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Abstract linear partial differential equations related to size-structured population models with diffusion

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Abstract

We study abstract linear partial differential equations in Banach spaces and/or Banach lattices related to size-structured population models with spatial diffusion and their dual problems. We introduce mild solutions through semigroup theory and characteristic method and investigate differentiability of mild solutions. Existence of a unique mild solution is shown. Also, a comparison result is obtained as well as the boundedness of mild solutions is investigated in the Banach lattice setting. Furthermore, we consider the dual problems, and then we introduce weak solutions and establish their uniqueness.

Keywords: Size-structured populations with diffusion, characteristic curves, semigroup, mild solutions, dual problems, weak solutions
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1. Size-structured population models with diffusion

Let us consider a biological population living in a habitat $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $p(s,t,x)$ be the population density of size $s \in [0, s_\dagger]$ at time $t \in [0,T]$ in position $x \in \Omega$, where $s_\dagger \in (0,\infty)$ is the finite maximum size, $T \in (0,\infty)$ is a given time. As usual, the spatial diffusion is represented by Laplacian $k\Delta$ with diffusion coefficient $k > 0$ and we assume the individuals do not move outside of Ω through the boundary $\partial\Omega$. Denote by $g(s,t)$ the growth rate of the individuals of size s and time t. Let $\mu(s,t,x)$ and $\beta(s,t,x)$ be the mortality and reproduction rates, respectively, of size s

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at time t in position x. Let us denote by $f(s,t,x)$ and $C(t,x)$ the inflows of s-size and zero-size individuals, respectively, from outside of the environment. Put $\Omega_T := (0,T)\times\Omega$, $Q := (0,s_\dagger)\times\Omega$, $\mathcal{Q}_T := (0,s_\dagger)\times(0,T)\times\Omega$ and $\Sigma_T := (0,s_\dagger)\times(0,T)\times\partial\Omega$. Size-structured population models with diffusion is formulated as follows:

$$\left.\begin{aligned}
&\partial_t p + \partial_s(g(s,t)p) = k\Delta p(s,t,x) - \mu(s,t,x)p(s,t,x) + f(s,t,x),\\
&\qquad \text{in } \mathcal{Q}_T,\\
&g(0,t)p(0,t,x) = C(t,x) + \int_0^{s_\dagger} \beta(s,t,x)p(s,t,x)\,ds, \quad \text{in } \Omega_T,\\
&\frac{\partial p}{\partial \nu}(s,t,x) = 0, \quad \text{on } \Sigma_T,\\
&p(s,0,x) = p_0(s,x), \quad \text{in } Q.
\end{aligned}\right\} \tag{1}$$

We assume that the mortality rate $\mu(s,t,x)$ has the form $\mu(s,t,x) = \mu_0(s,t) + \widetilde{\mu}(s,t,x)$ with the natural mortality rate $\mu_0(s,t)$ independent of position x and the other factor $\widetilde{\mu}(s,t,x)$ depending on position x. The natural mortality $\mu_0(s,t)$ is nonnegative but not assumed bounded while $\widetilde{\mu}(s,t,x)$ is bounded but not assumed nonnegative. We also assume the reproduction rate $\beta(s,t,x)$ is bounded.

Let $\mathcal{A}$ be the realization of Laplacian $k\Delta$ in $L^q(\Omega)$, $q\in(1,\infty)$, with the Neumann boundary condition, that is

$$\begin{aligned}
&D(\mathcal{A}) = \left\{ \phi \in W^{2,q}(\Omega) \;\middle|\; \frac{\partial \phi}{\partial \nu}(x) = 0 \text{ a.e. on } \partial\Omega \right\} \subset L^q(\Omega)\\
&\mathcal{A}\phi = k\Delta\phi \quad \text{for } \phi \in D(\mathcal{A}).
\end{aligned}$$

Recall that $\mathcal{A}$ generates an analytic semigroup $\{\mathcal{T}(t) \mid t \geq 0\}$ in $L^q(\Omega)$ and there exists $M>0$ and $\omega\in\mathbb{R}$ such that $\|\mathcal{T}(t)\phi\|_{L^q(\Omega)} \leq Me^{\omega t}\|\phi\|_{L^q(\Omega)}$ for all $\phi\in L^q(\Omega)$. See e.g. [2, 5, 13]. Define the bounded linear operators $\mathcal{M}(s,t)$ and $\mathcal{B}(s,t)$ in $L^q(\Omega)$ by

$$[\mathcal{M}(s,t)\phi](x) = \widetilde{\mu}(s,t,x)\phi(x), \quad [\mathcal{B}(s,t)\phi](x) = \beta(s,t,x)\phi(x)$$

for $\phi\in L^q(\Omega)$ and let $[C(t)](x) := C(t,x)$. Then (1) can be transformed to the following problem in $X = L^q(\Omega)$:

$$
\left.\begin{aligned}
&\partial_t p + \partial_s(g(s,t)p) = [\mathcal{A} - \mu_0(s,t)I - \mathcal{M}(s,t)]p(s,t) + f(s,t),\\
&\qquad (s,t) \in \overline{S}_T,\\
&g(0,t)p(0,t) = C(t) + \int_0^{s_\dagger} \mathcal{B}(s,t)p(s,t)\,ds, \quad t \in [0,T],\\
&p(s,0) = p_0(s), \quad s \in [0, s_\dagger],
\end{aligned}\right\} \tag{2}
$$

where unknown $p(s,t)$ is an $L^q(\Omega)$-valued function.

Size-structured population models without diffusion have been studied in [1, 3, 6, 7, 8, 9, 10, 11] etc. Webb [15] has studied structured population models with age, size and position in connection with semigroup theory, where the reproduction process is described by individuals of age zero and each size s. Compared with [15], our models are focused rather on size-structure and the reproduction process is described by individuals of size zero. Also, our models are inhomogeneous type with the growth rate depending on size and time while [15] deals with the homogeneous models with the growth rate depending only on size.

We develop the abstract theory of partial differential equations in Banach spaces or/and Banach lattices. This enables us to treat size-structured population models with spatial diffusion in rigorous and unified way.

The paper is organized as follows. Section 2 is devoted to the setting of the problem of abstract partial differential equations in Banach spaces with suitable assumptions. In Section 3, we introduce mild solutions through semigroup theory and characteristic methods, and then we derive some properties of mild solutions. We show the existence of a unique nonnegative mild solution in Section 4, where nonnegative is described by the positive cone in ordered Banach space. We establish a comparison result and the boundedness properties in Banach lattice setting. We also investigate dual problems in Section 5 and show the existence of solutions to the dual problems. In Section 6, we introduce weak solutions and establish the uniqueness of the weak solution.

