mu}_k^{(m)}(\xi) - \hat{\mu}_k^{(m-1)}(\xi)| + \left| \left(\varphi\left(\beta_m^k |H_m \pi_{s_m}^d R_m(\xi)|\right) - 1 \right) \hat{\mu}_k^{(m-1)}(\xi) \right|, \end{aligned}$$

arguing as in the proof of (2.7), we see that

$$\begin{aligned} |\hat{\mu}_k^{(m)}(\xi) - \hat{\eta}_k^{(m)}(\xi)| &\leq C(\log \rho)\|h\|_{\Delta_1}\|\Omega\|_1 \left(\beta_m^{k+1} |L_m(\xi)| \right)^{\alpha_m} \quad (2.28) \\ &\leq CA \left(\beta_m^{k+1} |L_m(\xi)| \right)^{\alpha_m}. \end{aligned}$$

We also have the following:

$$|\hat{\mu}_k^{(m)}(\xi)| \leq CA \left(\beta_m^k |L_m(\xi)| \right)^{-\alpha_m}, \quad (2.29)$$

$$\begin{aligned} |\hat{\eta}_k^{(m)}(\xi)| &\leq C(\log \rho) \|h\|_{\Delta_1} \|\Omega\|_1 \left(\beta_m^k |L_m(\xi)| \right)^{-\alpha_m} \\ &\leq CA \left(\beta_m^k |L_m(\xi)| \right)^{-\alpha_m}. \end{aligned} \quad (2.30)$$

We can prove the estimate (2.29) arguing as in the proof of (2.6). The definition of $\eta_k^{(m)}$ and (2.27) imply the first inequality of (2.30).

We have only to prove (2.4) with $j = m$ for $p \in (1 + \theta, 2]$, since the estimate (2.4) for $p > 2$ follows from interpolation between the estimate (2.4) for $p \in (1 + \theta, 2]$ and the obvious estimate $\|(\mu^{(m)})^*(f)\|_\infty \leq CA\|f\|_\infty$. Let

$$g_m(f)(x) = \left(\sum_{k \leq L} \left| v_k^{(m)} * f(x) \right|^2 \right)^{1/2},$$

where $v_k^{(m)} = \mu_k^{(m)} - \eta_k^{(m)}$. Then, we see that

$$(\tilde{\mu}^{(m)})^*(f) \leq g_m(f) + (\eta^{(m)})^*(|f|), \quad (2.31)$$

where $(\tilde{\mu}^{(m)})^*(f) = \sup_{k \leq L} |\mu_k^{(m)} * f|$. Note that to prove (2.4) with $j = m$ it suffices to prove it with $(\tilde{\mu}^{(m)})^*$ in place of $(\mu^{(m)})^*$. Since we have (2.26) and (2.31), to show (2.4) with $j = m$ it suffices to prove $\|g_m(f)\|_p \leq CAB^{2/p}\|f\|_p$ for $p \in (1 + \theta, 2]$. Let

$$U_\epsilon^{(m)}(f) = \sum_{k \leq L} \epsilon_k v_k^{(m)} * f,$$

where $\epsilon = \{\epsilon_k\}$, $\epsilon_k = 1$ or -1 . Then, we shall show that

$$\left\| U_\epsilon^{(m)}(f) \right\|_p \leq CAB^{2/p} \|f\|_p \quad (2.32)$$

for $p \in (1 + \theta, 2]$, where C is independent of ϵ . The desired estimate follows from (2.32) by a well-known property of Rademacher's functions.

To prove (2.32) we use the following:

Lemma 2.9 *Let $\{p_j\}_1^\infty$ be a sequence of real numbers defined by $p_1 = 2$ and $1/p_{j+1} = 1/2 + (1 - \theta)/(2p_j)$ for $j \geq 1$. Then, we have*

$$\left\| U_\epsilon^{(m)}(f) \right\|_{p_j} \leq C_j AB^{2/p_j} \|f\|_{p_j} \quad \text{for } j \geq 1.$$

We can see that $1/p_j = (1 - a^j)/(1 + \theta)$, where $a = (1 - \theta)/2$. Thus $\{p_j\}$ is decreasing and converges to $1 + \theta$. We can prove Lemma 2.9 by (2.26)–(2.30).

