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Singular integrals associated with functions of finite type and extrapolation

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Summary: We consider a singular integral along a submanifold of finite type. We prove a certain $L^p$ estimate for the singular integral, which is useful in applying an extrapolation method that shows $L^p$ boundedness of the singular integral under a sharp condition of the kernel.

1 Introduction

Let $B(0, 1) = \{x \in \mathbb{R}^n : |x| < 1\}$ and let $\Phi : B(0, 1) \to \mathbb{R}^d$ be a smooth function. We assume that $\Phi$ is of finite type at the origin, that is, for any $\xi \in S^{d-1}$ (the unit sphere in $\mathbb{R}^d$) there exists a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $|\alpha| \geq 1$ and $\partial^\alpha_x \Phi(x, \xi)|_{x=0} \neq 0$, where $\partial^\alpha_x = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$.

Let a function $\Omega$ in $L^1(S^{n-1})$ satisfy

$$\int_{S^{n-1}} \Omega(\theta) \, d\sigma(\theta) = 0,$$

where $d\sigma$ denotes the Lebesgue surface measure on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. Throughout this note we assume $n \geq 2$. Let $\Delta_s, s \geq 1,$ denote the collection of functions $h$ on $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$ satisfying

$$\|h\|_{\Delta_s} = \sup_{j \in \mathbb{Z}} \left( \int_{2^j}^{2^{j+1}} |h(t)|^s \, dt / t^s \right)^{1/s} < \infty,$$

where $\mathbb{Z}$ denotes the set of integers. We define

$$\omega(h, t) = \sup_{|s| < tR/2} \int_R^{2R} |h(r-s) - h(r)| \, dr / r, \quad t \in (0, 1],$$

where the supremum is taken over all $s$ and $R$ such that $|s| < tR/2$ (see [6, 12]). For $\eta > 0$, let $\Lambda^\eta$ denote the family of functions $h$ satisfying

$$\|h\|_{\Lambda^\eta} = \sup_{t \in (0, 1]} t^{-\eta} \omega(h, t) < \infty.$$
Define a space $\Lambda_s^\eta = \Delta_s \cap \Lambda^\eta$ and set $\|h\|_{\Lambda_s^\eta} = \|h\|_{\Delta_s} + \|h\|_{\Lambda^\eta}$ for $h \in \Lambda_s^\eta$.

We consider a singular Radon transform of the form:

$$T(f)(x) = \text{p.v.} \int_{B(0,1)} f(x - \Phi(y)) K(y) \, dy$$

(1.2)

$$= \lim_{\epsilon \to 0} \int_{1>|y|>\epsilon} f(x - \Phi(y)) K(y) \, dy$$

for an appropriate function $f$ on $\mathbb{R}^d$, where $K(y) = h(|y|)|\Omega(y')|y^{1-n}$, $y' = |y|^{-1}y$, $h \in \Delta_1$. See Stein [13], Fan, Guo, and Pan [4], Al-Salman and Pan [1] and also [2, 5, 14] for this singular integral and related topics.

In the previous works, the operator $T$ was studied under the condition that $h$ is a constant function. In this note, we consider the operator $T$ under a more general condition on $h$. We shall prove the following:

**Theorem 1.1** Let $q \in (1, 2]$, $\Omega \in L^q(S^{n-1})$ and $h \in \Lambda_1^\eta$ for some $\eta > 0$. Suppose that $\Omega$ satisfies the condition (1.1). Let $T$ be defined as in (1.2). Then we have

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq C_p(q-1)^{-1}\|h\|_{\Lambda_1^\eta} \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$, where the constant $C_p$ is independent of $q, h$ and $\Omega$.

Let $L \log L(S^{n-1})$ denote the Zygmund class of the functions $F$ on $S^{n-1}$ satisfying

$$\int_{S^{n-1}} |F(\theta)| \log(2 + |F(\theta)|) \, d\sigma(\theta) < \infty.$$  

Then, as an application of Theorem 1.1 and extrapolation, we have the following theorem.

**Theorem 1.2** Let $h \in \Lambda_1^\eta$ for some $\eta > 0$. Suppose that $\Omega$ is in $L \log L(S^{n-1})$ and satisfies the condition (1.1). Let $T$ be as in (1.2). Then we have

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$.

The extrapolation argument that proves Theorem 1.2 from Theorem 1.1 can be found in [8, 9, 10, 11] (see also [15, Chap. XII, pp. 119–120]). If the function $h$ is assumed to be a constant function in Theorem 1.2, we have a result of Al-Salman and Pan shown in [1] (see [1, Theorem 1.1]); so we can give a different proof of the result by applying Theorem 1.1 and extrapolation. Relevant results can be found in [8, 9, 10, 11].

In Section 2, we shall prove Theorem 1.1. Consider a singular integral of the form

$$S(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - P(y))h(|y|)|\Omega(y')|y^{1-n} \, dy,$$

where $P(y)$ is a polynomial mapping from $\mathbb{R}^n$ to $\mathbb{R}^d$ satisfying $P(-y) = -P(y)$ ($P \neq 0$), $h \in \Delta_s$ for $s \in (1, 2]$ and $\Omega$ is a function in $L^q(S^{n-1})$, $q \in (1, 2]$, satisfying (1.1). Then, it has been proved that

$$\|S(f)\|_{L^p(\mathbb{R}^d)} \leq C_p(q-1)^{-1}(s-1)^{-1}\|\Omega\|_{L^q(S^{n-1})} \|h\|_{\Delta_s} \|f\|_{L^p(\mathbb{R}^d)}$$
for all $p \in (1, \infty)$, where the constant $C_p$ is independent of $q, s, \Omega, h$ and the polynomial components of $P$ if they are of fixed degree (see [8, Theorem 1]). Outline of our proof of Theorem 1.1 is similar to that of the proof for [8, Theorem 1]. We apply methods of [4] to obtain some basic estimates. We need to assume that $h \in L_1^q$ for some $\eta > 0$ to prove certain Fourier transform estimates. As in [8] (see also [9, 10]), a key idea of the proof of Theorem 1.1 is to apply a Littlewood–Paley decomposition adapted to an appropriate lacunary sequence depending on $q$ for which $\Omega \in L^q(S^{n-1})$.

In Section 3, we shall give analogs of Theorems 1.1 and 1.2 for a maximal singular integral operator related to $T$. In what follows we also write $\|f\|_{L^p(\mathbb{R}^d)} = \|f\|_p$ and $\|\Omega\|_{L^q(S^{n-1})} = \|\Omega\|_q$. Throughout this note, the letter $C$ will be used to denote non-negative constants which may be different in different occurrences.

## 2 Proof of Theorem 1.1

Let $M$ be a positive integer. We write $\Phi(y) = (\Phi_1(y), \ldots, \Phi_d(y))$. Let $P_j(y)$ be the Taylor polynomial of $\Phi_j(y)$ at the origin defined by

$$P_j(y) = \sum_{|\alpha| \leq M-1} \frac{1}{\alpha!} (\partial^\alpha \Phi_j)(0)y^\alpha,$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$ and $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $y = (y_1, \ldots, y_n)$. We write $P(y) = (P_1(y), P_2(y), \ldots, P_d(y))$ and

$$P(y) = \sum_{j=1}^{\ell} Q_j(y), \quad Q_j(y) = \sum_{|\gamma| = N(j)} a_\gamma y^\gamma \quad (a_\gamma \in \mathbb{R}^d),$$

where $0 = N(1) < N(2) < \cdots < N(\ell)$, $Q_j \neq 0$ for $j \geq 2$. Let $\beta_m = \rho^{N(m)}$ and $\alpha_m = r(q-1)/(q N(m))$ for $2 \leq m \leq \ell$, where $r = 4^{-1} \min(1, \eta)$, $\rho \geq 2$. Also, let $\beta_{\ell+1} = \rho^M$ and $\alpha_{\ell+1} = \epsilon_0(q-1)/q$ for some $\epsilon_0 \in (0, 1/4)$. The positive integer $M$ and the positive number $\epsilon_0$ will be specified in the following (see Lemma 2.4 below).