2. Abstract problems

Let X be a Banach space with norm $\|\cdot\|$. Let $s_\dagger \in (0,\infty)$ and $T \in (0,\infty)$ be fixed and set $S_T := (0, s_\dagger) \times (0,T)$ and $\overline{S}_T := [0, s_\dagger] \times [0,T]$. We consider

(2) as the abstract partial differential equation in X:

$$\left.\begin{aligned}
&\partial_t p + \partial_s(g(s,t)p) = [\mathcal{A} - \mu_0(s,t)I - \mathcal{M}(s,t)]p(s,t) + f(s,t),\\
&\qquad \text{a.e. } (s,t) \in S_T,\\
&g(0,t)p(0,t) = C(t) + \int_0^{s_\dagger} \mathcal{B}(s,t)p(s,t)\,ds, \quad \text{a.e. } t \in (0,T),\\
&p(s,0) = p_0(s), \quad \text{a.e. } s \in (0,s_\dagger).
\end{aligned}\right\} \tag{3}$$

Here unknown $p(s,t)$ is an X-valued function and the following hypotheses are imposed:

(H1) $g : \overline{S}_T \to [0,\infty)$ is continuous, $g(s,t) > 0$ for $(s,t) \in S_T$, $g(s,t)$ is of C^1-class with respect to $s \in (0,s_\dagger)$ for each $t \in [0,T]$ and there exists a constant $L_g > 0$ such that $|\partial_s g(s,t)| \le L_g$.
In addition, we assume that $g(s,t)$ satisfies one of the following four cases:
Case 1 : $g(0,t) > 0$ and $g(s_\dagger,t) > 0$ for each $t \in [0,T]$,
Case 2 : $g(0,t) > 0$ and $g(s_\dagger,t) = 0$ for each $t \in [0,T]$,
Case 3 : $g(0,t) = 0$ and $g(s_\dagger,t) > 0$ for each $t \in [0,T]$,
Case 4 : $g(0,t) = 0$ and $g(s_\dagger,t) = 0$ for each $t \in [0,T]$.
(Note that we do not consider the other cases such as $g(0,t) \not\equiv 0$ but $g(0,t_0) = 0$ for some $t_0 \in [0,T]$ or $g(s_\dagger,t) \not\equiv 0$ but $g(s_\dagger,t_1) = 0$ for some $t_1 \in [0,T]$.)

(H2) $\mu_0 \in L^1_{loc}((0,s_\dagger) \times [0,T])$, $\mu_0(s,t) \ge 0$ a.e. $(s,t) \in S_T$.

(H3) $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $\{\mathcal{T}(t) \mid t \ge 0\}$ in X.

(H4) $\mathcal{M},\ \mathcal{B} \in L^\infty(S_T; \mathcal{L}(X))$, where $\mathcal{L}(X)$ is the space of all bounded linear operators in X.

(H5) $f \in L^1(S_T; X)$, $C \in L^1(0,T;X)$, and $p_0 \in L^1(0,s_\dagger;X)$.

We extend the function $g(s,t)$ on $\mathbb{R} \times [0,T]$ by taking $g(s,t) = g(0,t)$ for $(s,t) \in (-\infty,0) \times [0,T]$ and $g(s,t) = g(s_\dagger,t)$ for $(s,t) \in (s_\dagger,\infty)$. Then, it is shown that g has the following Lipschitz property:

$$|g(s_1,t) - g(s_2,t)| \le L_g|s_1 - s_2| \quad \text{for } s_1,\ s_2 \in \mathbb{R},\ t \in [0,T].$$

Thus the characteristic curve $s(t) =: \varphi(t;t_0,s_0)$ through $(s_0,t_0) \in \mathbb{R} \times [0,T]$ is defined by the unique solution of the differential equation

$$\frac{d}{dt}s(t) = g(s(t),t), \quad t \in [0,T], \qquad s(t_0) = s_0 \in \mathbb{R}. \tag{4}$$

Let $z_0(t) := \varphi(t; 0, 0)$ and $z_1(t) := \varphi(t; T, s_\dagger)$. Then the following holds corresponding to the four cases in (H1).

1. In Case 1, it follows that $z_0(t) > 0$ for $t > 0$ and $z_1(t) < s_\dagger$ for $t < T$. For $(s, t) \in \overline{S}_T$ satisfying $s \leq z_0(t)$, there exists a unique $\tau_0 \in [0, T]$ satisfying the relation

$$\varphi(\tau_0; t, s) = 0, \quad \text{or equivalently,} \quad \varphi(t; \tau_0, 0) = s. \tag{5}$$

   Then the initial time $\tau_0(t, s)$ for $(s, t) \in \overline{S}_T$ is defined by

$$\tau_0(t, s) = \begin{cases} \tau_0, & s \leq z_0(t), \\ 0, & s > z_0(t). \end{cases} \tag{6}$$

   Similarly, for $(s, t) \in \overline{S}_T$ satisfying $s \geq z_1(t)$, there exists a unique $\tau_1 \in [0, T]$ such that

$$\varphi(\tau_1; t, s) = s_\dagger \quad \text{or equivalently,} \quad \varphi(t; \tau_1, s_\dagger) = s. \tag{7}$$

   Then the final time $\tau_1(t, s)$ for $(s, t) \in \overline{S}_T$ is defined by

$$\tau_1(t, s) = \begin{cases} \tau_1, & s \geq z_1(t), \\ T, & s < z_1(t). \end{cases} \tag{8}$$

2. In Case 2, it follows that $z_0(t) > 0$ for $t > 0$ and $z_1(t) \equiv s_\dagger$. In this case, there is no $\tau_1 \in [0, T]$ satisfying (7) unless $s = s_\dagger$ and the final time is defined as $\tau_1(t, s) = T$ for $(s, t) \in \overline{S}_T$ satisfying $s < s_\dagger$. The initial time $\tau_0(t, s)$ is defined as (6) for $(s, t) \in \overline{S}_T$.
3. In Case 3, $z_0(t) \equiv 0$, $z_1(t) < s_\dagger$ for $t < T$. In this case, there is no $\tau_0 \in [0, T]$ satisfying (5) unless $s = 0$ and the initial time is defined as $\tau_0(t, s) = 0$ for $(s, t) \in \overline{S}_T$ such that $s > 0$. The final time $\tau_1(t, s)$ is defined as (8) for $(s, t) \in \overline{S}_T$.
4. In Case 4, $z_0(t) \equiv 0$, $z_1(t) \equiv s_\dagger$. In this case, the final time is defined as $\tau_1(t, s) = T$ for $(s, t) \in \overline{S}_T$ such that $s < s_\dagger$ and the initial time is defined as $\tau_0(t, s) = 0$ for $(s, t) \in \overline{S}_T$ such that $s > 0$.