Proof: Define

$$U_j^{(m)}(f) = \sum_{k=-\infty}^L \epsilon_k S_{j+k} \left(v_k^{(m)} * S_{j+k}(f) \right),$$

where $S_k = S_k^{(m)}$ (the operators $S_k^{(m)}$ are as in the proof of Proposition 2.2). Then, $U_\epsilon^{(m)} = \sum_j U_j^{(m)}$. Arguing as in the proof of (2.19), and using Plancherel's theorem and the estimates (2.27)–(2.30), we have

$$\left\| U_j^{(m)}(f) \right\|_2 \leq CA \min \left(1, \beta_m^{-\alpha_m(|j|-2)} \right) \|f\|_2, \quad (2.33)$$

and hence $\left\| U_\epsilon^{(m)}(f) \right\|_2 \leq \sum_j \|U_j^{(m)}(f)\|_2 \leq CAB \|f\|_2$. This proves the assertion of Lemma 2.9 for $j = 1$.

We now assume the estimate of Lemma 2.9 for $j = s$ and prove it for $j = s + 1$. By induction, this will complete the proof of Lemma 2.9. From the estimate (2.31), it follows that

$$(\tilde{v}^{(m)})^*(f) \leq (\tilde{\mu}^{(m)})^*(|f|) + (\eta^{(m)})^*(|f|) \leq g_m(|f|) + 2(\eta^{(m)})^*(|f|),$$

where $(\tilde{v}^{(m)})^*(f) = \sup_{k \leq L} |\nu_k^{(m)}| * |f|$. By our assumption we have $\|g_m(f)\|_{p_s} \leq CAB^{2/p_s} \|f\|_{p_s}$. This estimate and (2.26) imply

$$\begin{aligned} \left\| (\tilde{v}^{(m)})^*(f) \right\|_{p_s} &\leq \|g_m(|f|)\|_{p_s} + 2 \left\| (\eta^{(m)})^*(|f|) \right\|_{p_s} \\ &\leq CAB^{2/p_s} \|f\|_{p_s}. \end{aligned} \quad (2.34)$$

Arguing as in the proof of (2.25), and using (2.27), (2.33) and (2.34), we can now obtain the estimate of Lemma 2.9 for $j = s + 1$. This completes the proof of Lemma 2.9. $\square$

Let $p \in (1 + \theta, 2]$ and let $\{p_j\}_1^\infty$ be as in Lemma 2.9. Then, we can find a positive integer N such that $p_{N+1} < p \leq p_N$. The estimate (2.32) now follows from interpolation between the estimates of Lemma 2.9 for $j = N$ and $j = N + 1$. This finishes the proof of (2.4) for $j = m$. By induction, this completes the proof of Proposition 2.1.

Proof of Theorem 1.1: By taking $\rho = 2^{q'}$ in Proposition 2.2 we see that

$$\|T_{2^{q'}}^{(m)}(f)\|_p \leq C_\theta (q - 1)^{-1} \|h\|_{\Lambda_1^q} \|\Omega\|_q \|f\|_p$$

for $p \in (1 + \theta, (1 + \theta)/\theta)$. This completes the proof of Theorem 1.1, since $T = \sum_{m=1}^\ell T_\rho^{(m)}$ and $(1 + \theta, (1 + \theta)/\theta) \rightarrow (1, \infty)$ as $\theta \rightarrow 0$. $\square$

3 Estimates for maximal functions

Let

$$T^*(f)(x) = \sup_{\epsilon \in (0, 1)} \left| \int_{\epsilon < |y| < 1} f(x - \Phi(y)) K(y) dy \right|, \quad (3.1)$$

where K is as in (1.2). Then, we have an analog of Theorem 1.1 for the maximal operator T^* .

Theorem 3.1 *Let $\Omega \in L^q(S^{n-1})$, $q \in (1, 2]$ and $h \in \Lambda_1^\eta$ for some $\eta > 0$. Suppose that Ω satisfies (1.1). Then*

$$\|T^*(f)\|_{L^p(\mathbb{R}^d)} \leq C_p(q-1)^{-1} \|h\|_{\Lambda_1^\eta} \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$, where C_p is independent of q , h and Ω .

