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ABSTRACT

This paper formulates a discrete-time dynamic traffic assignment model and, under certain conditions, shows the existence and uniqueness of network equilibrium. Several theoretical issues need to be tackled. In discrete time traffic flow, the inflow to a link (or cell) in a particular discrete time period does not all necessarily exit within the same time period. We consider how flow is passed from one link and time period to the next, and the corresponding costs. Under the proposed model, flow departing within a discrete time period may experience different link travel times in different discrete time periods, even if the flow chooses a single route. Route travel time must then be defined so that route and OD costs are meaningful. To this end, *quasi-real route travel time* is defined. Based on this definition, a quasi-equilibrium condition for dynamic traffic assignment is proposed; a semi-dynamic analogue of user equilibrium. The existence and uniqueness of this equilibrium solution are proven.

Keywords: dynamic user equilibrium, quasi-real travel time, unique network flow, link-based formulation, discrete time

1. INTRODUCTION

Traffic conditions in most cities vary significantly within a day, so that static traffic assignment models often cannot represent the time-dependent congestion adequately for transportation network analysis. Even if the day is divided into several periods and static traffic assignment is made in each such *time slice*, the dynamics are not well described. Especially at peak times, not all of the demand can arrive at its destination within a single assignment, and the congestion should carry over to the next period. However, in the framework of static traffic assignment all demands reach their destinations in each time-slice and the traffic still *en route* is not dealt with between periods.

Continuous-time dynamic traffic assignment (DTA), which describes detailed dynamic traffic queues and congestion in continuous time, is theoretically preferable for modeling such scenarios. However, we face difficulties when applying continuous-time DTA to real networks. In particular it is difficult to obtain sufficiently accurate and detailed OD matrix data to describe the dynamic traffic in continuous time. In cases without high resolution OD data, dynamic traffic assignment must be reasonable in discrete time.

Typically, continuous-time DTA cannot necessarily be solved analytically in continuous time, and is in fact computed numerically in discrete time, with the discrete time interval chosen to be small. However, in the case of more coarse grained OD data, we will use a relatively larger discrete time period, e.g. 15 minutes or 30 minutes. This intermediate situation between the plausible application of time-sliced static assignment and continuous-time DTA is sometimes called “semi-dynamic traffic assignment”. Thus, we have the 4 models with increasing time resolution: (i) time-sliced static traffic assignment, (ii) traffic assignment with large discrete time or semi-dynamic traffic assignment, (iii) discrete-time DTA (with small interval) and (iv) continuous-time DTA. The capability of network modeling is different among these four DTA models. In general, the continuous-time DTA models satisfy the requirements of traffic flow theory e.g. the principle of FIFO; but uniqueness of network equilibrium flows is not necessarily guaranteed (Iryo, 2011). On the other hand, semi-dynamic models are closer to static traffic assignment, somewhat sharing the formulation and properties of network equilibrium. Crucially, it is possible to establish uniqueness of network equilibrium flows for semi-dynamic traffic assignment. If we have detailed dynamic OD data and desire high resolution dynamic traffic states, we must compute (iii) discrete-time DTA using small time intervals or (iv) continuous-time DTA. The latter is theoretically preferable, while the former is more often practically applicable. In cases having less data, or less demanding requirements of model outputs, (i) time-sliced static traffic assignment and (ii) semi-dynamic traffic assignment are more reasonable alternatives. Notwithstanding the computational cost of calculating continuous-time or discrete-time DTA, these models are not necessarily appropriate for coarse OD data. Meanwhile, time-sliced static traffic assignment completely neglects flow propagation between time periods. Particularly for peak period analysis, flow propagation is usually significant and semi-dynamic traffic assignment should be applied.

This classification of existing research highlights the position of the model presented in this paper in relation to time-sliced static traffic assignment. It does not distinguish other important contributions to the field of DTA such as the whole link and point queue models, or the representation of spill back (for example Gentile *et al*., 2007).

Development of different DTA models is important because we have to apply the models under the various circumstances and with variety of requirements. We can choose an appropriate model according to the circumstances and requirements under which it used, with respect to the purpose, data accuracy, computational cost and other parameters. Semi-dynamic DTA models contribute to this range of network models.

Peeta and Ziliaskopoulos (2001) provide a helpful review of the numerous past studies on DTA. Research continues apace, with new models and algorithms being developed in works such as Han et al. (2011), Szeto et al. (2011), Ban et al. (2008), Han (2007) and others. For practical applications, computational obstacles cannot be ignored. Route-based formulations require a tremendous number of variables for realistic networks. Furthermore, as noted above, continuous-time DTA computed using small time intervals requires high resolution OD matrices and detailed treatment of network flows across many time intervals; it is computationally demanding. In this paper, a link-based formulation of a discrete-time dynamic user equilibrium (UE) (or semi-dynamic user equilibrium) model with a unique solution is developed. So far, Kuwahara and Akamatsu (1993), Ran *et al*. (1996) and Li *et al*. (2000) have formulated link-based models for continuous time dynamic UE. The time discretization was examined in Wie *et al*. (2002).

Semi-dynamic traffic assignment models with UE (dynamic user equilibrium with large discrete time period) have been developed by Fujita *et al*. (1988, 1989), Miyagi and Makimura (1991), and Akamatsu *et al*. (1998). Fujita et al. (1988) and Miyagi and Makimura (1991) tackled flow propagation between large discrete time periods by modifying the OD demand in the next discrete time period, and formulated network user equilibrium in each discrete time period as an optimization problem with elastic demand. Uniqueness of the network flows was guaranteed under mild conditions. This flow propagation via elastic OD demands is somewhat coarse, motivating Fujita *et al*. (1989) to propose a link-based flow propagation model, formulated as a variational inequality. However, uniqueness of this model was not examined. Akamatsu *et al*. (1998) adopted the vertical queue to represent flow propagation, and formulated network user equilibrium as an optimization problem, having unique network flows. In this model, queues carry over to the next period, but queue propagation is simplified so that each queue jumps to the destination after it passes a bottleneck. Thus, in previous semi-dynamic traffic assignment models, flow propagation is rather coarse or uniqueness of equilibrium network flows is not guaranteed. A semi-dynamic traffic assignment model with practical flow propagation and unique equilibrium network flows remains to be developed.

Despite these existing works on discrete-time DTA or semi-dynamic traffic assignment, theoretical issues remain. The definition of UE is not necessarily unambiguous in the context of discrete-time DTA, because route travel time is not necessarily uniquely defined. By comparison, the concept of UE in continuous-time DTA is clear. Under the UE condition, no user can reduce his travel time he actually takes by changing his route unilaterally in the absence of departure time choice. The travel time is the time difference between departure time and actual arrival time. We shall call this the "real" route travel time. In discrete-time DTA, real route travel time is open to discussion; other relevant route travel times can also be determined. In traffic simulation, route travel time is often calculated to be the sum of link travel times at the moment of departure i.e. at the present moment without any evolution of the traffic flow. This travel time shall be called the "present" route travel time in this paper. The user does not necessarily act (only) on the present travel time, because he actually experiences the real route travel time ex post facto. Network equilibrium based on the present travel time is called "user optimal", rather than UE. In continuous time DTA, the real route travel time is clearly unique for each route and departure time. By contrast, it is not obvious how to define real route travel time for discrete-time DTA. All inflow to a link (in a discrete time period) does not necessarily exit that link within the same discrete time period, so the inflow will enter the downstream link in different time periods and hence will be ascribed different travel times. Thus, demand departing in one discrete time may experience different travel times, raising the question of how route travel time should be defined for this demand. We address this question and define "quasi-real" route travel time. On this basis, quasi-real-time-based dynamic user equilibrium (qDUE) is formulated according to the semi-dynamic model of Nakayama (2009). The existence and uniqueness of the qDUE model are examined.

2. NOTATION AND ASSUMPTIONS

Notations used in this paper are as follows:

x_{ijt} = the inflow to link *ij*, which connects node *i* and node *j*, in time period *t*
z_{ijt} = the outflow from link *ij* in time period *t*
y_{ijt} = the residual flow on link *ij* in time period *t*
x_{ijnt} = the inflow to link *ij* in time period *t* whose destination is node *n*,
z_{ijnt} = the outflow from link *ij* in time period *t* whose destination is node *n*
d_{int} = the travel demand whose destination is node *n* which departs node *i* in time period *t*
c_{ijt} = the travel time on link *ij* in time period *t*
τ_{int} = the minimum travel time from node *i* to node *n* in time period *t*
$\mathbf{x}$ = the vector of the link inflows (= $\{x_{ijnt}\}$)
$\mathbf{z}$ = the vector of the link outflows (= $\{z_{ijnt}\}$)
$\mathbf{c}$ = the vector of the link travel times (= $\{c_{ijt}\}$)
$\boldsymbol{\tau}$ = the vector of the minimum travel times (= $\{\tau_{int}\}$)
N = the set of nodes
N_{-n} = the set of nodes except node *n*
D = the set of destination nodes ($D \subset N$)
T = the set of (discrete) time periods
L = the length of the (discrete) time period
A = the set of links
A_{-n} = the set of links whose start node is not node *n*
N_i^{out} = the set of end nodes of the links that are connected from node *i*
N_i^{in} = the set of start nodes of the links that connect to node *i*

Assumptions are as follows:

A1 Time is discretized into intervals of length *L*.
A2 Demand and flow variables are similarly discretized. Within each time period,
  A2.1 link travel time does not change
  A2.2 demand/flow depart a node at a constant rate
  A2.3 demand/flow associated with different OD movements are treated equally.
A3 Route choice is made based on “quasi-real travel time” (defined later).
A4 Users experience the link travel time at the time period of their entry into the link (even if they do not traverse the whole of the link)

As assumption A1 shows, the time period [t_c, t_c+L) is represented by the discrete time *t* in this study, where t_c is ordinary continuous time and *t* is discrete time. Only when we define the flow on a link, imagine the time is continuous in the time period [t_c, t_c+L) of the discrete time period *t*. Consider a single link, with demand, *d*, departing in the period [t_c, t_c+L). The flow is assumed to depart at constant rate d/L throughout the time period (see sub-assumption A2.2). Then, not all of the flow will completely traverse the link within this time period; the part of the flow that does not exit is called **residual flow**. Assumptions on the residual flow are as follows:

A5 The residual flow on a link exits from the link in the next (run off) time period. Therefore the entire link inflow runs off in (at most) two successive time periods.
A6 The residual flow is a continuous and increasing function of the inflow on a link.
A7 In the next (run off) time period, the residual flow is treated as demand originating at the end node of the link (the start node of the next link), keeping its original destination node.