Remark 2.1. In Case 3 or Case 4, there is no reproduction process since every characteristic curve starts from time $t = 0$, and so the boundary condition described by the second equation in (3) is ignored and we read (3) as

$$
\left.\begin{aligned}
&\partial_t p + \partial_s(g(s,t)p) = [\mathcal{A} - \mu_0(s,t)I - \mathcal{M}(s,t)]p(s,t) + f(s,t),\\
&\qquad \text{a.e. } (s,t) \in S_T,\\
&p(s,0) = p_0(s), \quad \text{a.e. } s \in (0,s_\dagger).
\end{aligned}\right\} \tag{9}
$$

3. Mild solutions

For $\gamma \in L^1_{loc}((0,s_\dagger)\times[0,T])$, we set

$$
\Pi_\gamma(\sigma,u;t,s) = \exp\left(\int_u^\sigma \gamma(\varphi(\eta;t,s),\eta)\,d\eta\right), \ \forall \sigma, u \in [\tau_0,\tau_1] \tag{10}
$$

for $(s,t) \in \overline{S}_T$, where $\tau_0 := \tau_0(t,s)$ and $\tau_1 := \tau_1(t,s)$. Suppose now that $p(s,t)$ satisfies (3) in a strict sense. For $(s,t) \in \overline{S}_T$, put

$$
u(\sigma;t,s) := \Pi_\gamma(\sigma,\tau_0;t,s)p(\varphi(\sigma;t,s),\sigma) \quad \text{for } \sigma \in (\tau_0,\tau_1). \tag{11}
$$

Putting $s(\sigma) = \varphi(\sigma;t,s)$, we have

$$
\begin{aligned}
\frac{d}{d\sigma}u(\sigma;t,s) &= \Pi_\gamma(\sigma,\tau_0;t,s)\big[\partial_t p(s(\sigma),\sigma) + \partial_s p(s(\sigma),\sigma)g(s(\sigma),\sigma)\big]\\
&\quad + \Pi_\gamma(\sigma,\tau_0;t,s)\gamma(s(\sigma),\sigma)p(s(\sigma),\sigma)\\
&= \Pi_\gamma(\sigma,\tau_0;t,s)\big[\mathcal{A} - \mathcal{M}(s(\sigma),\sigma)\big]p(s(\sigma),\sigma)\\
&\quad + \Pi_\gamma(\sigma,\tau_0;t,s)\big[\gamma(s(\sigma),\sigma) - \mu_0(s(\sigma),\sigma) - \partial_s g(s(\sigma),\sigma)\big]p(s(\sigma),\sigma)\\
&\quad + \Pi_\gamma(\sigma,\tau_0;t,s)f(s(\sigma),\sigma).
\end{aligned} \tag{12}
$$

Taking $\gamma(s,t) = \mu_0(s,t) + \partial_s g(s,t)$ in (12) leads to

$$
\frac{d}{d\sigma}u(\sigma;t,s) = [\mathcal{A} - \mathcal{M}(s(\sigma),\sigma)]u(\sigma;t,s) + \Pi_\gamma(\sigma,\tau_0;t,s)f(s(\sigma),\sigma). \tag{13}
$$

We then employ a mild solution $u(\cdot;t,s) \in C([\tau_0,\tau_1];X)$ to (13) defined by the variation of constants formula:

$$
\begin{aligned}
u(\eta;t,s) &= \mathcal{T}(\eta-\tau_0)u(\tau_0;t,s)\\
&\quad + \int_{\tau_0}^{\eta} \mathcal{T}(\eta-\sigma)\Big[-\mathcal{M}(s(\sigma),\sigma)u(s(\sigma),\sigma) + \Pi_\gamma(\sigma,\tau_0;t,s)f(s(\sigma),\sigma)\Big]\,d\sigma
\end{aligned} \tag{14}
$$

for $\eta \in [\tau_0, \tau_1]$. In Case 1 or Case 2, for a.e. $s \in (0, z_0(t))$, it follows from (11) and (14) that

$$\begin{aligned}
&\Pi_\gamma(t, \tau_0; t, s)p(s, t) = u(t; t, s) \\
&= \mathcal{T}(t - \tau_0)p(0, \tau_0) + \int_{\tau_0}^{t} \mathcal{T}(t - \sigma)\Big[-\mathcal{M}(s(\sigma), \sigma)\Pi_\gamma(\sigma, \tau_0; t, s)p(s(\sigma), \sigma) \\
&\qquad\qquad\qquad\qquad + \Pi_\gamma(\sigma, \tau_0; t, s)f(s(\sigma), \sigma)\Big]\, d\sigma,
\end{aligned}$$

where $\tau_0 := \tau_0(t, s)$. Then since Π_γ is defined by (10), we have

$$\begin{aligned}
p(s, t) &= \mathcal{T}(t - \tau_0)\Pi_{-\gamma}(t, \tau_0; t, s)\frac{1}{g(0, \tau_0)}\Big[C(\tau_0) + \int_0^{s_\dagger} \mathcal{B}(s, \tau_0)p(s, \tau_0)\, ds\Big] \\
&\quad + \int_{\tau_0}^{t} \mathcal{T}(t - \sigma)\Pi_{-\gamma}(t, \sigma; t, s)\Big[-\mathcal{M}(s(\sigma), \sigma)p(s(\sigma), \sigma) + f(s(\sigma), \sigma)\Big]\, d\sigma
\end{aligned}$$

for a.e. $s \in (0, z_0(t))$. Similarly, it follows from (14) that

$$\begin{aligned}
p(s, t) &= \mathcal{T}(t)\Pi_{-\gamma}(t, 0; t, s)p_0(\varphi(0; t, s)) \\
&\quad + \int_0^{t} \mathcal{T}(t - \sigma)\Pi_{-\gamma}(t, \sigma; t, s)\Big[-\mathcal{M}(s(\sigma), \sigma)p(s(\sigma), \sigma) + f(s(\sigma), \sigma)\Big]\, d\sigma
\end{aligned}$$

for a.e. $s \in (z_0(t), s_\dagger)$. For simplicity, we set

$$F_p(t) := C(t) + \int_0^{s_\dagger} \mathcal{B}(s, t)p(s, t)\, ds, \tag{15}$$

$$G_p(s, t) := -\mathcal{M}(s, t)p(s, t) + f(s, t). \tag{16}$$

Recall that $\gamma(s, t) := \mu_0(s, t) + \partial_s g(s, t)$. From above consideration, we will define a mild solution to (3) as follows.