Let $T$ be as in Theorem 1.1. Let $E_k = \{x \in \mathbb{R}^d : \rho^k \leq |x| < \rho^{k+1}\}$, $k \in \mathbb{Z}$, $\rho \geq 2$. Then $T(f)(x) = \sum_{k=-\infty}^{1} \sigma_k * f(x)$, where $\{\sigma_k\}_{k=-\infty}^{1}$ is a sequence of Borel measures on $\mathbb{R}^d$ such that

$$\sigma_k * f(x) = \int_{E_k} f(x - \Phi(y)) K(y) dy. \quad (2.1)$$

Put $P^{(m)}(y) = \sum_{j=1}^{m} Q_j(y)$ for $m = 1, 2, \ldots, \ell$ and $P^{(\ell+1)}(y) = \Phi(y)$. Consider a sequence $\mu^{(m)} = (\mu_k^{(m)})_{k=-\infty}^{1}$ of positive measures on $\mathbb{R}^d$ such that

$$\mu_k^{(m)} * f(x) = \int_{E_k} f(x - P^{(m)}(y)) |K(y)| dy$$

for $m = 1, 2, \ldots, \ell + 1$. Note that $\mu_k^{(1)} = (\int_{E_k} |K(y)| dy) \delta_{\rho^k}$, where $\delta_a$ is Dirac’s delta function on $\mathbb{R}^d$ concentrated at $a$. Let $\sigma^{(m)} = (\sigma_k^{(m)})_{k=-\infty}^{1}$ be a sequence of Borel
measures on \( \mathbb{R}^d \) such that

\[
\sigma_k^{(m)} * f(x) = \int_{E_k} f \left( x - P^{(m)}(y) \right) K(y) \, dy,
\]

for \( m = 1, 2, \ldots, \ell + 1 \). We note that \( \sigma_k^{(1)} = 0 \) by (1.1) and

\[
(\sigma_k^{(m)} * f)(\xi) = \hat{f}(\xi) \int_{E_k} e^{-2\pi i \langle P^{(m)}(y), \xi \rangle} K(y) \, dy,
\]

where \( \hat{f} \) denotes the Fourier transform of \( f \). A similar formula holds for \( \mu_k^{(m)} \).

Let \( \{\gamma(j,k)\}_{k=1}^{r_j} \) be an enumeration of \( \{\gamma\}_{|\gamma|=N(j)} \) for \( 1 \leq j \leq \ell \). Define a linear mapping \( L_j \) from \( \mathbb{R}^d \) to \( \mathbb{R}^{r_j} \) by

\[
L_j(\xi) = (a_{\gamma(j,1)}, \xi), (a_{\gamma(j,2)}, \xi), \ldots, (a_{\gamma(j,r_j)}, \xi),
\]

for \( 1 \leq j \leq \ell \). Let \( L_{\ell+1} \) be the identity mapping on \( \mathbb{R}^d \). Let \( s_j = \text{rank } L_j \). For \( j \geq 2 \), there exist non-singular linear transformations \( R_j : \mathbb{R}^d \to \mathbb{R}^d \) and \( H_j : \mathbb{R}^{s_j} \to \mathbb{R}^{s_j} \) such that

\[
|H_j \pi_s^d R_j(\xi)| \leq |L_j(\xi)| \leq C |H_j \pi_s^d R_j(\xi)|,
\]

where \( \pi_s^d(\xi) = (\xi_1, \ldots, \xi_{s_j}) \) is the projection and \( C \) is a constant depending only on \( r_j \) (see [5]).

Let \( \varphi \) be a function in \( C^\infty(\mathbb{R}) \) satisfying \( \varphi(r) = 1 \) for \( |r| < 1/2 \) with support in \( \{|r| \leq 1\} \). Define a sequence \( \tau^{(m)} = \{\tau_k^{(m)}\}_{k=-\infty}^{1} \) of Borel measures by

\[
\tau_k^{(m)}(\xi) = \hat{\sigma}_k^{(m+1)}(\xi) \Phi_{k,m+1}(\xi) - \hat{\sigma}_k^{(m)}(\xi) \Phi_{k,m}(\xi)
\]

for \( m = 1, 2, \ldots, \ell \), where

\[
\Phi_{k,m}(\xi) = \prod_{j=m+1}^{\ell+1} \varphi \left( \beta_j^k |H_j \pi_s^d R_j(\xi)| \right)
\]

if \( 1 \leq m \leq \ell \) and \( \Phi_{k,\ell+1} = 1 \). Then \( \sigma_k = \sigma_k^{(\ell+1)} = \sum_{m=1}^{\ell} \tau_k^{(m)} \). We note that

\[
\Phi_{k,m+1}(\xi) \varphi \left( \beta_k^m |H_{m+1} \pi_s^d R_{m+1}(\xi)| \right) = \Phi_{k,m}(\xi) \quad (1 \leq m \leq \ell).
\]

For \( 1 \leq m \leq \ell \), let \( T^{(m)}_\rho(f) = \sum_{k=-\infty}^{-1} \tau_k^{(m)} * f \). Then \( T = \sum_{m=1}^{\ell} T^{(m)}_\rho \).

For a sequence \( \nu = \{\nu_k\}_{k=-\infty}^{1} \) of finite Borel measures on \( \mathbb{R}^d \), let \( \nu^*(f)(x) = \sup_k ||\nu_k * f(x)|| \), where \( |\nu_k| \) denotes the total variation. We consider the maximal operators \( (\mu^{(m)})^* \) (1 \leq m \leq \ell + 1). We also write \( (\mu^{(\ell+1)})^* = \mu^*_\rho \).

Let \( \theta \in (0, 1) \). For \( p \in (1, \infty) \) let \( p' = p/(p - 1) \) and \( \delta(p) = |1/p - 1/p'| \). Then we prove the following two propositions.
Proposition 2.1 Let \( p > 1 + \theta \) and \( 1 \leq j \leq \ell + 1 \). Then we have
\[
\left\| (\mu^{(j)})(f) \right\|_{L^p(\mathbb{R}^d)} \leq C(\log \rho)\|h\|_{\Lambda^q_1} \|\Omega\|_{L^q(S^{n-1})} B^{2/p} \|f\|_{L^p(\mathbb{R}^d)},
\]  
where \( B = \left(1 - \rho^{-\beta / q'}\right)^{-1} \) for some positive constant \( \kappa \) such that
\[
(1 - \beta_m^{-\beta/k_m})^{-1} \leq B
\]
for all \( m \) with \( 2 \leq m \leq \ell + 1 \). The constant \( C \) is independent of \( q \in (1, 2] \), \( h \in \Lambda^q_1 \), \( \Omega \in L^q(S^{n-1}) \) and \( \rho \).

Proposition 2.2 Let \( p \in (1 + \theta, (1 + \theta)/\theta) \) and \( 1 \leq m \leq \ell \). Then
\[
\| T^{(m)}(f) \|_{L^p(\mathbb{R}^d)} \leq C(\log \rho)\|h\|_{\Lambda^q_1} \|\Omega\|_{L^q(S^{n-1})} B^{1+\delta(p)} \|f\|_{L^p(\mathbb{R}^d)},
\]
where \( B \) is as in Proposition 2.1 and the constant \( C \) is independent of \( q \in (1, 2] \), \( h \in \Lambda^q_1 \), \( \Omega \in L^q(S^{n-1}) \) and \( \rho \).

We can easily derive Theorem 1.1 from Proposition 2.2. Proposition 2.1 is used to prove Proposition 2.2. To prove Proposition 2.2 we also need the following.

Lemma 2.3 Let \( q \in (1, 2] \), \( \Omega \in L^q(S^{n-1}) \), \( h \in \Lambda^q_1 \) and \( A = (\log \rho)\|h\|_{\Lambda^q_1} \|\Omega\|_q \). Let \( \tau^{(m)}_k \) be as in (2.2). Then, for \( 1 \leq m \leq \ell \) we have
\[
\|\tau^{(m)}_k\| = |\tau^{(m)}_k|_{(\mathbb{R}^d)} \leq c_1 A, \tag{2.5}
\]
\[
|\hat{\tau^{(m)}_k}(\xi)| \leq c_2 A \left(\beta^k_{m+1} |L_{m+1}(\xi)|\right)^{-\alpha_{m+1}}, \tag{2.6}
\]
\[
|\hat{\tau^{(m)}_k}(\xi)| \leq c_3 A \left(\beta^{k+1}_{m+1} |L_{m+1}(\xi)|\right)^{-\alpha_{m+1}}, \tag{2.7}
\]
for all \( k \in \mathbb{Z} \) satisfying \( k \leq L \) with some constants \( c_i \) (\( 1 \leq i \leq 3 \)), where \( L \) is a negative integer, \( L \leq -4 \), which will be determined in Lemma 2.4 below.