As mentioned stated

We clarify this approach with a simple example network consisting of two links in Figure 1.

(a) The network has a single OD with demand d_{13t} from node 1 to 3 in time period t.
(b) The demand departs uniformly at the rate d_{13t}/L (from a22) and hence the inflow to link 12 in time period t is $x_{12\mathrm{t}} = d_{13t}$. The travel time on link 12 in time period t is $\mathrm{c}_{12} = c_{12t}(x_{12t})$ as will be mentioned in assumption A9.
(c) Not all of the demand flow will traverse the entire link in time period t, leaving some residual flow, y_{12t}, on link 12. The magnitude of y_{12t} will depend on the link travel time. Part of the demand flow, $z_{12t} = x_{12t} - y_{12t}$, does traverse the whole of link 12 and reaches link 23 in time period t. This outflow from link 12 in time t is the inflow to link 23, $z_{12t} = x_{23t}$. This part of the flow experiences cost $c_{12t}+ c_{23t} = c_{12}(x_{12t})+ c_{23}(x_{23t})$ due to assumption A4.
(d) In time period t the inflow to link 23 is x_{23t}; part of this, z_{23t}, reaches the destination, leaving residual flow, y_{23t}.
(e) In the next period, t+1, the residual flow y_{12t} departs node 2 at a uniform rate (by assumption A7) Since there is no other demand flow originating at node 2, $x_{23(t+1)} = y_{12t}$, which experiences total cost $c_{12t} + c_{23(t+1)}$.
(f) Some of $x_{23(t+1)}$ reaches the destination within time period $t + 1$, but there is also some residual flow, $y_{23(t+1)}$, left on link 23. This residual flow will start the next time period at node 3 as assumed in assumption A7, and therefore will have arrived immediately at its destination with no further costs needing to be computed.

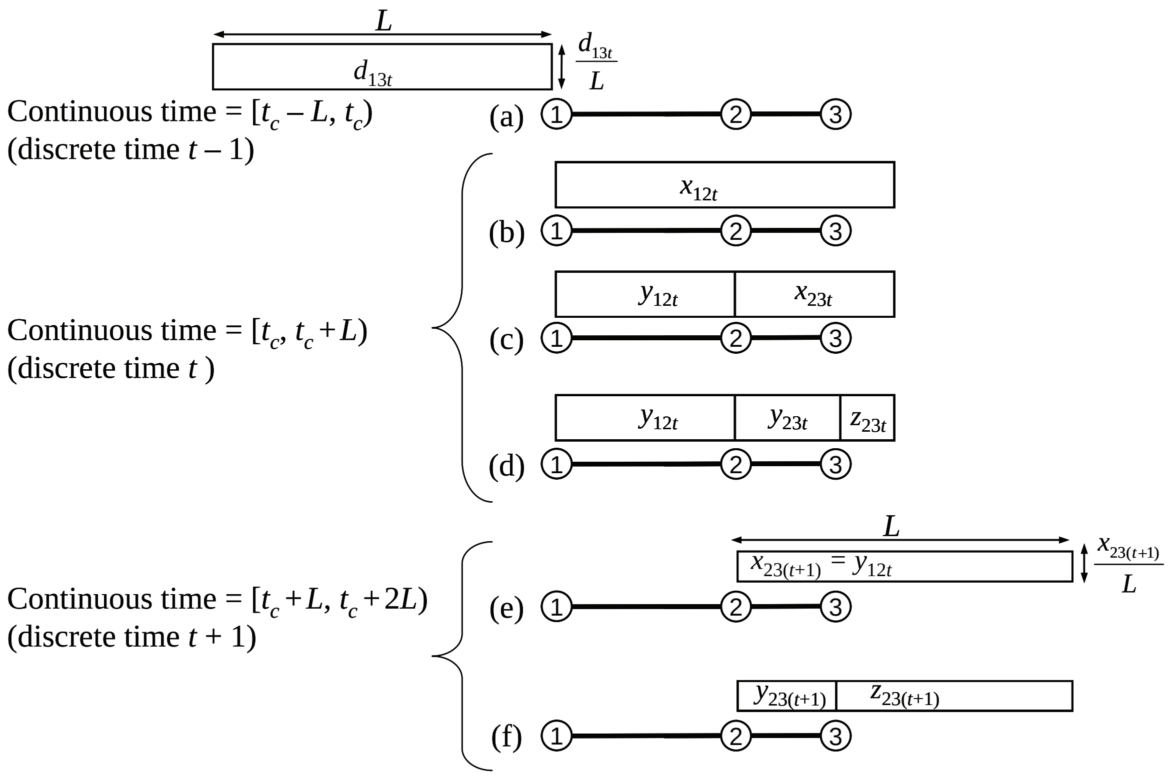


Figure 1: Residual flow

As illustrated above, the magnitude of $(x_{12t} - y_{12t}) = x_{23t}$ experience $(c_{12t} + c_{23t})$ while y_{12t} take $c_{12t} + c_{23(t+1)}$. Therefore, the (route) travel time of d_{13t} $(=x_{12t})$ is assumed to be the mean experienced travel time: $c_{12t} + (1 - y_{12t}/x_{12t})\, c_{23t} + (y_{12t}/x_{12t})\, c_{23(t+1)}$ for this example. This route travel time shall be called the **quasi-real route travel time**.[1] Assumption A2 states that we assume all of the demand, d_{int}, experiences the same travel time, even if they do not individually; the users of d_{int} are not distinguished. We ascribe the quasi-real route travel time to all of the demand, d_{int}.

Assumption A5 means that we only need to track flow across (the nearest) two successive time periods. The model could be extended to accommodate the flow clearing over three or more time periods, but this would be cumbersome. In order for this assumption to be reasonable as an approximation to the 'real' (continuous time) flow behavior, the length of time period L should be less than the maximum link travel time. This reinforces the notion that the model is intended to be used with a relatively coarse time discretization. Links could of course be (artificially) divided into smaller components if this assumption were limiting.

In general, unlike in Figure 1, the inflow to link ij will be heading to many different destinations. Part of the inflow, x_{ijt}, becomes the residual flow, y_{ijt}, which travels in the next time period. Destination information is preserved and the residual flow is treated as demand from node j in the next period. Assumption A7 arises from this requirement.

Assumptions on the link travel time is as follows:

A8 Link travel time is expressed as a function of its inflow.
A9 The link travel time function is continuous, strictly increasing and positive.

Assumptions A8 and A9 play an important role in the proof of uniqueness of the network equilibrium flows. However, the link travel time function based on A8 and A9, and the way residual flow is dealt with A5 and A7 does not ensure that we satisfy a strict and rigorous description of dynamic traffic flow. As mentioned previously, this model is intended to offer a representation that improves on the simple approach of time-sliced UE and is appropriate for a coarse discretization of time. We sacrifice the requirements of standard dynamic traffic flow theory in order to define a tractable model that guarantees uniqueness of the equilibrium flows, as will be stated later.

3. FORMULATIONS

3.1 Flow Conservation

According to assumption A5, the inflow exits the link in (at most) two successive time periods. That part of the inflow exiting the link within the same time period we are calling the outflow, while the other part is the residual flow. However, as assumption A4 states, all of the inflow experiences the link travel time at the time period of entry, even though only the outflow part exits in that time period. Furthermore, the outflow, z_{ijt}, does not include any residual flow from the previous time period, $y_{ij(t-1)}$. Any previous residual flow, $y_{ij(t-1)}$, will exist as demand from node j in time period t as mentioned in assumption A7. Therefore,

$$z_{ijt} = x_{ijt} - y_{ijt} \quad \forall ij \in A,\ \forall t \in T \tag{1}$$

[1] A more obvious term may be **mean real travel time**, however, the model in this study is deterministic, rather than stochastic. To avoid confusion, quasi-real travel time is adopted.

Consider all network flows with destination node n. At node i (with $i \neq n$) and time t we can specify the flow conservation condition as follows:

$$\sum_{k \in N_i^{in}} z_{kint} + q_{int} = \sum_{j \in N_i^{out}} x_{ijnt} \tag{2}$$

where

$$q_{int} = d_{int} + \sum_{k \in N_i^{in}} y_{kin(t-1)} = d_{int} + \sum_{k \in N_i^{in}} [x_{kin(t-1)} - z_{kin(t-1)}] \tag{3}$$

The first equation (2) states that inflows plus flow originating at node i in time period t must balance with outflows. The flow originating from node i at time t comprises new demand flow, plus residual flow on links flowing into node i from the previous time period. The departing flow includes the residual flow according to assumption A7.