Definition 3.1. (In Case 1 or Case 2) By a *mild solution* to (3), we mean a function $p \in L^\infty(0, T; L^1(0, s_\dagger; X))$ satisfying the following relation:

for a.e. $t \in (0, T)$,

$$p(s,t) = \begin{cases} \mathcal{T}(t-\tau_0)\Pi_{-\gamma}(t,\tau_0;t,s)\dfrac{F_p(\tau_0)}{g(0,\tau_0)} \\ \qquad + \displaystyle\int_{\tau_0}^{t} \mathcal{T}(t-\sigma)\Pi_{-\gamma}(t,\sigma;t,s)G_p(\varphi(\sigma;t,s),\sigma)\,d\sigma \\ \qquad a.e.\ s \in (0, z_0(t)), \\ \mathcal{T}(t)\Pi_{-\gamma}(t,0;t,s)p_0(\varphi(0;t,s)) \\ \qquad + \displaystyle\int_{0}^{t} \mathcal{T}(t-\sigma)\Pi_{-\gamma}(t,\sigma;t,s)G_p(\varphi(\sigma;t,s),\sigma)\,d\sigma \\ \qquad a.e.\ s \in (z_0(t), s_\dagger), \end{cases} \tag{17}$$

where $\tau_0 = \tau_0(t,s)$ is defined by (6);

(In Case 3 or Case 4) By a *mild solution* to (3) (or precisely, (9)), we mean a function $p \in L^\infty(0,T;L^1(0,s_\dagger;X))$ satisfying the following relation:

$$\begin{aligned} p(s,t) = {} & \mathcal{T}(t)\Pi_{-\gamma}(t,0;t,s)p_0(\varphi(0;t,s)) \\ & + \int_0^t \mathcal{T}(t-\sigma)\Pi_{-\gamma}(t,\sigma;t,s)G_p(\varphi(\sigma;t,s),\sigma)\,d\sigma \quad a.e.\ (s,t) \in S_T. \end{aligned}$$

Remark 3.1. If $p \in L^\infty(0,T;L^1(0,s_\dagger;X))$ is a mild solution to (3), then it is continuous along the characteristic curve $\varphi(\cdot;t,s)$ for a.e. $(s,t) \in S_T$. In fact, for a.e. $(s,t) \in S_T$ satisfying $s \in (0, z_0(t))$,

$$\begin{aligned} p(\varphi(\eta;t,s),\eta) = {} & \mathcal{T}(\eta-\tau_0)\Pi_{-\gamma}(\eta,\tau_0;t,s)\frac{F_p(\tau_0)}{g(0,\tau_0)} \\ & + \int_{\tau_0}^{\eta} \mathcal{T}(\eta-\sigma)\Pi_{-\gamma}(\eta,\sigma;t,s)G_p(\varphi(\sigma;t,s),\sigma)\,d\sigma \end{aligned} \tag{18}$$

and for a.e. $(s,t) \in S_T$ satisfying $s \in (z_0(t), s_\dagger)$,

$$\begin{aligned} p(\varphi(\eta;t,s),\eta) = {} & \mathcal{T}(\eta)\Pi_{-\gamma}(\eta,0;t,s)p_0(\varphi(0;t,s)) \\ & + \int_{0}^{\eta} \mathcal{T}(\eta-\sigma)\Pi_{-\gamma}(\eta,\sigma;t,s)G_p(\varphi(\sigma;t,s),\sigma)\,d\sigma. \end{aligned} \tag{19}$$

Here we used the relation $\tau_0(\eta,\varphi(\eta;t,s)) = \tau_0(t,s)$ and $\varphi(\sigma;\eta,\varphi(\eta;t,s)) = \varphi(\sigma;t,s)$. In (18) and (19), the right hand sides are continuos in η from $[\tau_0(t,s), \tau_1(t,s)]$ into X.

We now introduce a class of functions which are X-valued C^k functions along characteristic curves ($k = 0, 1, \cdots$):

$$C^k_\varphi(S_T; X) := \left\{ p \in L^1(S_T; X) \;\middle|\; \begin{array}{l} \eta \mapsto p(\varphi(\eta; t, s), \eta) \text{ is a } C^k \text{ function} \\ \text{from } (\tau_0(t,s), \tau_1(t,s)) \text{ into } X \\ \text{for a.e. } (s,t) \in S_T \end{array} \right\}.$$

Note that Remark 3.1 says that any mild solution of (3) belongs to $C^0_\varphi(S_T; X)$.

Next, we shall consider a differentiability property of mild solutions. The derivative along the characteristic curve φ is defined by

$$D_\varphi p(s,t) := \lim_{h\to 0} \frac{1}{h} \big[p(\varphi(t+h; t, s), t+h) - p(s,t) \big] \quad \text{in } X$$

Note that the following relation:

$$\begin{aligned} D_\varphi p(\varphi(\eta; t, s), \eta) &= \lim_{h\to 0} \frac{1}{h} \left[p(\varphi(\eta + h; \eta, \varphi(\eta; t, s), \eta + h)) - p(\varphi(\eta; t, s), \eta) \right] \\ &= \lim_{h\to 0} \frac{1}{h} \left[p(\varphi(\eta + h; t, s), \eta + h) - p(\varphi(\eta; t, s), \eta) \right] \\ &= \frac{d}{d\eta} p(\varphi(\eta; t, s), \eta) \end{aligned}$$

Then we have the following characterization of mild solutions.

Theorem 3.1. *Assume that $\mathcal{A}$ is the generator of an analytic semigroup. Let $\gamma(s,t) := \mu_0(s,t) + \partial_s g(s,t)$. In* Case 1 *or* Case 2*, any mild solution $p \in L^\infty(0, T; L^1(0, s_\dagger; X))$ of* (3) *satisfies $p \in C^1_\varphi(S_T; X)$ and*

$$\left. \begin{aligned} & D_\varphi p(s,t) = \mathcal{A} p(s,t) - \gamma(s,t) p(s,t) - \mathcal{M}(s,t) p(s,t) + f(s,t), \\ & \qquad \text{a.e. } (s,t) \in S_T, \\ & g(0,t) p(0,t) = C(t) + \int_0^{s_\dagger} \mathcal{B}(s,t) p(s,t)\, ds, \quad \text{a.e. } t \in (0,T), \\ & p(s,0) = p_0(s), \quad \text{a.e. } s \in (0, s_\dagger); \end{aligned} \right\} \qquad (20)$$

where $p(0,t)$ and $p(s,0)$ are understood as the limit along the characteristic curve:

$$\begin{aligned} p(0,t) &:= \lim_{\eta \to t} p(\varphi(\eta; t, 0), \eta) \quad \text{in } X \quad \text{a.e. } t \in (0,T), \\ p(s,0) &:= \lim_{\eta \to +0} p(\varphi(\eta; 0, s), \eta) \quad \text{in } X \quad \text{a.e. } s \in (0, s_\dagger). \end{aligned}$$