To prove Lemma 2.3 we need the following two lemmas.

Lemma 2.4 Let \( 1 < q \leq 2 \), \( \Omega \in L^q(S^{n-1}) \), \( h \in \Lambda^q_1 \) and let \( \sigma_k \) be as in (2.1). Then, there exist a positive integer \( M \), a positive number \( \epsilon_0 \in (0, 1/4) \) and a negative integer \( L, L \leq -4 \), such that
\[
|\hat{\sigma}_k(\xi)| \leq C(\log \rho) \left(|\xi|\rho^{kM}\right)^{-\epsilon_0/q'} \|h\|_{\Lambda^q_1} \|\Omega\|_q
\]
for \( k \leq L \). The constants \( M, \epsilon_0, L \) and \( C \) are independent of \( \rho, q, h \) and \( \Omega \).

Lemma 2.5 Let \( \rho \geq 2 \), \( k \in \mathbb{Z} \), \( 1 < q \leq 2 \), \( h \in \Lambda^q_1 \) and \( \Omega \in L^q(S^{n-1}) \). Let \( P \) be a real-valued polynomial on \( \mathbb{R}^n \) of degree \( m \geq 1 \). Write
\[
P(x) = \sum_{|\alpha|=m} a_{\alpha} x^{\alpha} + Q(y),
\]
where \( \text{deg} \, Q \leq m - 1 \) if \( Q \neq 0 \). Then there exists a constant \( C > 0 \) independent of \( \rho, k, q, h, \Omega \) and the coefficients of the polynomial \( P \) such that

\[
\left| \int_{\rho^k \leq |y| < \rho^{k+1}} \exp(iP(x))h(|x|)\Omega(x')|x|^{-n} \, dx \right| \\
\leq C(\log \rho)\|h\|_{\Lambda^q_1}\|\Omega\|_q \left( \rho^{km} \sum_{|\alpha|=m} |a_{\alpha}| \right)^{-\tau/(mq')} ,
\]

where \( \tau = 4^{-1} \min(1, \eta) \).

We can prove Lemma 2.5 similarly to the proof of Lemma 2.4 of [4]. To prove Lemma 2.4 we need the following two results, which can be found in [4].

**Lemma 2.6** Let \( \Phi : B(0, 1) \to \mathbb{R}^d \) be smooth and of finite type at the origin. Define \( G_m : B(0, 1) \times S^{d-1} \to \mathbb{R} \) by

\[
G_m(x, \xi) = \sum_{|\alpha|=m} \langle \xi, \partial_x^\alpha \Phi(x) \rangle x^\alpha a! 
\]

for \( m \geq 1 \). Then, there exist constants \( R, \delta \in (0, 1/4) \) and a mapping \( \ell \) from \( S^{d-1} \) to a finite set of positive integers such that

\[
\sup_{\xi \in S^{d-1}} \int_{|x| \leq R} |G_\ell(x, \xi)|^{-\delta} \, dx < \infty.
\]

**Lemma 2.7** Let \( \psi, \phi \in C^\infty(\mathbb{R}) \) be real-valued. Let \( s \in (0, 1] \) and \( a, b \in \mathbb{R} \) with \( a < b \). Suppose that \( \phi \) is compactly supported and that

\[
|\langle \partial_x^k \psi(x) \rangle| \leq s \quad \text{for } x \in [a, b], \\
|\langle \partial_x^{(k+1)} \psi(x) \rangle| \leq 1 \quad \text{for } x \in [a-s, b+s].
\]

where \( k \) is a positive integer. Then, there exists a positive constant \( C \) depending only on \( k \) and \( \phi \) such that

\[
\left| \int_a^b \exp(i\lambda \psi(x))\phi(x) \, dx \right| \leq C|\lambda|^{-\epsilon/k} \int_{a-s}^{b+s} |\langle \partial_x^k \psi(x) \rangle|^{-\epsilon(1+1/k)} \, dx
\]

for all \( \lambda \in \mathbb{R} \setminus \{0\} \) and \( \epsilon \in (0, 1] \).

Define a function \( F \) on an appropriate subinterval of \( \mathbb{R}_+ \) by \( F(t) = \langle \xi, \Phi(tx) \rangle \) for fixed \( \xi \in S^{d-1} \) and \( x \in B(0, 1) \). Then, we note that \( (d/dt)^m F(t) = t^{-m} G_m(tx, \xi) \), where \( G_m \) is as in Lemma 2.6.

**Proof of Lemma 2.4:** Take an integer \( v \geq 1 \) and \( a \in [2, 4] \) such that \( \rho = a^v \). Let \( \Phi, \delta, R \) and \( \ell(\xi) \) be as in Lemma 2.6. Put \( \ell_0 = \max_{\xi \in S^{d-1}} \ell(\xi) \). Let \( L \) be a negative integer such that

\[
|\langle \partial_x^L \rangle \xi', \Phi(\rho^k s \theta) \rangle| < 1/2
\]
for $1 \leq \ell \leq \ell_0 + 1, \ s \in [1, \rho], \ r \in (0, 5), \ \xi' \in S^{d-1}$ and $\theta \in S^{n-1}$ whenever $k \leq L$ and such that $2^{k+2} < R$ if $k \leq L$. Then, when $\xi \in \mathbb{R}^d \setminus \{0\}$ and $k \leq L$, we write

$$\hat{\sigma}_k(\xi) = \sum_{j=0}^{v-1} \int_{\rho^k a^j}^{\rho^k a^{j+1}} \int_{S^{n-1}} \exp(-2\pi i \langle \xi, \Phi(r\theta) \rangle) \ h(r) \Omega(\theta) \ d\sigma(\theta) \ dr/r$$

$$= \sum_{j=0}^{v-1} \int_{1}^{a} \int_{S^{n-1}} \exp(-2\pi i \langle \xi, \Phi(\rho^k a^j r\theta) \rangle) \ h(\rho^k a^j r) \Omega(\theta) \ d\sigma(\theta) \ dr/r.$$

Let $\phi \in C^\infty(\mathbb{R})$ satisfy $\text{supp}(\phi) \subset (0, 10^{-9})$, $\phi \geq 0$, $\int \phi(s) \ ds = 1$. Define $h_j(r) = \int_{s<r/2} h(\rho^k a^j \ (r-s)) \phi_u(s) \ ds$, $r > 0$, where $\phi_u(s) = u^{-1} \phi(u^{-1}s)$, $u > 0$. Then, if $u < 1$,

$$\int_{1}^{a} |h(\rho^k a^j r) - h_j(r)| \ dr/r \leq C \omega(h, u). \quad (2.8)$$

We take $u = (|\xi| \rho^{kM})^{-\zeta/q'}$ for a suitable $M$ with $M \geq \ell_0$ and $\zeta > 0$, which will be specified below. We assume $|\xi| \rho^{kM} \geq 1$ for the moment. Define

$$s_k(\xi) = \sum_{j=0}^{v-1} \int_{1}^{a} \int_{S^{n-1}} \exp(-2\pi i \langle \xi, \Phi(\rho^k a^j r\theta) \rangle) \ h_j(r) \Omega(\theta) \ d\sigma(\theta) \ dr/r.$$

Then, by (2.8)

$$|\hat{\sigma}_k(\xi) - s_k(\xi)| \leq C(\log \rho) \|\Omega\|_1 \omega(h, u) \quad (2.9)$$

$$\leq C(\log \rho) \|\Omega\|_1 \|h\|_{\Lambda^V} (|\xi| \rho^{kM})^{-\eta/\zeta/q'},$$

where we have used the fact that $v \approx \log \rho$.