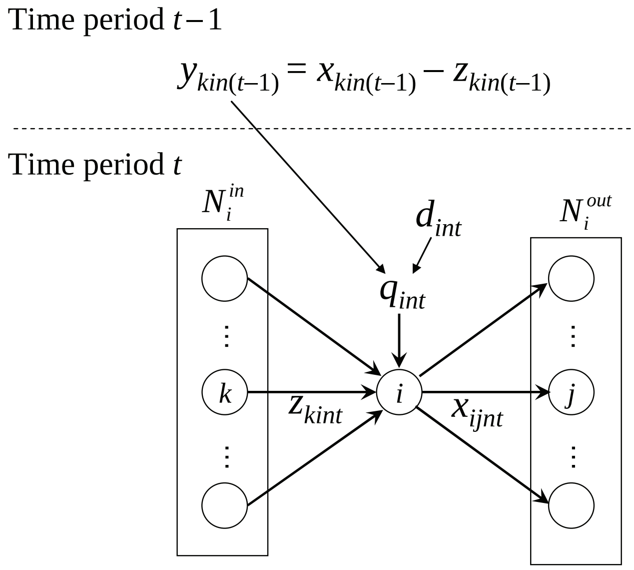


Figure 2: Flow conservation at note *i*

3.2 Residual Flows

Assumption A2 states

The residual flow on a link is determined by the inflow and the link travel time (which is a function of inflow). Let $y_{ij}(x_{ijt})$ denote the residual flow function of link ij, so that $y_{ijt} = y_{ij}(x_{ijt})$. The residual flow function $y_{ij}(\cdot)$ is increasing and continuous, and clearly, $0 \leq y_{ij}(x_{ijt}) = y_{ijt} \leq x_{ijt}$ $(\forall ij \in A, \forall t \in T)$.

In the simple example of Figure 1, the quasi-real route travel time is $c_{12t} + (1 - y_{12t}/x_{12t})\, c_{23t} + (y_{12t}/x_{12t})\, c_{23(t+1)}$. To define the quasi-real travel time, $y_{ij}(x_{ijt})/x_{ijt}$ have to be clarified. If $x_{ijt} = 0$, $y_{ij}(x_{ijt})/x_{ijt}$ does not have the finite value. In addition, it is natural that $y_{ij}(x_{ijt})/x_{ijt}$ is continuous in $x_{ijt} \geq 0$. Then, assume that $\lim_{x \to +0} [x - y_{ij}(x)]/x = t_{0,ij}/L$ $(\forall ij \in A)$, where $t_{0,ij}$ is the free-flow travel time on link ij. Let r_{ijt} be defined as follows:

$$r_{ijt} = \begin{cases} \dfrac{z_{ijt}}{x_{ijt}} = \dfrac{x_{ijt} - y_{ijt}}{x_{ijt}} & \text{if } x_{ijt} > 0 \\[2ex] \dfrac{t_{0,ij}}{L} & \text{if } x_{ijnt} = 0 \end{cases} \tag{4}$$

Thus, r_{ijt} is continuous in $x_{ijt} \geq 0$. Also, r_{ijt} represents the ratio of inflow which exits link ij within time period t and $1 - r_{ijt}$ stands for the ratio of the residual flow. As shown later, this is playing an important role in the proof of existence of equilibrium network flow. From (1), $z_{ijt} = x_{ijt} - y_{ij}(x_{ijt})$. Therefore, z_{ijt} is also a continuous of x_{ijt}, and $0 \le z_{ijt} \le x_{ijt}$ ($\forall ij \in A$, $\forall t \in T$).

The link inflow does not necessarily consist of inflows whose destination is a single destination node. Inflow is treated without any distinction according to assumption A2. Therefore, the following equation holds:

$$\frac{z_{ijnt}}{x_{ijnt}} = \frac{z_{ijt}}{x_{ijt}} \quad \text{if } x_{ijt}, x_{ijnt} > 0 \quad \forall ij \in A_{-n}, \forall n \in D, \forall t \in T \tag{5}$$

This means that y_{ijnt} and z_{ijnt} are uniquely determined when x_{ijnt} is given.

3.3 Formulation

In continuous-time DTA studies, the experienced (route) travel time is equilibrated. In some link-based continuous-time DTA models, such as Kuwahara & Akamatsu (1993), Ran *et al*. (1996) and Li *et al*. (2000), UE is modeled as

$$c_{ij}(t_c) + \tau_{jn}(t_c + c_{ij}(t_c)) - \tau_{in}(t_c) = 0 \ \text{ if } x_{ijn}(t_c) > 0 \tag{6}$$
$$c_{ij}(t_c) + \tau_{jn}(t_c + c_{ij}(t_c)) - \tau_{in}(t_c) \ge 0 \ \text{ if } x_{ijn}(t_c) = 0 \tag{7}$$

where $c_{ij}(t_c)$ denotes the travel time on link ij experienced by flow departing from node i at (continuous) time t_c, $x_{ijn}(t_c)$ is the inflow to link ij at the time t_c whose destination is node n, and $\tau_{in}(t_c)$ is the minimum travel time from node i to node n (destination) at time t_c. Note that $c_{ij}(t_c)$, $\tau_{in}(t_c)$, and $x_{ijn}(t_c)$ are used only in the description of the continuous-time DTA, while c_{ijt}, τ_{int}, and x_{ijnt} are employed in the discrete-time DTA. The conditions (6) and (7) mean that link ij is on the route which has the minimum travel time.

In this study, some part of the link inflow cannot exit the link within one time period, and departs from the end of the link in the next period. The other part of the flow does exit the link and enters the next link in the current time period. As described earlier, the quasi-real route travel time is adopted in this study as the basis for route choice and hence underlies network equilibrium. To derive the minimum value of the quasi-real route travel times, the following is proposed:

$$\mu_{ijnt} \equiv r_{ijt}\tau_{jnt} + \left(1 - r_{ijt}\right)\tau_{jn(t+1)} . \tag{8}$$

In the above equation, the weighted average of the minimum travel time from node j and node n in time period t and t + 1 is used. In the remainder of this paper, the minimum value of quasi-real route travel times is called the quasi-real minimum travel time. In the next section we show that the quasi-real minimum travel time can be derived using the above equation.

By discretizing (6) and (7) and using (8), a link-based formulation of discrete-time dynamic UE is given as

$$c_{ijt} + \mu_{ijnt} - \tau_{int} = 0 \quad if\ x_{ijnt} > 0 \tag{9}$$
$$c_{ijt} + \mu_{ijnt} - \tau_{int} \ge 0 \quad if\ x_{ijnt} = 0 . \tag{10}$$

This means that a link is on the route which has the minimum of (quasi-real) route travel time. The above equilibrium is formulated as the complementarity problem for which the following holds:

$$x_{ijnt}\left(c_{ijt} + \mu_{ijnt} - \tau_{int}\right) = 0 \qquad \forall ij \in A_{-n}, n \in D, t \in T \tag{11}$$
$$x_{ijnt} \ge 0,\ c_{ijt} + \mu_{ijnt} - \tau_{int} \ge 0 \qquad \forall\ ij \in A_{-n}, n \in D, t \in T \tag{12}$$
$$\tau_{int}\left(u_{int} - q_{int} - v_{int}\right) = 0 \qquad \forall\ ij \in A_{-n}, n \in D, t \in T \tag{13}$$
$$\tau_{int} \ge 0,\ u_{int} - q_{int} - v_{int} \ge 0 \qquad \forall\ ij \in A_{-n}, n \in D, t \in T, \tag{14}$$

where

$$u_{int} = \sum_{j \in N_i^{out}} x_{ijnt} \qquad \forall ij \in A_{-n},\ n \in D,\ t \in T \tag{15}$$

$$v_{int} = \sum_{k \in N_i^{in}} z_{kint} \qquad \forall i \in N_{-n},\ n \in D,\ t \in T \tag{16}$$

$$q_{int} = d_{int} + \sum_{k \in N_i^{in}} (x_{kint-1} - z_{kint-1}). \qquad \forall i \in N_{-n},\ n \in D,\ t \in T \tag{17}$$

Equations (13) and (14) represent the flow conservation. The original flow conservation is expressed as an equation, but the complementarity form is used for consistent formulation. The (quasi-real) minimum travel time, τ_{int}, is greater than 0, and $u_{int} = q_{int} + v_{int}$ by (13).

3.4 Quasi-real minimum travel time and qDUE

The definition of the weighted average of minimum travel times (or quasi-real route travel time) in (8) is worth clarifying. Figure 3 shows an example of the quasi-real travel time on a route which consists of 3 (series connected) links.

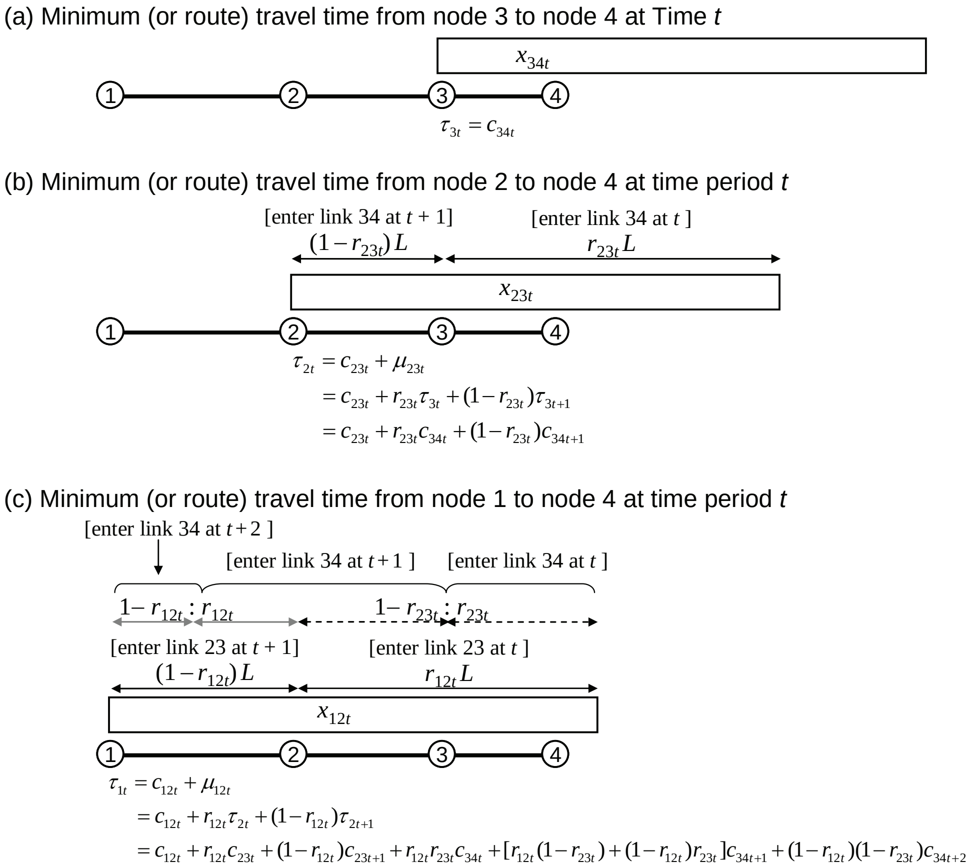


Figure 3. Quasi-real travel time

The quasi-real route travel time is equal to the minimum value of quasi-real route travel times in this case because there is only one route. As written above, $r_{ijt} = z_{ijt}/x_{ijt}$ and $1 - r_{ijt} = y_{ijt}/x_{ijt}$, that is, the ratio of the residual flow to inflow is $1 - r_{ijt}$. The quasi-real minimum travel time from node 3 to node 4 in time period t, τ_{34t}, is equal to the travel time on link 34, c_{34t}, because $\mu_{344t} = 0$ from (8). Next, the quasi-real minimum travel time from node 2 to node 4 is given as