Conversely, if $p \in L^\infty(0,T;L^1(0,s_\dagger;X)) \cap C^1_\varphi(S_T;X)$ satisfies (20)*, then p is a mild solution to* (3)*. In* Case 3 *or* Case 4*, the similar facts hold without the second equation in* (20)*.*

Proof. We prove the theorem just in Case 1 or Case 2. For a.e. $(s,t) \in S_T$ satisfying $s \in (0, z_0(t))$, the relation (18) holds. Since $\{\mathcal{T}(t) \mid t \geq 0\}$ is an analytic semigroup, we find that the right hand side of (18) is continuously differentiable in η as a mapping from (τ_0, τ_1) into X and we find $p(\varphi(\eta;t,s),\eta) \in D(\mathcal{A})$ and

$$\begin{aligned}
&D_\varphi p(\varphi(\eta;t,s),\eta) = \frac{d}{d\eta} p(\varphi(\eta;t,s),\eta) \\
&= \mathcal{A}\mathcal{T}(\eta-\tau_0)\Pi_{-\gamma}(\eta,\tau_0;t,s)\frac{F_p(\tau_0)}{g(0,\tau_0)} \\
&\quad - \gamma(\varphi(\eta;t,s),\eta)\mathcal{T}(\eta-\tau_0)\Pi_{-\gamma}(\eta,\tau_0;t,s)\frac{F_p(\tau_0)}{g(0,\tau_0)} + G_p(\varphi(\eta;t,s),\eta) \\
&\quad + \int_{\tau_0}^{\eta} \mathcal{A}\mathcal{T}(\eta-\sigma)\Pi_{-\gamma}(\eta,\sigma;t,s)G_p(\varphi(\sigma;t,s),\sigma)\,d\sigma \\
&\quad - \int_{\tau_0}^{\eta} \gamma(\varphi(\eta;t,s),\eta)\mathcal{T}(\eta-\sigma)\Pi_{-\gamma}(\eta,\sigma;t,s)G_p(\varphi(\sigma;t,s),\sigma)\,d\sigma \\
&= \mathcal{A}p(\varphi(\eta;t,s),\eta) - \gamma(\varphi(\eta;t,s),\eta)p(\varphi(\eta;t,s),\eta) + G_p(\varphi(\eta;t,s),\eta)
\end{aligned} \tag{21}$$

for all $\eta \in (\tau_0,\tau_1)$. For a.e. $(s,t) \in S_T$ satisfying $s \in (z_0(t), s_\dagger)$, the relation (19) holds. Again, since $\{\mathcal{T}(t) \mid t \geq 0\}$ is an analytic semigroup, the right hand side of (19) is continuously differentiable in η as a mapping from (τ_0,τ_1) into X and we find that $p(\varphi(\eta;t,s),\eta) \in D(\mathcal{A})$ and

$$\begin{aligned}
&D_\varphi p(\varphi(\eta;t,s),\eta) = \frac{d}{d\eta} p(\varphi(\eta;t,s),\eta) \\
&= \mathcal{A}p(\varphi(\eta;t,s),\eta) - \gamma(\varphi(\eta;t,s),\eta)p(\varphi(\eta;t,s),\eta) + G_p(\varphi(\eta;t,s),\eta)
\end{aligned} \tag{22}$$

for all $\eta \in (\tau_0,\tau_1)$. Furthermore, it follows from (18) and (19) that p satisfies

$$p(0,t) := \lim_{\eta\to t} p(\varphi(\eta;t,0),\eta) = \frac{F_p(t)}{g(0,t)} \quad \text{a.e.} \quad t \in (0,T). \tag{23}$$

$$p(s,0) := \lim_{\eta\to +0} p(\varphi(\eta;0,s),\eta) = p_0(s), \quad \text{a.e.} \quad s \in (0,s_\dagger). \tag{24}$$

(21), (22), (23) and (24) show that the mild solution p satisfies (20). Conversely, suppose that $p \in L^\infty(0,T;L^1(0,s_\dagger;X)) \cap C^1_\varphi(S_T;X)$ satisfies (20). Let $w(\sigma;t,x) := p(\varphi(\sigma;t,s),\sigma)$ for a.e. $\sigma \in (0,T)$ and a.e. $(s,t) \in S_T$. Then

$$\begin{aligned}\frac{d}{d\sigma} w(\sigma;t,s) &= D_\varphi p(\varphi(\sigma;t,s),\sigma) \\ &= \mathcal{A} w(\sigma;t,s) - \gamma(\varphi(\sigma;t,s),\sigma) w(\sigma;t,s) + G_p(\varphi(\sigma;t,s),\sigma).\end{aligned} \tag{25}$$

The solution of (25) on $[\tau_0,t]$ can be written as a variation-of-constants formula

$$\begin{aligned} w(t;t,s) =& \mathcal{T}(t-\tau_0)\Pi_{-\gamma}(t,\tau_0;t,s) w(\tau_0;t,s) \\ &+ \int_{\tau_0}^t \mathcal{T}(t-\sigma)\Pi_{-\gamma}(t,\sigma;t,s) G_p(\varphi(\sigma;t,s),\sigma)\, d\sigma.\end{aligned}$$

Since $w(t;t,s) = p(s,t)$, $w(\tau_0;t,s) = p(0,\tau_0) = F_p(\tau_0)/g(0,\tau_0)$ by the second equation of (20), and $w(0;t,s) = p_0(\varphi(0;t,s))$, we find that p is a mild solution of (3). □

For positivity, we need the following characterization.