By Lemma 2.7

$$\left| \int_{1}^{w} \exp(-2\pi i \langle \xi, \Phi(\rho^k a^j r\theta) \rangle) \ dt \right|$$

$$\leq C|\xi|^{-\epsilon/(\ell(\xi'))} \int_{1/2}^{a+1/2} \left| G_{\ell(\xi')}(\rho^k a^j r\theta, \xi') \right|^{-\epsilon(1+1/\ell(\xi'))} \ dr$$

for $w \in [1, a]$, where $\xi' = \xi/|\xi|$. Also, $|h_j(\xi)| \leq Cu^{-1} \|h\|_{\Delta_1}$, $\int_{1}^{a} |h_j(r)| \ dr/r \leq C\|h\|_{\Delta_1}$, $\int_{1}^{a} |h_j'(r)| \ dr/r \leq Cu^{-1} \|h\|_{\Delta_1}$. Therefore, applying integration by parts, we see that

$$\left| \int_{1}^{a} \exp(-2\pi i \langle \xi, \Phi(\rho^k a^j r\theta) \rangle) \ h_j(r) \ dr/r \right|$$

$$\leq Cu^{-1} \|h\|_{\Delta_1} |\xi|^{-\epsilon/(\ell(\xi'))} \int_{1/2}^{a+1/2} \left| G_{\ell(\xi')}(\rho^k a^j r\theta, \xi') \right|^{-\epsilon(1+1/\ell(\xi'))} \ dr/r.$$
Note that
\[
\int_{S^{n-1}} \left( \int_{1/2}^{a+1/2} \left| G_{\ell}(\xi')(\rho^k a^j r \theta, \xi') \right|^{-\epsilon(1+1/\ell(\xi'))} dr/r \right) |\Omega(\theta)| d\sigma(\theta)
\]
\[
\leq C(\rho^k a^j)^{-n} \int_{|x| \leq 2\rho^k a^j+1} \left| G_{\ell}(\xi')(x, \xi') \right|^{-2\epsilon} |\Omega(x')| dx =: I,
\]
where \( \epsilon \in (0, 1) \). Since \( 2\rho^k a^{j+1} < R \), by Hölder's inequality we have
\[
I \leq C(\rho^k a^j)^{-n} (\rho^k a^j)^{n/q} |\Omega_q \left( \int_{|x| \leq R} |G_{\ell}(\xi')(x, \xi')|^{-2\epsilon} dx \right)^{1/q'}.
\]
Therefore
\[
\sum_{j=0}^{v-1} (\rho^k a^j)^{-n} \int_{|x| \leq 2\rho^k a^{j+1}} \left| G_{\ell}(\xi')(x, \xi') \right|^{-2\epsilon} |\Omega(x')| dx 
\]
\[
\leq C \|\Omega\|_q \rho^{-kn/q'} \left( \sum_{j=0}^{v-1} a^{-jn/q'} \right) \left( \int_{|x| \leq R} |G_{\ell}(\xi')(x, \xi')|^{-2\epsilon} dx \right)^{1/q'} 
\]
\[
\leq C(\log \rho) \|\Omega\|_q \rho^{-kn/q'} \left( \int_{|x| \leq R} |G_{\ell}(\xi')(x, \xi')|^{-2\epsilon} dx \right)^{1/q'},
\]
since \( v \approx \log \rho \). Using these estimates, we have
\[
\left| \sum_{j=0}^{v-1} \int_{1}^{a} \int_{S^{n-1}} \exp \left( -2\pi i \langle \xi, \Phi(\rho^k a^j r \theta) \rangle \right) h_j(r) |\Omega(\theta)| d\sigma(\theta) dr/r \right| 
\]
\[
\leq C(\log \rho) u^{-1} \|h\|_{\Delta_1} \|\xi\|^{-\epsilon/\ell(\xi')} \|\Omega\|_q \rho^{-kn/q'} \left( \int_{|x| \leq R} |G_{\ell}(\xi')(x, \xi')|^{-2\epsilon} dx \right)^{1/q'},
\]
where \( C \) is independent of \( \epsilon, \rho, q, h \) and \( \Omega \). If we put \( \epsilon = \delta/(2q') \), then by Lemma 2.6 we have
\[
|s_k(\xi)| \leq CC_{\Phi}^{1/q'} (\log \rho) \|h\|_{\Delta_1} \|\Omega\|_q (|\xi|\rho^{kM})^{\delta/q'} (|\xi|\rho^{2kn(\xi')/\delta} - \delta/(2q'\ell(\xi'))).
\]
Therefore, if \( M \) is a positive integer such that \( M - 1 < 2n\ell_0/\delta \leq M \) and \( \zeta < \delta/(2\ell_0) \),
\[
|s_k(\xi)| \leq CC_{\Phi}^{1/q'} (\log \rho) \|h\|_{\Delta_1} \|\Omega\|_q (|\xi|\rho^{kM})^{-(\delta/(2\ell_0) - \zeta)/q'}.
\]
(2.10)
Combining (2.9) and (2.10), we can see that
\[
|\hat{s}_k(\xi)| \leq C(\log \rho) \|h\|_{\Delta_1} \|\Omega\|_q (|\xi|\rho^{kM})^{-\epsilon_0/q'},
\]
where \( \epsilon_0 = \min(\eta\zeta, \delta/(2\ell_0) - \zeta) \). If \( |\xi|\rho^{kM} \leq 1 \), the conclusion of Lemma 2.4 follows from the estimate \( |\hat{s}_k(\xi)| \leq C(\log \rho) \|h\|_{\Delta_1} \|\Omega\|_1 \) (see (2.14) below with \( m = \ell + 1 \)). This completes the proof of Lemma 2.4. □
Proof of Lemma 2.5: Let

\[ I(x) = \int_1^\rho \exp \left( i \left( (\rho^k t)^m \sum_{|\alpha|=m} a_{\alpha} x^\alpha + Q(\rho^k t x) \right) \right) h(\rho^k t) \, dt. \]

Note that

\[ \int_{\rho^k \leq |y| < \rho^{k+1}} \exp (i P(x)) h(|x|) \Omega(x') |x|^{-n} \, dx = \int_{S^{n-1}} \Omega(\theta) I(\theta) \, d\sigma(\theta). \]

Let \( a \in [2, 4] \) and \( \nu \geq 1 \) be as in the proof of Lemma 2.4. Decompose \( I(x) = \sum_{j=0}^{\nu-1} I_j(x) \), where

\[ I_j(x) = \int_1^a \exp \left( i \left( (\rho^k a_j t)^m \sum_{|\alpha|=m} a_{\alpha} x^\alpha + Q(\rho^k a_j t x) \right) \right) h(\rho^k a_j t) \, dt. \]

Let \( h_j(t) = \int_{s<t/2} h(\rho^k a_j (t-s)) \phi_u(s) \, ds \) be as in the proof of Lemma 2.4 and

\[ \tilde{I}_j(x) = \int_1^a \exp \left( i \left( (\rho^k a_j t)^m \sum_{|\alpha|=m} a_{\alpha} x^\alpha + Q(\rho^k a_j t x) \right) \right) h_j(t) \, dt. \]

Then by (2.8)

\[ |I_j(x) - \tilde{I}_j(x)| \leq C \omega(h, u), \quad 0 < u < 1. \]

So,

\[ \left| \int_{S^{n-1}} \Omega(\theta) I_j(\theta) \, d\sigma(\theta) - \int_{S^{n-1}} \Omega(\theta) \tilde{I}_j(\theta) \, d\sigma(\theta) \right| \]

\[ \leq \int_{S^{n-1}} |\Omega(\theta)||I_j(\theta) - \tilde{I}_j(\theta)| \, d\sigma(\theta) \]

\[ \leq C \omega(h, u) \|\Omega\|_1 \leq C \|h\|_{\Delta^\gamma} \|\Omega\|_1 u^\eta \]

for \( 0 \leq j \leq \nu - 1 \). Also, since \( |I(x)| \leq C(\log \rho) \|h\|_{\Delta^1} \),

\[ \left| \int_{S^{n-1}} \Omega(\theta) I(\theta) \, d\sigma(\theta) \right| \leq C(\log \rho) \|h\|_{\Delta^1} \|\Omega\|_1. \]