$$\tau_{24t} = c_{23t} + \mu_{234t} = c_{23t} + r_{23t}\ \tau_{34t} + (1 - r_{23t})\ \tau_{34t+1} = c_{23t} + r_{34t}\ c_{34t} + (1 - r_{34t})\ c_{34t+1} \tag{18}$$

due to (8) and (9). In the same manner, the minimum travel time from node 1 to node 4 is

$$c_{12t} + \mu_{124t} = c_{12t} + r_{12t}\ \tau_{24t} + (1 - r_{12t})\ \tau_{24t+1}$$
$$= c_{12t} + r_{12t}\ \tau_{23t} + (1 - r_{12t})\ \tau_{23t+1} + r_{12t}\, r_{23t}\, c_{34t} + [(1 - r_{12t})\, r_{23t} + r_{12t}\,(1 - r_{23t})]c_{34t+1} + (1 - r_{12t})\,(1 - r_{23t})\, c_{34t+2} \quad (19)$$

since $\tau_{24t} = c_{23t} + r_{23t}\ \tau_{34t} + (1 - r_{23t})\ \tau_{34t+1}$.

The departing demand flow at the rate of r_{12t} from node 1 travels on link 23 in time period t, and $(1 - r_{12t})$ of the demand runs in time period t +1. The flow at the rate of $r_{12t}\, r_{23t}$ departing node 1 traverses link 34 in time period t, $[(1 - r_{12t})\, r_{23t} + r_{12t}\,(1 - r_{23t})]$ of the flow travels in time period t+1, and $(1 - r_{12t})\,(1 - r_{23t})$ travels in time period t+2.

Thus, the quasi-real minimum travel time here is the expected value of the travel time. The quasi-real route travel time is expressed as an unwieldy combination of link travel times, but we do not necessarily deal with it in this way. Using the quasi-real minimum travel time, we can constitute it instantly. Furthermore, the quasi-real minimum travel time is determined endogenously in the complementarity problem as (11) through (14) shows.

Under assumption A2, each user does not recognize whether or not they exit a link within the time period. The user has knowledge of all link travel times in all times, but does not know if they are part of the residual flow. The user recognizes the minimum travel time between their origin and destination as an expected value because the minimum travel time varies in time. The definition of μ_{ijnt} in (8) represents this situation.

In this model, each (individual) user does not necessarily choose the minimum travel time route ex post facto; rather, he just chooses the “expected” route which has the minimum travel time (as an expected value). Also, each user “actually” chooses the expected route with the minimum travel time ex post facto. This is the same that the user chooses the route with the quasi-real minimum travel time. The traffic condition is not necessarily the ordinary dynamic UE in the sense that each user chooses the minimum travel time ex post facto as continuous-time dynamic UE models suppose. However, no user in this study can certainly reduce his quasi-real route travel time by changing his route choice unilaterally under assumption A3, and the traffic condition shares some properties with dynamic UE.

The quasi-real-time-based dynamic user equilibrium (qDUE) condition is that: at the qDUE, all routes used by those who depart in a time and travel between an OD pair has the quasi-real minimum travel time while all of their unused routes have the greater or equal time. The quasi-real minimum travel time is the minimum of quasi-real travel times, which can be interpreted as the “expected” travel time.

In the remainder of this section, the above qDUE condition that all routes used by those who depart in a time and travel between an OD pair has the quasi-real minimum travel time equal while all of their unused routes have greater or equal time is reached in the formulation of (11) through (14) is shown.
Let $n_{l,-1}$, $n_{l,-2}$,…, $n_{l,-m}$,…, $n_{l,-M_l}$ denote the nodes on route l which are used and M_l denote the number of links which consists of route l. When m = 1, that is, at the node which is the closest to the destination, $n_{l,-1}$, $\tau_{n_{l,-1}nt} = c_{n_{l,-1}nt}$ due to (8) and (9), and $\tau_{nnt} = \tau_{nnt+1} = 0$. Let χ_{lkt} denote the quasi-real route travel time of the flow who departs in time period t from node $n_{l,-k}$ to the destination node on route l. Assume that the quasi-real route travel time from node $n_{l,-k}$ to node n (the destination) on route l which is used, χ_{lkt}, is the quasi-real minimum travel times from node $n_{l,-k}$ to node n, $\tau_{n_{l,-k}nt}$.

According to equation (4),

$$\chi_{l(k+1)t} = r_{n_{l,-(k+1)}n_{l,-k}t}\,(c_{n_{l,-(k+1)}n_{l,-k}t} + \chi_{lkt}) + (1 - r_{n_{l,-(k+1)}n_{l,-k}t})\,(c_{n_{l,-(k+1)}n_{l,-k}t} + \chi_{lk(t+1)}). \quad (20)$$

As assumed above, $\chi_{lkt} = \tau_{n_{l,-k}nt}$, then

$$\chi_{l(k+1)t} = r_{n_{l,-(k+1)}n_{l,-k}t}\,(c_{n_{l,-(k+1)}n_{l,-k}t} + \tau_{n_{l,-k}nt}) + (1 - r_{n_{l,-(k+1)}n_{l,-k}t})\,(c_{n_{l,-(k+1)}n_{l,-k}t} + \tau_{n_{l,-k}nt}). \quad (21)$$

The right side of this equation represents the quasi-real minimum travel time from $n_{l,-(k+1)}$ to node n as (8) and (9) show. Thus, $\chi_{l(k+1)t} = \tau_{n_{l,-(k+1)}nt}$. This can be applied to any used route. By mathematical induction, we show that used route l has the quasi-real minimum travel time. The above can be applied to any used route. Therefore, all routes used by those who depart in a time and travel between an OD pair has the quasi-real minimum travel time.

An unused route between an OD pair includes at least one link without flow which departs in the time and travels between the OD pair. In the case of the unused route, we assume that $\chi_{l'kt} \geq \tau_{n_{l',-k}nt}$ on route l', which is unused. In a manner similar to that for the above mathematical induction, $\chi_{l'(k+1)t} \geq \tau_{n_{l',-(k+1)}nt}$ is derived, and we find that the quasi-real travel time of their unused routes is greater than or equal to the quasi-real minimum travel time.

3.5 Static UE and qDUE

In this sub-section, we consider the relationship between qDUE and static user equilibrium (UE). Consider static UE as describing a single time period, long enough that all routes are completed. Under static UE, each user takes the route with minimum travel time, and the equilibrium condition can be formulated as

$$x_{ijn1}\left(c_{ij1} + \tau_{jn1} - \tau_{in1}\right) = 0 \qquad \forall ij \in A_{-n},\ n \in D \quad (22)$$

$$x_{ijn1} \geq 0,\ c_{ij1} + \tau_{jn1} - \tau_{in1} \geq 0 \qquad \forall\ ij \in A_{-n},\ n \in D \quad (23)$$

Now, consider qDUE under the situation that the length of time period, L, approaches infinity. In this case, the set of time periods, $T = \{1\}$. Furthermore, $y_{ij1} = 0$ ($\forall ij \in A$) because all inflow traverses the link since the length of the time period is infinite. Then, $r_{ij1} = 1$ in $x_{ij1} \geq 0$ by (4), and hence $\mu_{ijn1} = \tau_{jn1}$ due to (8). Substituting $\mu_{ijn1} = \tau_{jn1}$ to (11), we obtain

$$x_{ijn1}\left(c_{ij1} + \tau_{jn1} - \tau_{in1}\right) = 0\ . \quad (24)$$

This is the same of the static user equilibrium condition of (22). Thus, qDUE approaches the static Wardrop's user equilibrium as $L \to \infty$. The limit $L \to 0$ is intended for future research.

4. THE EXISTENCE AND UNIQUENESS OF NETWORK EQUILIBRIUM, AND A SOLUTION ALGORITHM

In this section, the existence and uniqueness of qDUE which is formulated by the complementarity problem of (11) through (14) is proven. The proof of the existence and uniqueness of the network flow in this study is similar to those in Aashtiani and Magnanti (1981) and Wie *et al*. (2002). Also, an algorithm of qDUE is proposed.