By Theorem 3.1 and extrapolation we have the following result.

Theorem 3.2 *Let $\Omega \in L \log L(S^{n-1})$ and $h \in \Lambda_1^\eta$ for some $\eta > 0$. Suppose that Ω satisfies the condition (1.1). Let T^*f be defined as in (3.1) with the functions h and Ω . Then*

$$\|T^*(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$.

If the function h is identically 1, then Theorem 3.2 was shown in [1].

To prove Theorem 3.1, we use the following result.

Lemma 3.3 *Let $\theta \in (0, 1)$ and let positive numbers $A = (\log \rho) \|h\|_{\Lambda_1^\eta} \|\Omega\|_q$, $B = (1 - \rho^{-\theta\kappa/q'})^{-1}$ be as above. Define*

$$T_{m,\rho}^*(f)(x) = \sup_{k \leq L} \left| \sum_{j=k}^L \tau_j^{(m)} * f(x) \right| \quad (3.2)$$

for $1 \leq m \leq \ell$, where the measures $\tau_k^{(m)}$ are as in (2.2) and L is as in Lemma 1. Let $I_\theta = (2(1+\theta)/(\theta^2 - \theta + 2), (1+\theta)/\theta)$. Then, we have

$$\|T_{m,\rho}^*(f)\|_p \leq CA \left(B^{1+\delta(p)} + B^{2/p+1-\theta/2} \right) \|f\|_p$$

for $p \in I_\theta$, where C is independent of $q \in (1, 2]$, $\Omega \in L^q(S^{n-1})$, $h \in \Lambda_1^\eta$ and ρ .

This can be proved by results in Section 2.

Proof: Let $\tilde{T}_\rho^{(m)}(f) = \sum_{k \leq L} \tau_k^{(m)} * f$ be as in the proof of Proposition 2.2. Let φ_k be defined by

$$\hat{\varphi}_k(\xi) = \varphi \left(\beta_{m+1}^k |H_{m+1} \pi_{s_{m+1}}^d R_{m+1}(\xi)| \right),$$

where φ is as in the definition of $\tau_k^{(m)}$ in (2.2). We now decompose

$$\sum_{j=k}^L \tau_j^{(m)} * f = \varphi_k * \tilde{T}_\rho^{(m)}(f) - \varphi_k * \left(\sum_{j=-\infty}^{k-1} \tau_j^{(m)} * f \right) + (\delta - \varphi_k) * \left(\sum_{j=k}^L \tau_j^{(m)} * f \right),$$

where $k \leq L$ and $\delta = \delta_0$ is the delta function on $\mathbb{R}^d$ (see [3, 5]). Then, we have

$$T_{\rho,m}^*(f) \leq \sup_{k \leq L} \left| \varphi_k * \tilde{T}_\rho^{(m)}(f) \right| + \sum_{j=0}^{\infty} M_j^{(m)}(f), \quad (3.3)$$

where

$$M_j^{(m)}(f) = \sup_{k \leq L} \left| \varphi_k * \left(\tau_{k-j-1}^{(m)} * f \right) \right| + \sup_{k \leq L-j} \left| (\delta - \varphi_k) * \left(\tau_{j+k}^{(m)} * f \right) \right|.$$

From Proposition 2.2 it follows that

$$\left\| \sup_{k \leq L} \left| \varphi_k * \tilde{T}_\rho^{(m)}(f) \right| \right\|_p \leq CAB^{1+\delta(p)} \|f\|_p \quad (3.4)$$

for $p \in (1 + \theta, (1 + \theta)/\theta)$, and the estimate (2.21) implies that

$$\|M_j^{(m)}(f)\|_r \leq CAB^{2/r} \|f\|_r \quad \text{for } r > 1 + \theta. \quad (3.5)$$