Proposition 3.2. *Let $\alpha \in \mathbb{R}$ be given. Let $\gamma(s,t) := \mu_0(s,t) + \partial_s g(s,t)$. A function $p \in L^\infty(0,T;L^1(0,s_\dagger;X))$ is a mild solution of (3) if and only if the following hold:*

(In Case 1 or Case 2) *for a.e. $t \in (0,T)$,*

$$p(s,t) = \begin{cases} e^{-\alpha(t-\tau_0)}\mathcal{T}(t-\tau_0)\Pi_{-\gamma}(t,\tau_0;t,s)\dfrac{F_p(\tau_0)}{g(0,\tau_0)} \\ \quad + \displaystyle\int_{\tau_0}^t e^{-\alpha(t-\sigma)}\mathcal{T}(t-\sigma)\Pi_{-\gamma}(t,\sigma;t,s)\big[G_p(s(\sigma),\sigma) \\ \qquad\qquad + \alpha p(s(\sigma),\sigma)\big]\, d\sigma \quad \text{a.e. } s \in (0,z_0(t)), \\ e^{-\alpha t}\mathcal{T}(t)\Pi_{-\gamma}(t,0;t,s)p_0(\varphi(0;t,s)) \\ \quad + \displaystyle\int_0^t e^{-\alpha(t-\sigma)}\mathcal{T}(t-\sigma)\Pi_{-\gamma}(t,\sigma;t,s)[G_p(s(\sigma),\sigma) \\ \qquad\qquad + \alpha p(s(\sigma),\sigma)]\, d\sigma \quad \text{a.e. } s \in (z_0(t),s_\dagger), \end{cases} \tag{26}$$

where $\tau_0 = \tau_0(t,s)$ and $s(\sigma) = \varphi(\sigma;t,s)$;

(In Case 3 or Case 4) *for a.e. $(s,t) \in S_T$,*

$$\begin{aligned} p(s,t) &= e^{-\alpha t}\mathcal{T}(t)\Pi_{-\gamma}(t,0;t,s)p_0(\varphi(0;t,s)) \\ &\quad + \int_0^t e^{-\alpha(t-\sigma)}\mathcal{T}(t-\sigma)\Pi_{-\gamma}(t,\sigma;t,s)[G_p(s(\sigma),\sigma) + \alpha p(s(\sigma),\sigma)]\, d\sigma,\end{aligned} \tag{27}$$

where $s(\sigma) = \varphi(\sigma; t, s)$.

Proof. Let $p \in L^\infty(0, T; L^1(0, s_\dagger; X))$ be a mild solution to (3). As in (11), put $u(\sigma; t, s) := \Pi_\gamma(\eta, \tau_0; t, s)p(\varphi(\eta; t, s), \eta)$ for $\sigma \in [\tau_0, \tau_1]$. Then $u(\cdot; t, s) \in C([\tau_0, \tau_1]; X)$ as in Theorem 3.1 and it is shown that $u(\cdot; t, s)$ satisfies (14), which means that $u(\cdot; t, s)$ is a mild solution to (13). It is known from the abstract theory of evolution equations (see e.g. [14, Thm. B.22] that $u(\cdot; t, s)$ is a mild solution if and only if $u(\cdot; t, s)$ is an "integral solution" in the sense that $\int_{\tau_0}^{\sigma} u(\eta; t, s)\, d\eta \in D(\mathcal{A})$ and

$$\begin{aligned} u(\sigma; t, s) = u(\tau_0; t, s) + \mathcal{A}\left(\int_{\tau_0}^{\sigma} u(\eta; t, s)\, d\eta\right) \\ + \int_{\tau_0}^{\sigma} \Pi_\gamma(\eta, \tau_0; t, s) G_p(s(\eta), \eta)\, d\eta \end{aligned} \tag{28}$$

for $\sigma \in [\tau_0, \tau_1]$, where $s(\eta) := \varphi(\eta; t, s)$. It is obvious that (28) can be written as

$$\begin{aligned} u(\sigma; t, s) = u(\tau_0; t, s) + (\mathcal{A} - \alpha I)\left(\int_{\tau_0}^{\sigma} u(\eta; t, s)\, d\eta\right) \\ + \int_{\tau_0}^{\sigma} \Pi_\gamma(\eta, \tau_0; t, s)[G_p(s(\sigma), \sigma) + \alpha p(s(\sigma), \eta)]\, d\eta \end{aligned}$$

for $\sigma \in [\tau_0, \tau_1]$. This shows that $u(\cdot; t, s)$ is an integral solution to the differential equation

$$\begin{aligned} \frac{d}{d\sigma} u(\sigma; t, s) = (\mathcal{A} - \alpha I)\, u(\sigma; t, s) \\ + \Pi_\gamma(\sigma, \tau_0; t, s)[G_p(s(\sigma), \sigma) + \alpha p(s(\sigma), \sigma)]. \end{aligned} \tag{29}$$

Hence $u(\cdot; t, s)$ is also a mild solution to (29). Recall that $\mathcal{A} - \alpha I$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{-\alpha t}\mathcal{T}(t) \mid t \geq 0\}$. Then, $u(\cdot; t, s)$ is written by

$$\begin{aligned} u(\sigma; t, s) = e^{-\alpha t}\mathcal{T}(\sigma - \tau_0) u(\tau_0; t, s) \\ + \int_{\tau_0}^{\sigma} e^{-\alpha(t-\eta)}\mathcal{T}(t - \eta)\Pi_\gamma(\eta, \tau_0; t, s)[G_p(s(\eta), \eta) + \alpha p(s(\eta), \eta)]\, d\eta \end{aligned} \tag{30}$$

for $\sigma \in [\tau_0, \tau_1]$. From (30), we can deduce that p satisfies (26) in Case 1 or Case 2, and (27) in Case 3 or Case 4. The converse is also true. $\square$

4. Existence results

Let X be an ordered Banach space with positive cone X_+, that is, X_+ is a convex cone with vertex 0 and the order relation $x \le x'$ is defined by $x' - x \in X_+$ for $x,\ x' \in X$. For positivity of solutions, we impose the following conditions.

(H6) The semigroup $\{\mathcal{T}(t) \mid t \ge 0\}$ generated by $\mathcal{A}$ is a positive semigroup in X, that is, $\mathcal{T}(t)X_+ \subset X_+$ for $t \ge 0$.

(H7) The operator $\mathcal{B}(s,t)$ in (H4) is a positive operator, i.e., $\mathcal{B}(s,t)X_+ \subset X_+$ for a.e. $(s,t) \in S_T$.

(H8) $f \in L^1(S_T; X_+)$, $C \in L^1(0,T;X_+)$, and $p_0 \in L^1(0, s_\dagger; X_+)$.

Concerning the existence of mild solutions, we have

Theorem 4.1. *In addition to* (H1)–(H5)*, we assume* (H6)–(H8) *hold. Then there exists a unique mild solution* $p \in L^\infty(0,T; L^1(0,s_\dagger;X))$ *to* (3)*, which satisfies* $p(s,t) \in X_+$ *a.e.* $(s,t) \in S_T$ *and*

$$\|p(\cdot,t)\|_{L^1(0,s_\dagger;X)} \le \widetilde{C}_T \left\{ \|p_0\|_{L^1(0,s_\dagger;X)} + \|C\|_{L^1(0,T;X)} + \|f\|_{L^1(S_T;X)} \right\} \tag{31}$$

for some $\widetilde{C}_T > 0$ *depending on* $\|\mathcal{B}\|_{L^\infty(S_T;\mathcal{L}(X))}$ *and* $\|\mathcal{M}\|_{L^\infty(S_T;\mathcal{L}(X))}$ *as well as* M, ω, T.