4.1. Existence of qDUE

A vector x^* is the solution of the complementarity problem, $x^* \cdot F(x^*) = 0$, $x^* \geq 0$, $F(x^*) \geq 0$, if and only if x^* satisfies the fixed point problem, $x^* = \max[0, x^* - F(x^*)]$, where $\max[0, x] = x$ if $x \geq 0$; otherwise, 0 (Harker and Pang, 1990). A fixed point problem that is equivalent to the complementarity problem as described in the previous section is

$$x_{ijnt} = \max\left[0, x_{ijnt} - c_{ijt} - \mu_{ijnt} + \tau_{int}\right] \tag{25}$$

$$\tau_{int} = \max\left[0, \tau_{int} - u_{int} + q_{int} + v_{int}\right] . \tag{26}$$

The Brouwer's fixed point problem is employed to prove the existence of equilibrium, but the Brouwer's fixed point problem can be applied to the problem with a finite domain. A finite domain is set to apply the Brouwer's fixed point problem. Define finite K_1 and K_2 which are satisfied with the following inequalities:

$$K_1 > \max\left[q_{int} + v_{int} \middle| \forall i \in N_{-n}, n \in D, t \in T\right] \tag{27}$$

$$K_2 > \max\left[c_{ijt} + \mu_{ijnt} \middle| \forall ij \in A_{-n}, n \in D, t \in T\right]. \tag{28}$$

Let Ω denote the finite closed convex set for which $0 \le x_{ijnt} \le K_1$ and $0 \le \tau_{jnt} \le K_2$ hold ($\forall i \in N_{-n}$, $\forall j \in N$, $\forall n \in D$, $\forall t \in T$). The fixed point problem with the finite domain ($\mathbf{x}$, $\boldsymbol{\tau}$) $\in \Omega$ for which the following equations hold is introduced:

$$x_{ijnt} = \min\left\{K_1, \max\left[0, x_{ijnt} - c_{ijt} - \mu_{ijnt} + \tau_{int}\right]\right\} \tag{29}$$

$$\tau_{int} = \min\left\{K_2, \max\left[0, \tau_{int} - u_{int} + q_{int} + v_{int}\right]\right\} \tag{30}$$

To examine the existence of the solution of the original complementarity problem of the qDUE model given as (11) to (14), the relationship between the original complementarity problem and the fixed point problem expressed as (29) and (30) is investigated. As shown above, the original complementarity problem and the fixed point problem described by (25) and (26) are equivalent. When a fixed point ($\mathbf{x}$, $\boldsymbol{\tau}$) of (29) and (30) does not lie on the boundary of Ω, that is, $x_{ijnt} \ne K_1$ and $\tau_{int} \ne K_2$, solving (29) and (30) is equivalent to solving the original fixed point problem of (25) and (26). This is shown below using reductio ad absurdum.

Assume $\tau_{int} = K_2$. Then by (28), $K_2 > c_{ijt} + \mu_{ijnt}$, and so $-c_{ijt} - \mu_{ijnt} + \tau_{int} > 0$, giving $\max[0, x_{ijnt} - c_{ijt} - \mu_{ijnt} + \tau_{int}] > x_{ijnt}$. This implies $x_{ijnt} = K_1$ from (29). Therefore, $u_{int} \ge x_{ijnt} = K_1$ by (15). Since $u_{int} > q_{int} + v_{int}$ ($-u_{int} + q_{int} + v_{int} < 0$) from (27) and $u_{int} \ge x_{ijnt} = K_1$,

$$\tau_{int} - u_{int} + q_{int} + v_{int} < \tau_{int}. \tag{31}$$

Then, $\max[0, \tau_{int} - u_{int} + q_{int} + v_{int}] < \tau_{int}$. In order that $K_2 = \tau_{int} = \min\{K_2, \max[0, \tau_{int} - u_{int} + q_{int} + v_{int}]\}$ holds, i.e. (30) holds, it is necessary to show that $K_2 \le \max[0, \tau_{int} - u_{int} + q_{int} + v_{int}]$. However, as stated above, $\max[0, \tau_{int} - u_{int} + q_{int} + v_{int}] < \tau_{int}$ (= K_2). Thus, we find the contradiction. The above can be applied to any τ_{int}. Therefore, $\tau_{int} \ne K_2$ ($\forall i \in N_{-n}$, $\forall n \in D$, $\forall t \in T$).

Next, suppose $x_{ijnt} = K_1$. As (31) shows, $\tau_{int} - u_{int} + q_{int} + v_{int} < \tau_{int}$. Then, τ_{int} must be zero in order that (30) holds. Furthermore,

$$x_{ijnt} - c_{ijt} - \mu_{ijnt} + \tau_{int} = x_{ijnt} - c_{ijt} - \mu_{ijnt} < x_{ijnt} \tag{32}$$

because $c_{ijt} > 0$ and $\mu_{ijnt} \ge 0$ by $\tau_{int} = 0$. In order that (29) holds, x_{ijnt} must be 0. This contradicts the assumption of $x_{ijnt} = K_1$. The above can be applied to any x_{ijnt}. Therefore, $x_{ijnt} \ne K_1$ ($\forall i \in N_{-n}$, $\forall j \in N$, $\forall n \in D$, $\forall t \in T$). Thus, the solution of (29) and (30) is equivalent to that of (25) and (26).

Now, we establish the existence of the solution of the fixed point problem of (29) and (30). Let $\mathbf{F}$ denote the vector-valued function whose component function is (29) or (30). $\mathbf{F}$ is the vector-valued function of $\mathbf{x}, \boldsymbol{\tau}, \mathbf{c}, \boldsymbol{\mu}, \mathbf{u}$, $\mathbf{v}$ and $\mathbf{q}$, where $\boldsymbol{\mu} = \{\mu_{ijnt}\}$, $\mathbf{u} = \{u_{int}\}$, $\mathbf{v} = \{v_{int}\}$ and $\mathbf{q} = \{q_{int}\}$. Each of $\mathbf{c}, \boldsymbol{\mu}, \mathbf{u}$, $\mathbf{v}$ and $\mathbf{q}$ is also the function of $\mathbf{x}$ and $\boldsymbol{\tau}$, that is, $\mathbf{c}(\mathbf{x})$, $\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\tau}, \mathbf{z}(\mathbf{x},\mathbf{y}(\mathbf{x})))$, $\mathbf{u}(\mathbf{x})$, $\mathbf{v}(\mathbf{z}(\mathbf{x},\mathbf{y}(\mathbf{x})))$ and $\mathbf{q}(\mathbf{x}, \mathbf{z}(\mathbf{x},\mathbf{y}(\mathbf{x})))$, where $\mathbf{y}(\mathbf{x})$ is the vector-valued function the component function of which is $y_{ij}(x_{ijt})$. Due to assumption A9, $\mathbf{c}(\mathbf{x})$ is continuous. The vector-valued function of residual flows, $\mathbf{y}(\mathbf{x})$, is continuous as assumption A6 shows, and the vector-valued function of $\mathbf{z}$ (= $\mathbf{x} - \mathbf{y}(\mathbf{x})$) is continuous.

Therefore, we find $\mathbf{u}(\mathbf{x}), \mathbf{v}(\mathbf{z}(\mathbf{x},\mathbf{y}(\mathbf{x})))$ and $\mathbf{q}(\mathbf{x}, \mathbf{z}(\mathbf{x},\mathbf{y}(\mathbf{x})))$ are continuous from (15), (16) and (17). Furthermore, $\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\tau}, \mathbf{z}(\mathbf{x},\mathbf{y}(\mathbf{x})))$ is also continuous because of (4), (5), (8) and $\lim_{x\to+0} [x - y_{ij}(x)]/x = t_{0,ij}/L$. Therefore, $\mathbf{F}$ is continuous with respect to $\mathbf{x}$ and $\boldsymbol{\tau}$. Clearly, $\mathbf{F}(\mathbf{x}, \boldsymbol{\tau}) \in \Omega$ from (29) and (30).

According to the Brouwer's fixed point theorem, the existence of the solution of the problem, $(\mathbf{x}, \boldsymbol{\tau}) = \mathbf{F}(\mathbf{x}, \boldsymbol{\tau})$, is proven. As described above, the solution of (29) and (30) is equivalent to that of the complementarity problem of (11) through (14). Therefore, the existence of the solution of the above qDUE model is guaranteed.

4.2. Uniqueness of qDUE

In this section, the uniqueness of the solution of the qDUE model expressed as (11) through (14) is proven using reductio ad absurdum. Assume that the model has two different solutions, $(\mathbf{x}^*, \boldsymbol{\tau}^*)$ and $(\mathbf{x}^\circ, \boldsymbol{\tau}^\circ)$, where $\mathbf{x}^* = \{x^*_{ijnt}\}, \boldsymbol{\tau}^* = \{\tau^*_{int}\}$, $\mathbf{x}^\circ = \{x^\circ_{ijnt}\}$ and $\boldsymbol{\tau}^\circ = \{\tau^\circ_{int}\}$. From (11), the following equations hold:

$$x^*_{ijnt}\left(c^*_{ijt} + \mu^*_{ijnt} - \tau^*_{int}\right) = 0 \tag{33}$$

$$x^\circ_{ijnt}\left(c^\circ_{ijt} + \mu^\circ_{ijnt} - \tau^\circ_{int}\right) = 0 \ , \tag{34}$$

where c^*_{ijt} and c°_{ijt} are the travel time on link ij in time period t at $\mathbf{x} = \mathbf{x}^*$ and $\mathbf{x} = \mathbf{x}^\circ$, respectively, and μ^*_{ijnt} and μ°_{ijnt} are the weighted average minimum travel times at $(\mathbf{x}^*, \boldsymbol{\tau}^*)$ and $(\mathbf{x}^\circ, \boldsymbol{\tau}^\circ)$, respectively. Summing the above two equations yields

$$\begin{aligned}\left(x^*_{ijnt} - x^\circ_{ijnt}\right)\left(c^*_{ijt} - c^\circ_{ijt} + \mu^*_{ijnt} - \mu^\circ_{ijnt} - \tau^*_{int} + \tau^\circ_{int}\right) + x^\circ_{ijnt}\left(c^*_{ijt} + \mu^*_{ijnt} - \tau^*_{int}\right) \\ + x^*_{ijnt}\left(c^\circ_{ijt} + \mu^\circ_{ijnt} - \tau^\circ_{int}\right) = 0\end{aligned} . \tag{35}$$