Since

$$M_j^{(m)}(f) \leq \left(\sum_{k \leq L-j} \left| (\delta - \varphi_k) * \left(\tau_{j+k}^{(m)} * f \right) \right|^2 \right)^{1/2} + \left(\sum_{k \leq L} \left| \varphi_k * \left(\tau_{k-j-1}^{(m)} * f \right) \right|^2 \right)^{1/2},$$

arguing as in [5, p. 820] and using the estimates (2.6) and (2.7) along with Plancherel's theorem, we have

$$\|M_j^{(m)}(f)\|_2 \leq CA\beta_{m+1}^{-\alpha_{m+1}j} \left(1 - \beta_{m+1}^{-2\alpha_{m+1}} \right)^{-1/2} \|f\|_2. \quad (3.6)$$

We note that for any $p \in I_\theta$ there exists a number $r \in (1 + \theta, 2(1 + \theta)/\theta)$ such that $1/p = (1 - \theta)/r + \theta/2$. Therefore, interpolating between (3.5) and (3.6), we have

$$\|M_j^{(m)}(f)\|_p \leq CAB^{2(1-\theta)/r} \left(1 - \beta_{m+1}^{-2\alpha_{m+1}} \right)^{-\theta/2} \beta_{m+1}^{-\alpha_{m+1}\theta j} \|f\|_p. \quad (3.7)$$

From (3.3), (3.4) and (3.7), it follows that

$$\|T_{\rho,m}^*(f)\|_p \leq CA \left(B^{1+\delta(p)} + B^{2(1-\theta)/r+1} \left(1 - \beta_{m+1}^{-2\alpha_{m+1}} \right)^{-\theta/2} \right) \|f\|_p$$

for $p \in I_\theta$. Using $\left(1 - \beta_{m+1}^{-2\alpha_{m+1}} \right)^{-1} \leq B$ and $2(1 - \theta)/r + \theta/2 + 1 = 2/p + 1 - \theta/2$ in this estimate, we can obtain the conclusion of Lemma 3.3. $\square$

Proof of Theorem 3.1: Let

$$T_\rho^*(f)(x) = \sup_{\epsilon \in (0, \rho^{L+1})} \left| \int_{\epsilon < |y| < \rho^{L+1}} f(x - \Phi(y)) K(y) dy \right|.$$

Then, we have

$$T^*(f)(x) \leq T_\rho^*(f)(x) + J_\rho(f)(x), \quad (3.8)$$

where $J_\rho(f)(x) = \int_{\rho^{L+1} \leq |y| < 1} |f(x - \Phi(y))| |K(y)| dy$. We note that

$$T_\rho^*(f) \leq T_{0,\rho}^*(f) + \mu_\rho^*(|f|), \quad (3.9)$$

where $\mu_\rho^* = (\mu^{(\ell+1)})^*$ is as in Proposition 2.1 and $T_{0,\rho}^*(f)$ is defined by the formula in (3.2) with $\{\tau_j^{(m)}\}_{j \leq L}$ replaced by the sequence $\{\sigma_j\}_{j \leq L}$ of measures in (2.1). Since $T_{0,\rho}^*(f) \leq \sum_{m=1}^\ell T_{m,\rho}^*(f)$, using Lemma 3.3 with $\rho = 2^{q'}$, we see that

$$\|T_{0,2^{q'}}^*(f)\|_p \leq C_\theta (q-1)^{-1} \|h\|_{\Lambda_1^\eta} \|\Omega\|_q \|f\|_p \quad (3.10)$$

for $p \in I_\theta$. Also, by Proposition 2.1 with $\rho = 2^{q'}$ we have

$$\|\mu_{2^{q'}}^*(|f|)\|_p \leq C_\theta (q-1)^{-1} \|h\|_{\Lambda_1^\eta} \|\Omega\|_q \|f\|_p \quad (3.11)$$

for $p \in I_\theta$. Note that

$$\int_{\rho^{L+1} \leq |y| < 1} |K(y)| dy \leq C(\log \rho) \|h\|_{\Delta_1} \|\Omega\|_1.$$

Therefore, it is easy to see that

$$\|J_{2^{q'}}(f)\|_p \leq C(q-1)^{-1} \|h\|_{\Lambda_1^\eta} \|\Omega\|_q \|f\|_p \quad (3.12)$$

for $p \in I_\theta$. Since $I_\theta \rightarrow (1, \infty)$ as $\theta \rightarrow 0$, by (3.8)–(3.12) we obtain the conclusion of Theorem 3.1. $\square$

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