Now, we assume that \( b := \rho^{km} \sum_{|\alpha|=m} |a_{\alpha}| \geq 1 \) and put \( u = (a^m b)^{-1/(4mq')} \). Then, as in the proof of Lemma 2.4, an integration by parts argument implies that

\[ \left| \tilde{I}_j(x) \right| \leq C u^{-1} \|h\|_{\Delta^1} \left( (\rho^k a_j^m)^{-1/m} \sum_{|\alpha|=m} a_{\alpha} x^\alpha \right), \]

since

\[ \int_1^w \exp \left( i \left( (\rho^k a_j t)^m \sum_{|\alpha|=m} a_{\alpha} x^\alpha + Q(\rho^k a_j t x) \right) \right) \, dt \]

\[ \leq C \left( (\rho^k a_j^m)^{-1/m} \sum_{|\alpha|=m} a_{\alpha} x^\alpha \right) \]
for \( w \in [1, a] \), which follows from van der Corput’s lemma. We also have \(|\tilde{I}_j(x)| \leq C\|h\|_{\Delta_1}\). Combining this with (2.13), we have

\[
|\tilde{I}_j(x)| \leq Cu^{-1}\|h\|_{\Delta_1} \min \left( 1, \left( \rho^k a^j \right)^m \sum_{|\alpha|=m} a_{\alpha} x^\alpha \right)^{-1/(2mq')} \]

and hence by Hölder’s inequality and [7, Corollary 1]

\[
\int_{S^{n-1}} \Omega(\theta) \tilde{I}_j(\theta) d\sigma(\theta) \leq \int_{S^{n-1}} |\Omega(\theta) \tilde{I}_j(\theta)| d\sigma(\theta) \leq \|\Omega\|_q \|\tilde{I}_j\|_{q'} \\
\leq Cu^{-1}\|h\|_{\Delta_1} \|\Omega\|_q \left( \int_{S^{n-1}} \left( \rho^k a^j \right)^m \sum_{|\alpha|=m} a_{\alpha} \theta^\alpha \right)^{-1/(2m)} \right)^{1/q'} \\
\leq C\|h\|_{\Delta_1} \|\Omega\|_q \left( \rho^k \sum_{|\alpha|=m} |a_{\alpha}| \right)^{-1/(4mq')} .
\]

By this estimate and (2.11) we see that

\[
\int_{S^{n-1}} \Omega(\theta) I_j(\theta) d\sigma(\theta) \\
\leq C \left( \|h\|_{\Delta_1} \|\Omega\|_1 + \|h\|_{\Delta_1} \|\Omega\|_q \right) \left( \rho^k \sum_{|\alpha|=m} |a_{\alpha}| \right)^{-\tau/(mq')},
\]

where \( \tau = 4^{-1} \min(1, \eta) \). Thus

\[
\int_{S^{n-1}} \Omega(\theta) I_j(\theta) d\sigma(\theta) \leq \sum_{j=0}^{\nu-1} \int_{S^{n-1}} \Omega(\theta) I_j(\theta) d\sigma(\theta) \\
\leq C(\log \rho) \left( \|h\|_{\Delta_1} \|\Omega\|_1 + \|h\|_{\Delta_1} \|\Omega\|_q \right) \left( \rho^k \sum_{|\alpha|=m} |a_{\alpha}| \right)^{-\tau/(mq')},
\]

if \( \rho^k \sum_{|\alpha|=m} |a_{\alpha}| \geq 1 \). Along with (2.12), this implies the conclusion of Lemma 2.5. □

**Proof of Lemma 2.3:** We easily see that

\[
\|\sigma^{(m)}_k\| \leq C\|\Omega\|_1 \int_{\rho^k}^{\rho^{k+1}} |h(r)| dr/r \leq C(\log \rho)\|h\|_{\Delta_1} \|\Omega\|_1 \tag{2.14}
\]

for \( 1 \leq m \leq \ell + 1 \). By (2.14) and (2.2) we have

\[
\|\tau^{(m)}_k\| \leq C(\log \rho)\|h\|_{\Delta_1} \|\Omega\|_1 \tag{2.15}
\]

for \( 1 \leq m \leq \ell \). By (2.15) and Hölder’s inequality we have (2.5).
Let \( k \leq L \), where \( L \) is as in Lemma 2.4. By Lemmas 2.4 and 2.5 we have \( |\hat{\sigma}_k^{(m)}(\xi)| \leq CA \left( \beta_m^k |L_m(\xi)| \right)^{-\alpha_m} \) for \( m = 2, \ldots, \ell + 1 \). Also, we note that \( |\Phi_k,m(\xi)| \) is bounded by \( C \left( \rho_{m+1}^k |L_{m+1}(\xi)| \right)^{-N} \) for all \( N > 0 \), when \( 1 \leq m \leq \ell \). Using these estimates and (2.14) in the definition of \( \tilde{\tau}_k^{(m)} \) in (2.2), we have (2.6).

To prove (2.7), we note that

\[
\left| \hat{\sigma}_k^{(m+1)}(\xi) - \hat{\sigma}_k^{(m)}(\xi) \right| \leq C(\log \rho) \| h \|_{\Delta_1} \| \Omega \|_1 \beta_{m+1}^{k+1} |L_{m+1}(\xi)|. \tag{2.16}
\]

Also, by (2.3) we see that

\[
\left| \Phi_{k,m+1}(\xi) - \Phi_{k,m}(\xi) \right| \leq C \beta_{m+1}^k |L_{m+1}(\xi)|. \tag{2.17}
\]

The estimates (2.14), (2.16) and (2.17) imply

\[
|\hat{\tau}_k^{(m)}(\xi)| \leq C(\log \rho) \| h \|_{\Delta_1} \| \Omega \|_1 \beta_{m+1}^{k+1} |L_{m+1}(\xi)|, \tag{2.18}
\]

since

\[
|\hat{\tau}_k^{(m)}(\xi)| \leq \left| \left( \hat{\sigma}_k^{(m+1)}(\xi) - \hat{\sigma}_k^{(m)}(\xi) \right) \Phi_{k,m+1}(\xi) \right| + \left| \left( \Phi_{k,m+1}(\xi) - \Phi_{k,m}(\xi) \right) \hat{\sigma}_k^{(m)}(\xi) \right|.
\]

By (2.15) we also have \( |\hat{\tau}_k^{(m)}(\xi)| \leq C(\log \rho) \| h \|_{\Delta_1} \| \Omega \|_1 \). This estimate and (2.18) imply (2.7). This completes the proof of Lemma 2.3.

\[\square\]

**Proof of Proposition 2.2:** Let \( \tilde{T}_\rho^{(m)}(f) = \sum_{k \leq L} \tilde{t}_k^{(m)} \ast f \) for \( 1 \leq m \leq \ell \), where \( L \) is as in Lemma 2.3. Then, to prove Proposition 2.2 it suffices to show a version of Proposition 2.2 for \( \tilde{T}_\rho^{(m)} \) with bounds similar to those for \( T_\rho^{(m)} \), since \( \| T_\rho^{(m)}(f) - \tilde{T}_\rho^{(m)}(f) \|_p \leq CA \| f \|_p \) for \( 1 \leq p \leq \infty \), where \( A \) is as in Lemma 2.3. Let \( \{ \psi_k \}_{k=1}^\infty \) be a sequence of non-negative functions in \( C^\infty(\mathbb{R}) \) such that each \( \psi_k \) is supported in \( [\beta_{m+1}^{-k} - 1, \beta_{m+1}^{-k} + 1] \), \( \sum_k \psi_k(t)^2 = 1 \) for \( t > 0 \) and

\[
|\frac{d}{dt} \psi_k(t)| \leq c_j |t|^{-j}, \quad j = 1, 2, \ldots,
\]

where the constants \( c_j \) are independent of \( \beta_{m+1} \). This is possible since \( \beta_{m+1} \geq 2 \). Let

\[
\left( S_k^{(m+1)}(f) \right) (\xi) = \psi_k \left( |H_{m+1}D_{x_{m+1}} R_{m+1}(\xi)| \right) \hat{f}(\xi).
\]