From (11) and (12) we see that

$$x^*_{ijnt} \geq 0 \ , \quad x^\circ_{ijnt} \geq 0 \ , \quad c^*_{ijt} + \mu^*_{ijnt} - \tau^*_{int} \geq 0 \ , \quad c^\circ_{ijt} + \mu^\circ_{ijnt} - \tau^\circ_{int} \geq 0 \, .$$

The second and third terms on the left-had side of (35) are not negative. Let $\vec{x}_{ijnt} = x^*_{ijnt} - x^\circ_{ijnt}$, $\vec{c}_{ijt} = c^*_{ijt} - c^\circ_{ijt}$, $\vec{\mu}_{ijnt} = \mu^*_{ijnt} - \mu^\circ_{ijnt}$, $\vec{\tau}_{int} = \tau^*_{int} - \tau^\circ_{int}$, and $\vec{q}_{ijnt} = q^*_{ijnt} - q^\circ_{ijnt}$. Using $\vec{x}_{ijnt}$, $\vec{c}_{ijt}$, $\vec{\mu}_{ijnt}$ and $\vec{\tau}_{int}$, the equation can be rewritten as

$$\vec{x}_{ijnt}\left(\vec{c}_{ijt} + \vec{\mu}_{ijnt} - \vec{\tau}_{int}\right) \leq 0 \ . \tag{36}$$

Similarly,

$$\vec{\tau}_{int}\left(\vec{u}_{int} - \vec{q}_{int} - \vec{v}_{int}\right) \leq 0 \tag{37}$$

can also be derived. We sum (36) with respect to $j \in N_i^{out}$, and add it to (37). Obtaining

$$\sum_{j \in N_i^{out}} \vec{x}_{ijnt}\left(\vec{c}_{ijt} + \vec{\mu}_{ijnt}\right) - \vec{\tau}_{int}\,\vec{q}_{int} - \vec{\tau}_{int} \sum_{k \in N_i^{in}} \vec{z}_{kint} \leq 0 \, . \tag{38}$$

Note that we used (15) and (16).

Substituting (8) for (38) yields

$$\sum_{j \in N_i^{out}} \left[\vec{\tau}_{jnt}\vec{z}_{ijnt} + \vec{\tau}_{jn(t+1)}\left(\vec{x}_{ijnt} - \vec{z}_{ijnt}\right)\right] - \vec{\tau}_{int}\,\vec{q}_{int} - \vec{\tau}_{int} \sum_{k \in N_i^{in}} \vec{z}_{kint} \leq 0 \, . \tag{39}$$

Then, summing (39) over i ($\in N$), we obtain

$$\sum_{ij \in A} \vec{x}_{ijnt}\,\vec{c}_{ijt} + \sum_{ij \in A} \vec{\tau}_{jnt+1}\left(\vec{x}_{ijnt} - \vec{z}_{ijnt}\right) - \sum_{i \in N} \vec{\tau}_{int}\,\vec{q}_{int} \leq 0 \, , \tag{40}$$

where $\vec{x}_{nknt}, \vec{z}_{nknt} = 0$ ($\forall k \in N_n^{out}$), which means that no flow exits from the destination, because $\sum_i \sum_j x_{ijnt}\ \mu_{ijnt} = \sum_i \sum_j \tau_{jnt}\ z_{ijnt} + \sum_i \sum_j \tau_{jnt+1}\ (x_{ijnt} - z_{ijnt})$, and $\sum_i \sum_j \tau_{jnt}\ z_{ijnt} = \sum_i\ \tau_{int} \sum_k z_{kint}$. Summing (40) with respect to $\forall n \in D$, $\forall t \in T$ gives

$$\sum_{ij\in A}\sum_{t\in T} \vec{c}_{ijt}\vec{x}_{ijt} + \sum_{ij\in A}\sum_{n\in D}\sum_{t\in T} \vec{\tau}_{jnt+1}\left(\vec{x}_{ijnt} - \vec{z}_{ijnt}\right) - \sum_{i\in N}\sum_{n\in D}\sum_{t\in T} \vec{\tau}_{int}\ \vec{q}_{int} \le 0\ , \tag{41}$$

where $\vec{x}_{ijt} = x^*_{ijt} - x^\circ_{ijt}$, $x^*_{ijt} = \sum_{n\in D} x^*_{ijnt}$, $x^\circ_{ijt} = \sum_{n\in D} x^\circ_{ijnt}$.

OD demands are constant, and $\vec{d}_{int} (= d^*_{int} - d^\circ_{int}) = 0$. By (15), $\vec{q}_{int} = \sum_k (\vec{x}_{kint-1} - \vec{z}_{kint-1})$. Substituting this for (41) gives

$$\sum_{ij\in A}\sum_{t\in T} \left(c^*_{ijt} - c^\circ_{ijt}\right)\left(x^*_{ijt} - x^\circ_{ijt}\right) \le 0\ \ . \tag{42}$$

This contradicts the convexity of the travel time functions as mentioned in assumption A9. Thus, the link inflows in each period are unique. Note that $\{x_{ijt} \mid \forall ij \in A,\ t \in T\}$ is unique, but x_{ijnt} is not necessarily unique.

4.3. Algorithm

There are various ways of solving the qDUE complementarity problem formulated in (11) through (14); we choose a simple approach, reformulating the CP (complementarity problem) using quadratic Fischer-Burmeister functions.

The Fischer-Burmeister function (Fischer, 1992), $\phi(x,y)$, is $x + y - \sqrt{x^2 + y^2}$. The function is (always) non-negative, $\phi(x,y) \ge 0$, and $\phi(x,y) = 0$ is identical to $x \ge 0$, $y \ge 0$ and $x\, y = 0$. Therefore, the complementarity problem of solving $x\, f(x) = 0$ s.t. $x \ge 0$ and $f(x) \ge 0$ is re-formulated as minimizing $\phi(x, f(x))$. The solution of minimizing $\phi(x, f(x))$ without constraints is identical to that of the original complementarity problem. However, the Fischer-Burmeister function, $\phi(x,y)$, is not differential at $(x, y) = (0, 0)$. In this study, the quadratic Fischer-Burmeister function, $\phi(x,y)^2$, which is differential, is used.

Now, the following Fischer-Burmeister-type functions are defined:

$$\phi_{ijnt}(\mathbf{x},\boldsymbol{\tau}) = x_{ijnt} + c_{ijt} + \mu_{ijnt} - \tau_{int} - \sqrt{{x_{ijnt}}^2 + \left(c_{ijt} + \mu_{ijnt} - \tau_{int}\right)^2} \tag{43}$$

$$\varphi_{int}(\mathbf{x},\boldsymbol{\tau}) = \tau_{int} + u_{int} - q_{int} - v_{int} - \sqrt{{\tau_{int}}^2 + \left(u_{int} - q_{int} - v_{int}\right)^2} \tag{44}$$

Eqation (43) is the Fischer-Burmeister function for the CP of (11) and (12) while equation (44) is for (13) and (14). As described above, we use the quadratic Fischer-Burmeister functions, and these FB functions allow us to define

$$L(\mathbf{x},\boldsymbol{\tau}) = \frac{1}{2}\sum_{i\in N_{-n}}\sum_{n\in D}\sum_{t\in T}\left[{\varphi_{int}}^2 + \sum_{j\in N_i^{out}} {\phi_{ijnt}}^2\right] \tag{45}$$

Clearly, $L(\mathbf{x}, \boldsymbol{\tau}) \ge 0$. The optimization problem to minimize $L(\mathbf{x}, \boldsymbol{\tau})$ is then

$$\min.\, L(\mathbf{x},\boldsymbol{\tau}) = \frac{1}{2}\sum_{i\in N_{-n}}\sum_{n\in D}\sum_{t\in T}\left[{\varphi_{int}}^2 + \sum_{j\in N_i^{out}} {\phi_{ijnt}}^2\right] \tag{46}$$

This minimization problem is unconstrained. A solution of (46) is identical to that of the complementarity problem (11) through (14). Many algorithms for unconstrained optimization problems have been developed. In this study we use a conjugate gradient method with the Polak-

Ribiere formula to solve the above optimization problem. Convergence of the Polak-Ribiere conjugate gradient method is guaranteed (Grippo & Lucidi, 1997).

5. EXAMPLE

5.1. Simple network example

In this section, the model is applied to a simple network at a simple setting, and we examine how the model works and whether or not the solution is unique. To reduce the number of variables for simplicity, two time periods of 60-min length (L = 60) are considered. The network has 6 nodes and 6 links as shown in Figure 4. Each link consists of a road part and a bottleneck part as shown in Figure 5. The link travel time function is

$$c_{ij}(x_{ijt}) = c_{ij}^r(x_{ijt}) + c_{ij}^b(x_{ijt}) \tag{47}$$

where $c_{ij}^r(\cdot)$ is the road part travel time function and $c_{ij}^b(\cdot)$ is the bottleneck part travel time function. The travel time of the road part, c_{ijt}^r, is given by the following standard BPR-type function:

$$c_{ij}^r(x_{ijt}) = t_{0,ij}\left[1 + 0.25\left(\frac{x_{ijt}}{C_{ij}}\right)^4\right] \tag{48}$$

where c_{ijt}^r is the travel time of the road part, $t_{0,ij}$ is the free-flow travel time, and C_{ij}^r is the capacity of the road part of the link. The travel time of the bottleneck part, c_{ijt}^b, is expressed as

$$c_{ij}^b(x_{ijt}) = \frac{\max\left[x_{ijt} - C_{ij}^b, 0\right]}{C_{ij}^b} \tag{49}$$

where C_{ij}^b is the capacity of the bottleneck part of link ij. $c_{ij}^b(x_{ijt})$ is a non-decreasing function, but is not necessarily strictly increasing. However, $c_{ij}^r(x_{ijt})$ is strictly increasing, and the link travel time, $c_{ij}(x_{ijt})$, is also strictly increasing. For simplicity, $C_{ij}^r = C_{ij}^b$ in this example. The capacity and free-flow travel times are given in Table 1. The residual flow is given as

$$y_{ijt} = y_{ij}(x_{ijt}) = \max[x_{ijt} - C_{ij}^b, 0] \ . \tag{50}$$

The travel demands given are illustrated in Table 2. The OD pairs are nodes 1 and 6, nodes 2 and 6, and nodes 3 and 6. The flow from node 2 has a route choice, but the others do not.