Proof. We prove the theorem only for Case 1 or Case 2. For Case 3 or Case 4, we just ignore the case $s \in (0, z_0(t))$ and the others are the same.

Set $E_{T,+} := L^\infty(0,T;L^1(0,s_\dagger;X_+))$. Put $\alpha = \|\mathcal{M}\|_{L^\infty(S_T;\mathcal{L}(X))}$ and define the mapping K_α on $E_{T,+}$ by

$$[K_\alpha p](s,t) = \begin{cases} e^{-\alpha(t-\tau_0)}\mathcal{T}(t-\tau_0)\Pi_{-\gamma}(t,\tau_0;t,s)\dfrac{F_p(\tau_0)}{g(0,\tau_0)} \\ \qquad + \displaystyle\int_{\tau_0}^t e^{-\alpha(t-\sigma)}\mathcal{T}(t-\sigma)\Pi_{-\gamma}(t,\sigma;t,s)[G_p(s(\sigma),\sigma) \\ \qquad\qquad + \alpha p(s(\sigma),\sigma)]\,d\sigma \quad \text{a.e. } s \in (0, z_0(t)), \\ e^{-\alpha t}\mathcal{T}(t)\Pi_{-\gamma}(t,0;t,s)p_0(\varphi(0;t,s)) \\ \qquad + \displaystyle\int_0^t e^{-\alpha(t-\sigma)}\mathcal{T}(t-\sigma)\Pi_{-\gamma}(t,\sigma;t,s)[G_p(s(\sigma),\sigma) \\ \qquad\qquad + \alpha p(s(\sigma),\sigma)]\,d\sigma \quad \text{a.e. } s \in (z_0(t), s_\dagger), \end{cases} \tag{32}$$

where $\tau_0 := \tau_0(t,s)$, $s(\sigma) := \varphi(\sigma;t,s)$ and $\gamma(s,t) := \mu_0(s,t) + \partial_s g(s,t)$.

In view of Proposition 3.2, we will seek a fixed point of K_α. To do this, we will show that K_α maps $E_{T,+}$ into itself and that K_α is a contraction mapping in $E_{T,+}$. First, note that $[K_\alpha p](s,t) \in X_+$ for $p \in E_{T,+}$ by the choice of α. By definition of $K_\alpha p$, we have

$$\begin{aligned}\|K_\alpha p(\cdot,t)\|_{L^1(0,s_\dagger;X)} &= \int_0^{z_0(t)} \|K_\alpha p(s,t)\|_X \, ds + \int_{z_0(t)}^{s_\dagger} \|K_\alpha p(s,t)\|_X \, ds \\ &\le K_1(t) + K_2(t) + K_3(t) + K_4(t),\end{aligned}$$

where

$$K_1(t) = \int_0^{z_0(t)} \left\| e^{-\alpha(t-\tau_0)} \mathcal{T}(t-\tau_0) \Pi_{-\gamma}(t,\tau_0;t,s) \frac{F_p(\tau_0)}{g(0,\tau_0)} \right\|_X ds,$$

$$K_2(t) = \int_0^{z_0(t)} \int_{\tau_0}^t \|e^{-\alpha(t-\sigma)} \mathcal{T}(t-\sigma) \Pi_{-\gamma}(t,\sigma;t,s) G_{p,\alpha}(s(\sigma),\sigma)\|_X \, d\sigma \, ds,$$

$$K_3(t) = \int_{z_0(t)}^{s_\dagger} \|e^{-\alpha t} \mathcal{T}(t) \Pi_{-\gamma}(t,0;t,s) p_0(\varphi(0;t,s))\|_X \, ds,$$

$$K_4(t) = \int_{z_0(t)}^{s_\dagger} \int_0^t \|e^{-\alpha(t-\sigma)} \mathcal{T}(t-\sigma) \Pi_{-\gamma}(t,\sigma;t,s) G_{p,\alpha}(s(\sigma),\sigma)\|_X \, d\sigma \, ds,$$

where $G_{p,\alpha}(s,t)$ is defined by

$$G_{p,\alpha}(s,t) = G_p(s,t) + \alpha p(s,t).$$

Recall that the semigroup $\{\mathcal{T}(t)\}$ satisfies $\|\mathcal{T}(t)\phi\|_X \le M e^{\omega t} \|\phi\|_X$ for $\phi \in X$ for some $M \ge 0$ and $\omega \in \mathbb{R}$. For $K_1(t)$, we use change of variable from s to σ by $\sigma = \tau_0(t,s)$. Since $ds/d\sigma = -g(0,\sigma)\Pi_{\partial_s g}(t,\sigma;\sigma,0)$, we have

$$\begin{aligned}K_1(t) &= \int_0^t \left\| e^{-\alpha(t-\sigma)} \mathcal{T}(t-\sigma) F_p(\sigma) \right\|_X \Pi_{-\mu_0}(t,\sigma;\sigma,0) \, d\sigma \\ &\le M e^{\omega T} \int_0^t \|F_p(\sigma)\|_X \, d\sigma.\end{aligned}$$

For $K_3(t)$, use change of variable from s to ξ by $\xi = \varphi(0;t,s)$. Since $ds/d\xi = \Pi_{\partial_s g}(t,0;t,s) = \Pi_{\partial_s g}(t,0;0,\xi)$, we obtain

$$K_3(t) = \int_0^{\varphi(0;t,s_\dagger)} \|e^{-\alpha t} \mathcal{T}(t) p_0(\xi)\|_X \Pi_{-\mu_0}(t,0;0,\xi) \, d\xi \le M e^{\omega T} \|p_0\|_{L^1(0,s_\dagger;X)}.$$

To estimate $K_2(t) + K_4(t)$, we use Fubini's theorem and change of variable from s to ξ by $\xi = \varphi(\sigma; t, s)$. Noting that $ds/d\xi = \Pi_{\partial_s g}(t, \sigma, t, s) = \Pi_{\partial_s g}(t, \sigma, \sigma, \xi)$, we have

$$
\begin{aligned}
&K_2(t) + K_4(t) \\
&= \int_0^t \int_{\varphi(t;\sigma,0)}^{s_\dagger} \| e^{-\alpha(t-\sigma)} \mathcal{T}(t-\sigma) \Pi_{-\gamma}(t, \sigma; t, s) G_{p,\alpha}(s(\sigma), \sigma) \|_X \, ds \, d\sigma \\
&= \int_0^t \int_0^{\varphi(\sigma;t,s_\dagger)} \| e^{-\alpha(t-\sigma)} \mathcal{T}(t-\sigma) G_{p,\alpha}(\xi, \sigma) \|_X \Pi_{-\mu_0}(t, \sigma; \sigma, \xi) \, d\xi \, d\sigma \\
&\leq M e^{\omega T} \int_0^t \| G_{p,\alpha}(\cdot, \sigma) \|_{L^1(0,s_\dagger;X)} \, d\sigma.
\end{aligned}
\tag{33}
$$