We also write \( S_k^{(m+1)} = S_k \). Put

\[
D_j^{(m)}(f) = \sum_{k=-\infty}^L S_{j+k}^{(m)} (\hat{t}_k^{(m)} \ast S_{j+k}(f)).
\]

Then \( \tilde{T}_\rho^{(m)} = \sum_j D_j^{(m)} \). Plancherel’s theorem and the estimates (2.5)–(2.7) imply that

\[
\left\| D_j^{(m)}(f) \right\|_2^2 \leq \sum_{k \leq L} C \int_{\Delta_{j+k}} |\hat{t}_k^{(m)}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \\
\leq CA^2 \min \left( 1, \beta_{m+1}^{-2\alpha_m+1(|j|-2)} \right) \sum_{k \leq L} \int_{\Delta_{j+k}} |\hat{f}(\xi)|^2 d\xi \\
\leq CA^2 \min \left( 1, \beta_{m+1}^{-2\alpha_m+1(|j|-2)} \right) \| f \|_2^2,
\]

where

\[
\Delta_{j+k} = \{ \xi \in \mathbb{R}^n : \beta_{m+1}^{-k} \leq |\xi| < \beta_{m+1}^{-k} + 2 \}
\]

and

\[
\rho_{m+1}^k = \left( \beta_{m+1}^{-2\alpha_m+1(|j|-2)} \right) \rho_{m+1}^{2\alpha_m-1}.
\]
where $\Delta(k) = \{\beta_m^{-k-1} \leq |R_{m+1}(\xi)| \leq \beta_m^{-k+1}\}$. Thus we have
\[ \|D_j^{(m)}(f)\|_2 \leq CA \min\left(1, \beta_m^{-\alpha_m(1-j+2)}\right) \|f\|_2. \quad (2.19) \]
By (2.19) we have
\[ \|\tilde{T}_\rho^{(m)}(f)\|_2 \leq \sum_j \|D_j^{(m)}(f)\|_2 \leq CAB \|f\|_2, \quad (2.20) \]
since $B \geq \left(1 - \beta_m^{-\alpha_m+1}\right)^{-1}$, where $B$ is as in Proposition 2.1.

Taking Proposition 2.1 for granted for the moment and recalling the definition of $\tau(m)_{k}$ in (2.2), by change of variables and a well-known theorem for $L^p$ boundedness of maximal functions (see [5, Section 6]) we have
\[ \left\|\left(\tau^{(m)}\right)^* (f)\right\|_p \leq C \left\|\left(\mu^{(m+1)}\right)^* (|f|)\right\|_p + C \left\|\left(\mu^{(m)}\right)^* (|f|)\right\|_p \leq C_p AB^{2/p} \|f\|_p \quad (2.21) \]
for $p > 1 + \theta$.

By (2.5), (2.21) and the proof of Lemma in [3, p. 544], we have the following.

**Lemma 2.8** Let $u \in (1 + \theta, 2]$, $1/v - 1/2 = 1/(2u)$. Then we have
\[ \left\|\left(\sum_{k \leq L} \tau_k^{(m)} * g_k\right)^2\right\|_{1/v} \leq (c_1 C_u)^{1/2} AB^{1/u} \left\|\sum_{k \leq L} |g_k|^2\right\|_{1/v}, \]
where the constants $c_1$ and $C_u$ are as in (2.5) and (2.21), respectively.

Also, the Littlewood–Paley theory implies that
\[ \|D_j^{(m)}(f)\|_p \leq c_p \left\|\left(\sum_{k \leq L} |\tau_k^{(m)} * S_{j+k}(f)|^2\right)^{1/2}\right\|_p, \quad (2.22) \]
\[ \left\|\left(\sum_k |S_k(f)|^2\right)^{1/2}\right\|_p \leq c_p \|f\|_p, \quad (2.23) \]
where $1 < p < \infty$ and $c_p$ is independent of $\beta_{m+1}$ and the linear transformations $R_{m+1}, H_{m+1}$.

Let $1 + \theta < p \leq 4/(3 - \theta)$. Then, there exists $u \in (1 + \theta, 2]$ such that $1/p = 1/2 + (1 - \theta)/(2u)$. Let $1/v - 1/2 = 1/(2u)$. Then, by (2.22), (2.23) and Lemma 2.8 we have
\[ \|D_j^{(m)}(f)\|_v \leq CAB^{1/u} \|f\|_v, \quad (2.24) \]
where $C$ is independent of $\rho$ and the linear transformations $R_i$, $H_i$, $2 \leq i \leq \ell + 1$, and $\rho = 1/p = \theta/2 + (1 - \theta)/2$ and interpolating between (2.19) and (2.24), we have

$$
\|D_j^{(m)}(f)\|_p \leq CAB^{(1-\theta)/u} \min \left(1, \beta_m^{\ell m+1}(|j|-2)\right) \|f\|_p,
$$

which implies that

$$
\|\tilde{T}_\rho^{(m)}(f)\|_p \leq \sum_j \|D_j^{(m)}(f)\|_p \leq CAB^{(1-\theta)/u} \left(1 - \beta_m^{\ell m+1}\right)^{-1} \|f\|_p \quad (2.25)
$$

$$
\leq CAB^{2/p} \|f\|_p.
$$

A duality and interpolation argument using (2.20) and (2.25) implies the conclusion of Proposition 2.2 with $T_\rho^{(m)}$ replaced by $\tilde{T}_\rho^{(m)}$, which proves Proposition 2.2.

We now prove Proposition 2.1 by induction on $j$. First, the inequality $(\mu^{(1)})^*(f)(x) \leq C(\log \rho)\|h\|_{\Delta_1}\|\Omega\|_1 |f(x - P(0))|$ implies the estimate (2.4) for $j = 1$. Next, we prove (2.4) for $j = m$ by assuming (2.4) for $j = m - 1$, $2 \leq m \leq \ell + 1$. Define a sequence $\eta^{(m)} = (\eta_k^{(m)})_{k=-\infty}^{-1}$ of Borel measures on $\mathbb{R}^d$ by

$$
\eta_k^{(m)}(\xi) = \varphi \left(\frac{\rho^k}{\rho_m} |H_m \pi^{d}_{sm} R_m(\xi)|\right) \hat{\mu}_k^{(m-1)}(\xi),
$$

where $\varphi \in C^\infty_0(\mathbb{R})$ is as in the definition of $\tau_k^{(m)}$ in (2.2). Then, from (2.4) with $j = m - 1$, it follows that

$$
\left\| (\eta^{(m)})^*(f) \right\|_p \leq C \left\| (\mu^{(m-1)})^*(f) \right\|_p \leq CAB^{2/p} \|f\|_p
$$

(2.26)

for $p > 1 + \theta$, where $A$, $B$ are as above. As in the proof of Lemma 2.3, we have

$$
\|\eta_k^{(m)}\| + \|\mu_k^{(m)}\| \leq C\|\mu_k^{(m-1)}\| + \|\mu_k^{(m)}\| \quad (2.27)
$$

$$
\leq C\|\Omega\|_1 \int_{\rho_k^{k+1}}^{\rho^{k+1}} |h(r)| \, dr/r
$$

$$\leq C(\log \rho)\|h\|_{\Delta_1}\|\Omega\|_1 \leq CA.
$$

Let $k \leq L$, where $L$ is as above. Since

$$
|\hat{\mu}_k^{(m)}(\xi) - \eta_k^{(m)}(\xi)|
$$

$$\leq |\hat{\mu}_k^{(m)}(\xi) - \hat{\mu}_k^{(m-1)}(\xi)| + \left| \left(\varphi \left(\frac{\rho^k}{\rho_m} |H_m \pi^{d}_{sm} R_m(\xi)|\right) - 1 \right) \hat{\mu}_k^{(m-1)}(\xi) \right|,
$$

arguing as in the proof of (2.7), we see that

$$
|\hat{\mu}_k^{(m)}(\xi) - \eta_k^{(m)}(\xi)| \leq C(\log \rho)\|h\|_{\Delta_1}\|\Omega\|_1 \left(\frac{\rho^k}{\rho_m} |L_m(\xi)|\right)^{\alpha_m}
$$

(2.28)

$$\leq CA \left(\frac{\rho^{k+1}}{\rho_m} |L_m(\xi)|\right)^{\alpha_m}.
$$
We also have the following:

\[
|\hat{\mu}_k^{(m)}(\xi)| \leq CA \left( \frac{c_k}{L^m(\xi)} \right)^{-\alpha_m}, \tag{2.29}
\]

\[
|\hat{\eta}_k^{(m)}(\xi)| \leq C(\log \rho) ||h||_{\Delta_1} ||\Omega||_1 \left( \frac{c_k}{L^m(\xi)} \right)^{-\alpha_m}
\leq CA \left( \frac{c_k}{L^m(\xi)} \right)^{-\alpha_m}. \tag{2.30}
\]

We can prove the estimate (2.29) arguing as in the proof of (2.6). The definition of \(\eta^{(m)}_k\) and (2.27) imply the first inequality of (2.30).