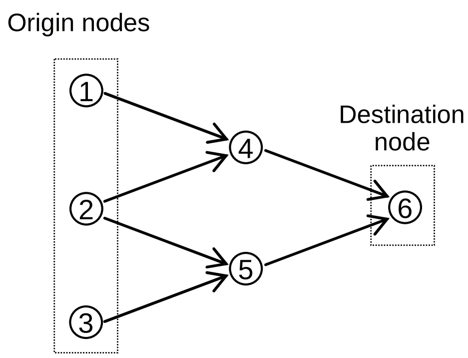


Figure 4: Example network

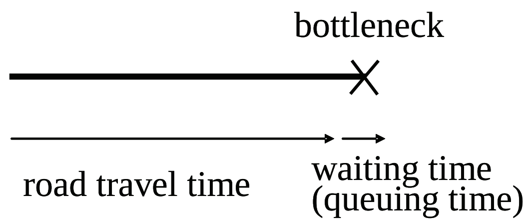


Figure 5: Link structure in the example

Table 1: Free-flow times and capacities in the example

	free-flow time	capacity
Link 14	10	150
Link 24	10	175
Link 25	10	125
Link 35	10	150
Link 46	10	200
Link 56	10	200

Table 2: Travel demands in the example

	Time 1	Time 2
1→6	70	60
2→6	350	300
3→6	70	60

Table 3 shows the results of the qDUE assignment for the above example. Part of the flow which departs at node 2 chooses link 24. This inflow to link 24 does not necessarily exit link 24 within time period 1. The amount of residual flow on link 24 in time period 1 is 14.8. They travel on link 46 in time period 2. The quasi-real travel time which takes link 24 in time period 1 is: 16.46 + (175.0/189.8)*26.88 + (14.8/189.8)*24.46 = 43.84. Similarly, the travel time of the flow which takes link 25 is 31.98 + (125.0/160.2)*11.36 + (35.2/160.2)*18.87 = 43.84. Thus, the qDUE is reached.

Table 3: The results of inflows, travel times and residual flows

	Time 1				Time 2			
	Link 24	Link 25	Link 46	Link 56	Link 24	Link 25	Link 46	Link 56
inflow	189.8	160.2	245.0	195.0	163.0	137.0	237.8	220.2
travel time	16.46	31.98	26.88	11.36	11.19	16.77	24.46	18.87
residual flow	14.8	35.2	45.0	0.0	0.0	12.0	37.8	20.2

To examine the uniqueness of the solution of the qDUE, the function, h, is introduced: $h(x_{241}, x_{242}) = (c_{241} + \mu_{4661} - c_{251} - \mu_{5661})^2 + (c_{242} + \mu_{4662} - c_{252} - \mu_{5662})^2$. The function, h, is non-negative, and (x_{241} , x_{242}) is at qDUE iff $h = 0$. Figures 6a and 6b show the function, h. In the figures, the x axis is x_{241}, the y axis is x_{242}, and the z axis is h. These figures illustrate the uniqueness of the solution.

The quasi-real route travel time is equilibrated in the model. The outflow from link 24 in time period 1 traverses link 46 in this time period, while the residual flow travels in time period 2. The former takes the travel time of $c_{241} + c_{461} = c_{24}(x_{241}) + c_{46}[x_{462}(x_{241})]$ as real (or actual) travel time, thus, the function solely of x_{241} in time period 1. On the other hand, the latter costs that of $c_{241} + c_{462} =$

$c_{24}(x_{241}) + c_{46}[x_{462}(x_{241}, x_{242})]$, thus, the function of x_{241} and x_{242}. Figure 7 shows the two real travel times and quasi-real travel time on the route which consists of link 24 and link 46 in time period 1 when $x_{242} = 150$. The x axis is the flow of x_{241}, and the y axis is the travel time. Thus, the quasi-real travel time locates between the real travel times of the outflow and residual flow.

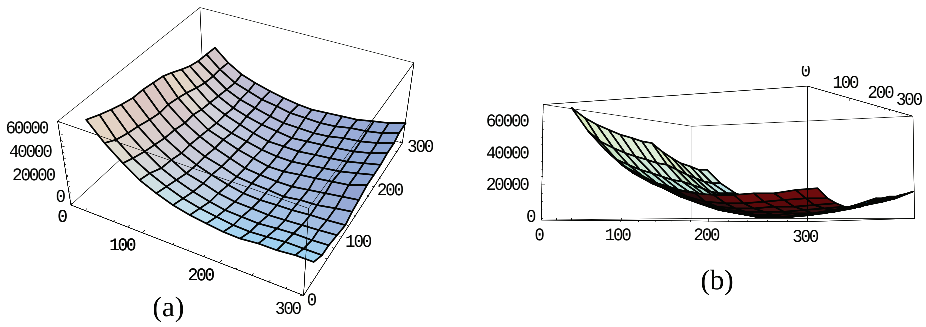


Figure 6: The shape of function *h* from the two different views

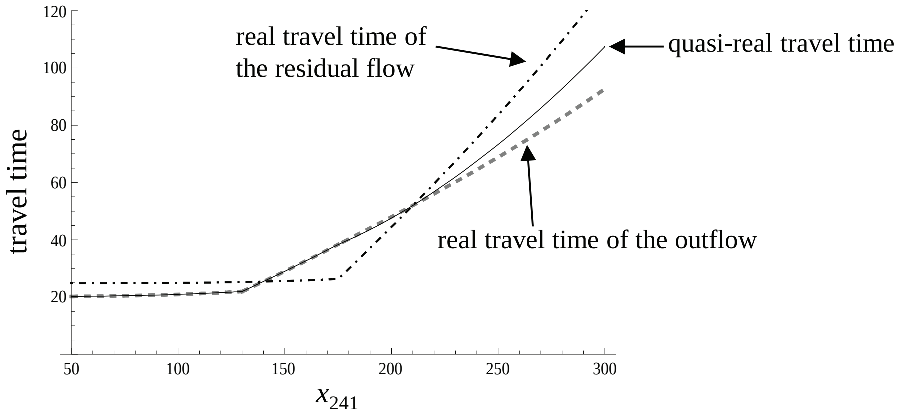


Figure 7: The two real travel times and quasi-real travel time

5.2. Application to Kanazawa Road Network

In this section, the model is applied to the Kanazawa road network. Figure 8 shows the Kanazawa road network, comprising 272 nodes and 964 links. Kanazawa is a local city and has a population of 450,000. The central business district is located at the city center, the destination for many commuters. The duration of a time period is set at 60 min, to be consistent with the available resolution of the OD demand data. Four periods during the morning peak are considered: period 1 is 6:00–7:00 AM, period 2 is 7:00–8:00 AM, period 3 is 8:00–9:00 AM, and period 4 is 9:00–10:00. The OD matrix in each period is derived from a previously conducted personal trip survey within the Kanazawa urban area. OD flow is assigned in the period of departure; flow departing at 6:50 that reaches the destination at 7:25 is included in the OD demand in period 1 (6:00–7:00). The travel times are given by standard BPR-type functions. The computation time for reaching convergence on an ordinary personal computer (CPU: Intel Core i7 2.80 GHz) is 24 min 30 sec. This suggests the model is practically applicable.

Figure 9 shows scatter plots comparing observed and computed link flows in each period. The observed link flows are the observed link flows which travel within the period. The correlation coefficients between observed and computed flows are between 0.83 and 0.88; the goodness-of-fit of the model to the Kanazawa road network is reasonable. The period of 6:00–7:00 is before the morning peak. The link flows in period 1 are fewer than those in period 2 and 3, and the link travel times in period 1 are close to free-flow travel times. There are not substantial residual flows from period 1 to period 2. Most drivers depart in period 2, 7:00–8:00, to travel to their workplace and traffic is most congested during this period. Figure 10(a) shows the residual flows on the links from period 2, 7:00–8:00, to period 3, 8:00–9:00 while Figure 10(b) those from period 3, 8:00–9:00, to

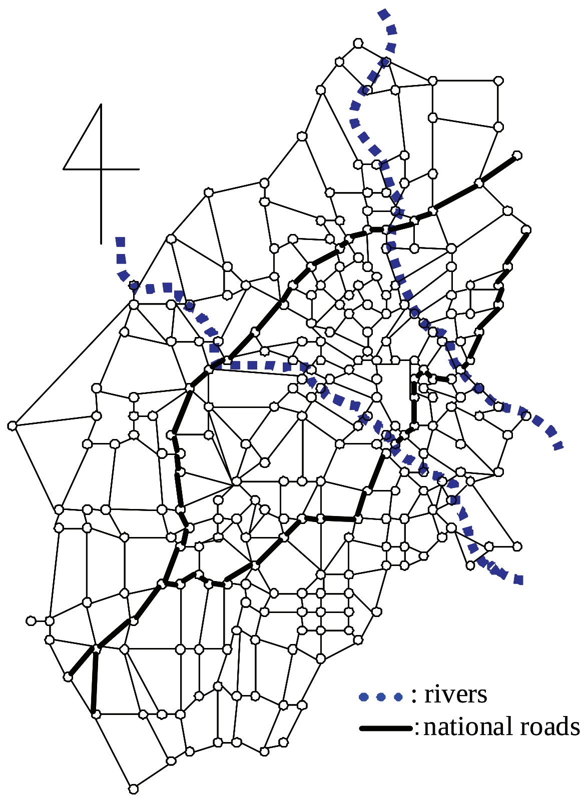


Figure 8: Kanazawa road network

period 4, 8:00–9:00. Most drivers are headed toward the city center (central business district), and the arrows in Figure 10(a) predominantly point toward the city center. Compared with Figure 10(b), Figure 9(a) shows that the residual flows from period 2 to period 3 are larger than those from period 3 to period 4. This is because the morning peak is heaviest during period 2. The morning peak still continues until period 3, 8:00–9:00. The starting time of work is commonly at or before 9:00, and the morning peak is over in period 4, 9:00–10:00. The link flows in period 4 are much less than those in period 3 as Figure 9(c) and (d).