We have only to prove (2.4) with \(j = m\) for \(p \in (1 + \theta, 2]\), since the estimate (2.4) for \(p > 2\) follows from interpolation between the estimate (2.4) for \(p \in (1 + \theta, 2]\) and the obvious estimate \(\|\hat{\mu}^{(m)}(f)\|_\infty \leq CA\|f\|_\infty\).

Let

\[
m_{m}(f)(x) = \left( \sum_{k \leq L} |\nu_k^{(m)} * f(x)|^2 \right)^{1/2},
\]

where \(\nu_k^{(m)} = \mu_k^{(m)} - \eta_k^{(m)}\). Then, we see that

\[
(\hat{\mu}^{(m)})^*(f) \leq m_{m}(f) + (\eta^{(m)})^*(|f|), \tag{2.31}
\]

where \((\hat{\mu}^{(m)})^*(f) = \sup_{k \leq L} |\mu_k^{(m)} * f|\). Note that to prove (2.4) with \(j = m\) it suffices to prove it with \((\hat{\mu}^{(m)})^*\) in place of \((\mu^{(m)})^*\). Since we have (2.26) and (2.31), to show (2.4) with \(j = m\) it suffices to prove \(\|m_{m}(f)\|_p \leq CAB^{2/p}\|f\|_p\) for \(p \in (1 + \theta, 2]\). Let

\[
U_{\epsilon}^{(m)}(f) = \sum_{k \leq L} \epsilon_k \nu_k^{(m)} * f,
\]

where \(\epsilon = \{\epsilon_k\}, \epsilon_k = 1\) or \(-1\). Then, we shall show that

\[
\left\|U_{\epsilon}^{(m)}(f)\right\|_p \leq CAB^{2/p}\|f\|_p \tag{2.32}
\]

for \(p \in (1 + \theta, 2]\), where \(C\) is independent of \(\epsilon\). The desired estimate follows from (2.32) by a well-known property of Rademacher’s functions.

To prove (2.32) we use the following:

**Lemma 2.9** Let \(\{p_j\}_1^\infty\) be a sequence of real numbers defined by \(p_1 = 2\) and \(1/p_{j+1} = 1/2 + (1 - \theta)/(2p_j)\) for \(j \geq 1\). Then, we have

\[
\left\|U_{\epsilon}^{(m)}(f)\right\|_{p_j} \leq C_j AB^{2/p_j}\|f\|_{p_j} \quad \text{for} \quad j \geq 1.
\]

We can see that \(1/p_{j} = (1 - a^j)/(1 + \theta)\), where \(a = (1 - \theta)/2\). Thus \(\{p_j\}\) is decreasing and converges to \(1 + \theta\). We can prove Lemma 2.9 by (2.26)–(2.30).
Proof: Define

\[ U_j^{(m)}(f) = \sum_{k=-\infty}^{L} \epsilon_k S_{j+k} \left( v_k^{(m)} \ast S_{j+k}(f) \right), \]

where \( S_k = S_k^{(m)} \) (the operators \( S_k^{(m)} \) are as in the proof of Proposition 2.2). Then, \( U_{\epsilon}^{(m)} = \sum_j U_j^{(m)} \). Arguing as in the proof of (2.19), and using Plancherel’s theorem and the estimates (2.27)–(2.30), we have

\[ \| U_j^{(m)}(f) \|_2 \leq CA \min \left( 1, \beta_m^{-\alpha_m(|j|^{-2})} \right) \| f \|_2, \]

and hence \( \| U_{\epsilon}^{(m)}(f) \|_2 \leq \sum_j \| U_j^{(m)}(f) \|_2 \leq CAB \| f \|_2 \). This proves the assertion of Lemma 2.9 for \( j = 1 \).

We now assume the estimate of Lemma 2.9 for \( j = s \) and prove it for \( j = s + 1 \). By induction, this will complete the proof of Lemma 2.9. From the estimate (2.31), it follows that

\[(\tilde{v}^{(m)})^*((f)) \leq (\tilde{\mu}^{(m)})^*((|f|)) + (\tilde{\eta}^{(m)})^*((|f|)) \leq g_m(|f|) + 2(\tilde{\eta}^{(m)})^*((|f|)),\]

where \((\tilde{v}^{(m)})^*((f)) = \sup_{k \leq \ell} |v_k^{(m)}| \ast f|\). By our assumption we have \( \| g_m(f) \|_{p_s} \leq CAB^{2/p} \| f \|_{p_s} \). This estimate and (2.26) imply

\[ \| (\tilde{v}^{(m)})^*((f)) \|_{p_s} \leq g_m(|f|) \| f \|_{p_s} + 2 \left( \| (\tilde{\eta}^{(m)})^*((|f|)) \|_{p_s} \right), \]

\[ \leq CAB^{2/p} \| f \|_{p_s}. \]

Arguing as in the proof of (2.25), and using (2.27), (2.33) and (2.34), we can now obtain the estimate of Lemma 2.9 for \( j = s + 1 \). This completes the proof of Lemma 2.9.

Let \( p \in (1 + \theta, 2] \) and let \( \{p_j\}_{1}^{\infty} \) be as in Lemma 2.9. Then, we can find a positive integer \( N \) such that \( p_{N+1} < p \leq p_N \). The estimate (2.32) now follows from interpolation between the estimates of Lemma 2.9 for \( j = N \) and \( j = N + 1 \). This finishes the proof of (2.4) for \( j = m \). By induction, this completes the proof of Proposition 2.1.

Proof of Theorem 1.1: By taking \( \rho = 2q^t \) in Proposition 2.2 we see that

\[ \| T_{2q^t}^{(m)}(f) \|_p \leq C_\theta (q - 1)^{-1} \| h \|_{\Lambda_1^q} \| \Omega \|_q \| f \|_p \]

for \( p \in (1 + \theta, (1 + \theta)/\theta) \). This completes the proof of Theorem 1.1, since \( T = \sum_{m=1}^\ell T_{\rho}^{(m)} \) and \( (1 + \theta, (1 + \theta)/\theta) \to (1, \infty) \) as \( \theta \to 0 \).

3 Estimates for maximal functions

Let

\[ T^*(f)(x) = \sup_{\epsilon \in (0,1)} \left| \int_{\epsilon < |y| < 1} f(x - \Phi(y))K(y)dy \right|, \]

(3.1)
where $K$ is as in (1.2). Then, we have an analog of Theorem 1.1 for the maximal operator $T^*$.  

**Theorem 3.1** Let $\Omega \in L^q(S^{n-1})$, $q \in (1, 2]$ and $h \in \Lambda_1^\eta$ for some $\eta > 0$. Suppose that $\Omega$ satisfies (1.1). Then  

$$
\|T^*(f)\|_{L^p(\mathbb{R}^d)} \leq C_p(q - 1)^{-1} \|h\|_{\Lambda_1^\eta} \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^p(\mathbb{R}^d)}
$$

for all $p \in (1, \infty)$, where $C_p$ is independent of $q$, $h$ and $\Omega$.

By Theorem 3.1 and extrapolation we have the following result.