The correlation coefficients between observed and computed flows in each period are more than 0.8, and the validity of the model is reasonable. However, Figure 9(c) implies that the link flows computed for period 3 (AM 8:00–9:00) underestimate observed flows. This is partially because of the inconsistency between OD matrix and observed link flows. In this paper, as described above, demand is assigned to the departure period regardless of when it reaches the destination. How to construct a semi-dynamic OD matrix, i.e. how to divide total demand into periods for this model, requires further examination. This underestimation may also result from the calculation of residual flows. In this application to Kanazawa road network, the residual flows, y_{jt}, is $x_{ijt}\, c_{ijt}/L$ as Figure 1 illustrates. However, this is not the only possible way to calculate residual flows. The residual flow of the simple network case in the previous section is determined using the bottleneck capacity. As assumption A6 states, we just assume that the residual flow is a continuous and increasing function of the inflow on a link. There remains some need to calibrate residual flows, which is left for future work.

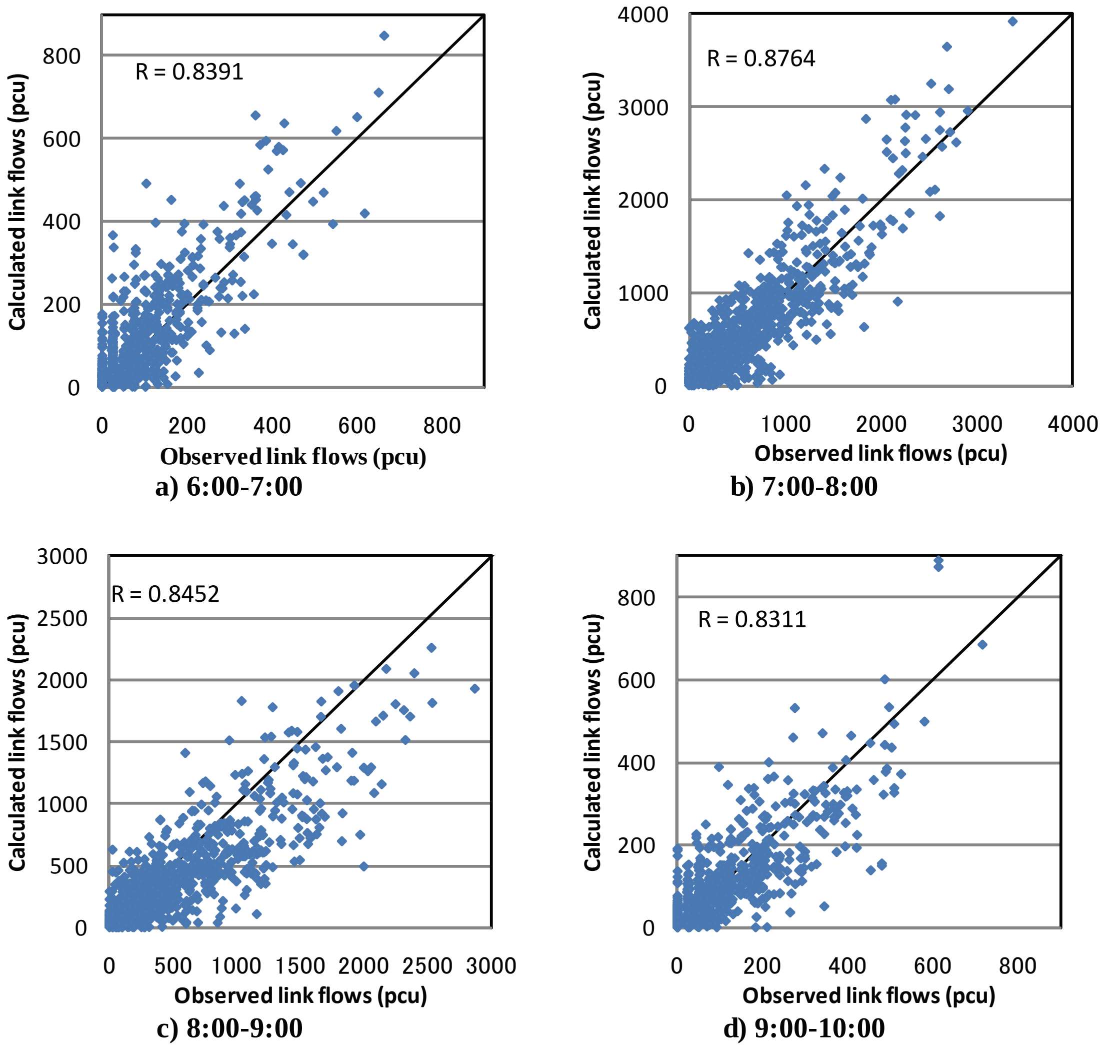
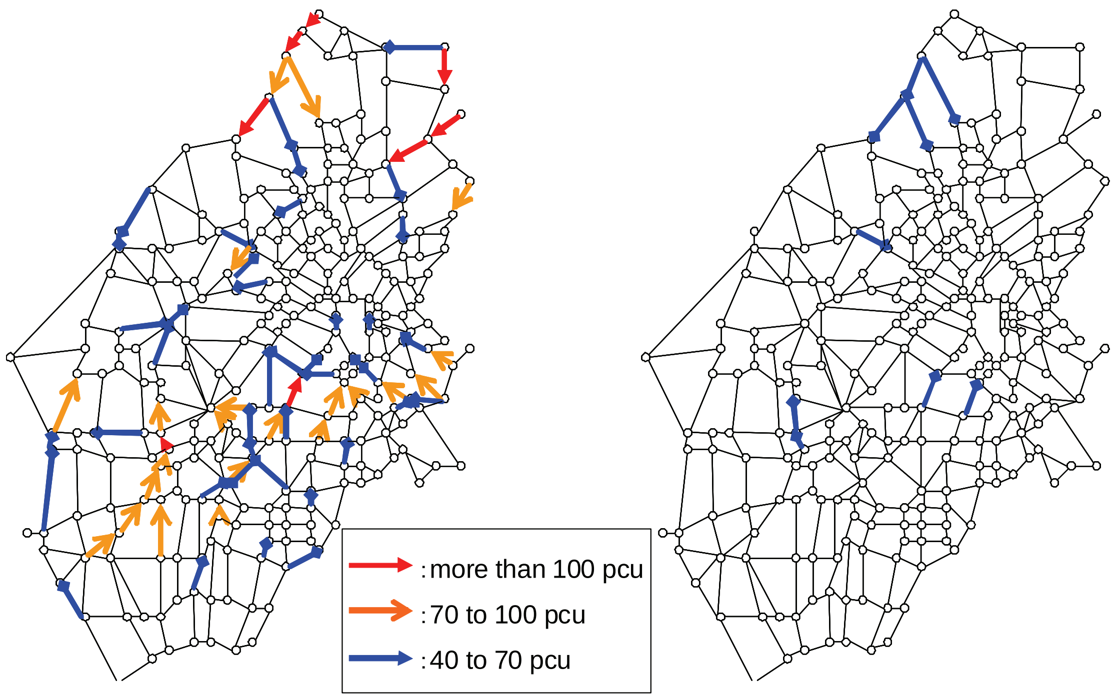


Figure 9: Scatter plots between the observed and calculated link flows in each period



a) From period 2 to 3 b) From period 3 to 4

Figure 10: Residual flows between the periods

6. CONCLUSIONS

Although continuous-time DTA is a theoretically preferable model, there are practical cases where discrete-time DTA (or semi-dynamic traffic assignment) is both a useful and reasonable approach.

However, discrete-time DTA has theoretical issues that need to be investigated. The link inflow in one discrete time period does not necessarily exit that link within the same time period. Therefore, on the next link the inflow may experience different travel times, despite having started as a contiguous 'block' of demand. In this way, the travel time (and minimum travel time) is disputable in discrete-time DTA. In this paper, the quasi-real minimum travel time is defined based on an assumption that inflow exits a link within two discrete time periods.

Based on this quasi-real travel time, a user equilibrium model for discrete-time dynamic network traffic assignment is proposed. For each OD pair, the quasi-dynamic user equilibrium condition is that all routes used by those who depart in one time period have the quasi-real minimum travel time while all of their unused routes have greater or equal time. This condition is modeled as a complementarity problem, having a link-based formulation using the quasi-minimum travel time. We confirm that the solution of this complementarity problem formulation satisfies the stated UE condition. Furthermore, the existence and uniqueness of a solution to the proposed model are proven.

An algorithm for the complementarity problem of the model is proposed based on the re-formulation approach. The non-constraint optimization problem is re-formulated using the quadratic Fischer-Burmeister function. The optimization problem is solved by the conjugate gradient method. The model is applied to a simple network and Kanazawa road network. As a result, the goodness-of-fit of the model to observed link flow data is found to be satisfactory, and the validity and applicability are confirmed for at least a medium sized network such as Kanazawa. In future work, a method of constitute a semi-dynamic OD data and an approach to calibrate the residual flows should be developed.

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