**Theorem 3.2** Let $\Omega \in L\log L(S^{n-1})$ and $h \in \Lambda_1^\eta$ for some $\eta > 0$. Suppose that $\Omega$ satisfies the condition (1.1). Let $T^* f$ be defined as in (3.1) with the functions $h$ and $\Omega$. Then  

$$
\|T^*(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}
$$

for all $p \in (1, \infty)$.

If the function $h$ is identically 1, then Theorem 3.2 was shown in [1].

To prove Theorem 3.1, we use the following result.

**Lemma 3.3** Let $\theta \in (0, 1)$ and let positive numbers $A = (\log \rho)\|h\|_{\Lambda_1^\eta} \|\Omega\|_q$, $B = (1 - \rho^{-\theta\kappa/q})^{-1}$ be as above. Define

$$
T_{m, \rho}^*(f)(x) = \sup_{k \leq L} \left| \sum_{j = k}^{L} \tau_{m, j}^{(m)} \ast f(x) \right|
$$

for $1 \leq m \leq \ell$, where the measures $\tau_{m, j}^{(m)}$ are as in (2.2) and $L$ is as in Lemma 1. Let $I_\theta = (2(1 + \theta)/(\theta^2 - \theta + 2), (1 + \theta)/\theta)$. Then, we have  

$$
\|T_{m, \rho}^*(f)\|_{p} \leq CA \left( B^{1+\delta(p)} + B^{2/p+1-\theta/2} \right) \|f\|_p
$$

for $p \in I_\theta$, where $C$ is independent of $q \in (1, 2]$, $\Omega \in L^q(S^{n-1})$, $h \in \Lambda_1^\eta$ and $\rho$.

This can be proved by results in Section 2.

**Proof:** Let $\tilde{T}_{\rho}^{(m)}(f) = \sum_{k \leq L} \tau_{k}^{(m)} \ast f$ be as in the proof of Proposition 2.2. Let $\varphi_k$ be defined by

$$
\hat{\varphi}_k(\xi) = \varphi \left( \beta_{m+1}^{k} |H_{m+1}^{1/2} R_{m+1}^{1} (\xi)| \right),
$$

where $\varphi$ is as in the definition of $\tau_{k}^{(m)}$ in (2.2). We now decompose

$$
\sum_{j = k}^{L} \tau_{j}^{(m)} \ast f = \varphi_k \ast \tilde{T}_{\rho}^{(m)}(f) - \varphi_k \ast \left( \sum_{j = -\infty}^{k-1} \tau_{j}^{(m)} \ast f \right) + (\delta - \varphi_k) \ast \left( \sum_{j = k}^{L} \tau_{j}^{(m)} \ast f \right),
$$
where \( k \leq L \) and \( \delta = \delta_0 \) is the delta function on \( \mathbb{R}^d \) (see \([3, 5]\)). Then, we have

\[
T_{\rho, m}^*(f) \leq \sup_{k \leq L} |\varphi_k * \hat{T}_{\rho}^{(m)}(f) + \sum_{k=0}^{\infty} M_j^{(m)}(f)|, \tag{3.3}
\]

where

\[
M_j^{(m)}(f) = \sup_{k \leq L} \left| \varphi_k * \left( \tau_{j-k+1}^{(m)} f \right) \right| + \sup_{k \leq L-j} \left| (\delta - \varphi_k) * \left( \tau_{j+k}^{(m)} f \right) \right|.
\]

From Proposition 2.2 it follows that

\[
\left\| \sup_{k \leq L} \varphi_k * \hat{T}_{\rho}^{(m)}(f) \right\|_p \leq C A B^{1+\delta(p)} \| f \|_p \tag{3.4}
\]

for \( p \in (1 + \theta, (1 + \theta)/\theta) \), and the estimate (2.21) implies that

\[
\| M_j^{(m)}(f) \|_r \leq C A B^{2/r} \| f \|_r \quad \text{for} \quad r > 1 + \theta. \tag{3.5}
\]

Since

\[
M_j^{(m)}(f) \leq \left( \sum_{k \leq L-j} \left| (\delta - \varphi_k) * \left( \tau_{j-k+1}^{(m)} f \right) \right|^2 \right)^{1/2} + \left( \sum_{k \leq L} \left| \varphi_k * \left( \tau_{j+k}^{(m)} f \right) \right|^2 \right)^{1/2},
\]

arguing as in \([5, p. 820]\) and using the estimates (2.6) and (2.7) along with Plancherel’s theorem, we have

\[
\| M_j^{(m)}(f) \|_2 \leq C A B^{-\alpha_m+\theta} \left( 1 - B^{-2\alpha_m} \right)^{-1/2} \| f \|_2. \tag{3.6}
\]

We note that for any \( p \in I_\theta \) there exists a number \( r \in (1 + \theta, 2(1 + \theta)/\theta) \) such that \( 1/p = (1 + \theta)/r + \theta/2 \). Therefore, interpolating between (3.5) and (3.6), we have

\[
\| M_j^{(m)}(f) \|_p \leq C A B^{2(1-\theta)/r} \left( 1 - B^{-2\alpha_m} \right)^{-\theta/2} \| f \|_p. \tag{3.7}
\]

From (3.3), (3.4) and (3.7), it follows that

\[
\| T_{\rho, m}^*(f) \|_p \leq C A \left( B^{1+\delta(p)} + B^{2(1-\theta)/r+1} \left( 1 - B^{-2\alpha_m} \right)^{-\theta/2} \right) \| f \|_p
\]

for \( p \in I_\theta \). Using \((1 - B^{-2\alpha_m})^{-1} \leq B \) and \( 2(1 - \theta)/r + \theta/2 + 1 = 2/p + 1 - \theta/2 \) in this estimate, we can obtain the conclusion of Lemma 3.3. \( \square \)

**Proof of Theorem 3.1:** Let

\[
T_{\rho}^*(f)(x) = \sup_{\rho \in (0, \rho^{L+1})} \left| \int_{|y| < \rho^{L+1}} f(x - \Phi(y)) K(y) \, dy \right|.
\]
Then, we have

$$T^*(f)(x) \leq T^*_\rho(f)(x) + J_\rho(f)(x), \quad (3.8)$$

where $J_\rho(f)(x) = \int_{\rho^{L+1} \leq |y| < 1} |f(x - \Phi(y))||K(y)| \, dy$. We note that

$$T^*_\rho(f) \leq T^*_0,\rho(f) + \mu^*_\rho(|f|), \quad (3.9)$$

where $\mu^*_\rho = (\mu(\ell + 1))^*$ is as in Proposition 2.1 and $T^*_0,\rho(f)$ is defined by the formula in (3.2) with $\{\tau^*_j\}_{j \leq L}$ replaced by the sequence $\{\sigma_j\}_{j \leq L}$ of measures in (2.1). Since $T^*_0,\rho(f) \leq \sum_{m=1}^\ell T^*_m,\rho(f)$, using Lemma 3.3 with $\rho = 2^{q'}$, we see that

$$\|T^*_0,2^{q'}(f)\|_p \leq C_\theta(q - 1)^{-1} \|h\|_{\Lambda^\theta_1} \|\Omega\|_q \|f\|_p \quad (3.10)$$

for $p \in I_\theta$. Also, by Proposition 2.1 with $\rho = 2^{q'}$ we have

$$\|\mu^*_\rho(|f|)\|_p \leq C_\theta(q - 1)^{-1} \|h\|_{\Lambda^\theta_1} \|\Omega\|_q \|f\|_p \quad (3.11)$$

for $p \in I_\theta$. Note that

$$\int_{\rho^{L+1} \leq |y| < 1} |K(y)| \, dy \leq C(\log \rho) \|h\|_{\Delta_1} \|\Omega\|_1.$$

Therefore, it is easy to see that

$$\|J_{2^{q'}}(f)\|_p \leq C(q - 1)^{-1} \|h\|_{\Lambda^\theta_1} \|\Omega\|_q \|f\|_p \quad (3.12)$$

for $p \in I_\theta$. Since $I_\theta \to (1, \infty)$ as $\theta \to 0$, by (3.8)–(3.12) we obtain the conclusion of Theorem 3.1.

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**References**


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