Here we have used the relation $\tau_0(t, \eta) = \sigma \iff \eta = \varphi(t; \sigma, 0)$ in the first equality in (33). From (15), (16), and the assumptions (H2), (H3), we find that

$$
\| F_p(\sigma) \|_X \leq \| C(\sigma) \|_X + \| \mathcal{B} \|_{L^\infty(S_T; \mathcal{L}(X))} \| p(\cdot, \sigma) \|_{L^1(0,s_\dagger;X)}.
$$
$$
\| G_{p,\alpha}(\cdot, \sigma) \|_{L^1(0,s_\dagger;X)} \leq 2\alpha \| p(\cdot, \sigma) \|_{L^1(0,s_\dagger;X)} + \| f(\cdot, \sigma) \|_{L^1(0,s_\dagger;X)}.
$$

Hence we have

$$
\begin{aligned}
\| K_\alpha p(\cdot, t) \|_{L^1(0,s_\dagger;X)} \leq \widetilde{C}_T \Big\{ &\int_0^t \| p(\cdot, \sigma) \|_{L^1(0,s_\dagger;X)} \, d\sigma \\
&+ \| p_0 \|_{L^1(0,s_\dagger;X)} + \| C \|_{L^1(0,T;X)} + \| f \|_{L^1(S_T;X)} \Big\}
\end{aligned}
\tag{34}
$$

for some constant $\widetilde{C}_T > 0$ depending on $\| \mathcal{B} \|_{L^\infty(S_T; \mathcal{L}(X))}$, $\| \mathcal{M} \|_{L^\infty(S_T; \mathcal{L}(X))}$, M, ω, and T. In particular, $K_\alpha p \in L^\infty(0, T; L^1(0, s_\dagger; X))$ and we find that $K_\alpha p \in E_{T,+}$. Next, as usual, we introduce an equivalent norm

$$
\| p \|_\lambda := \operatorname{ess\,sup}_{t \in (0,T)} e^{-\lambda t} \| p(\cdot, t) \|_{L^1(0,s_\dagger;X)}
$$

in $L^\infty(0, T; L^1(0, s_\dagger; X))$ for $\lambda > 0$. Let p_1, $p_2 \in E_{T,+}$. Considering $p := p_1 - p_2$, by linearity, $K_\alpha p$ satisfies (32) with $p_0 \equiv 0$, $C(t) \equiv 0$, $f(s, t) \equiv 0$ and hence by (34), we have

$$
\begin{aligned}
&e^{-\lambda t} \| K_\alpha p_1(\cdot, t) - K_\alpha p_2(\cdot, t) \|_{L^1(0,s_\dagger;X)} \\
&\quad \leq \widetilde{C}_T e^{-\lambda t} \int_0^t e^{\lambda \sigma} e^{-\lambda \sigma} \| p_1(\cdot, \sigma) - p_2(\cdot, \sigma) \|_{L^1(0,s_\dagger;X)} \, dt \leq \frac{\widetilde{C}_T}{\lambda} \| p_1 - p_2 \|_\lambda.
\end{aligned}
$$

This implies that K_α is a contraction in $E_{T,+}$ if $\lambda > 0$ is taken large enough. Thus, there exists a unique fixed point p in $E_{T,+}$ of K_α. Since the fixed point p satisfies (34) with $K_\alpha p(\cdot,t) = p(\cdot,t)$, using Gronwall's lemma, the estimate (31) is obtained. □

Remark 4.1. (i) Notice that if we just assume (H1)–(H5) without positivity, then (3) still admits a unique mild solution $p \in L^\infty(0,T;L^1(0,s_\dagger;X))$ satisfying the estimate (31).

(ii) If $\mathcal{A}$ is the generator of an analytic semigroup, then by virtue of Theorem 3.1, the mild solution $p(s,t)$ obtained by Theorem 4.1 is differentiable along the characteristic curves and satisfies (20).

Next, we shall consider the boundedness property of mild solutions.

Proposition 4.2. *Suppose $\mathcal{M}(s,t) = \mathbf{0}$, the zero operator in $\mathcal{L}(X)$ for a.e. $(s,t) \in S_T$. Suppose that $f \in L^\infty(S_T;X_+)$, $C \in L^\infty(0,T;X_+)$, and $p_0 \in L^\infty(0,s_\dagger;X_+)$. Then the mild solution $p \in L^\infty(0,T;L^1(0,s_\dagger;X_+))$ obtained by* Theorem 4.1 *satisfies $p \in L^\infty(S_T;X_+)$ and the following estimate holds:*
(in Case 1 or Case 2)

$$\|p(s,t)\|_X \le \widetilde{C}_T^\infty \left\{ \|p_0\|_{L^\infty(0,s_\dagger;X)} + \|C\|_{L^\infty(0,T;X)} + \int_0^t \|f(\cdot,\sigma)\|_{L^\infty(0,s_\dagger;X)} \right\}, \tag{35}$$

where $\widetilde{C}_T^\infty > 0$ is a constant depending on $L_g > 0$, $\underline{g}_T := \min_{t\in[0,T]} g(0,t) > 0$, $\|\mathcal{B}\|_{L^\infty(S_T;\mathcal{L}(X))}$, $s_\dagger$ and $T > 0$;
(in Case 3 or Case 4)

$$\|p(s,t)\|_X \le M e^{\omega T} e^{L_g T} \left\{ \|p_0\|_{L^\infty(0,s_\dagger;X)} + \int_0^t \|f(\cdot,\sigma)\|_{L^\infty(0,s_\dagger;X)}\, d\sigma \right\}. \tag{36}$$

Remark 4.2. If no positivity conditions on f, C, and p_0 are assumed, the same assertions hold for the corresponding mild solution without positivity.

Proof. Let $p \in L^\infty(0,T;L^1(0,s_\dagger;X_+))$ be the mild solution obtained by Theorem 4.1. Note that since $\mathcal{M}(s,t) = 0$, we have $G_p(s,t) = f(s,t)$. In Case 1 or Case 2, it follows from (17) that for a.e. $(s,t) \in S_T$ satisfying $s \in (0, z_0(t))$,

$$\|p(s,t)\|_X \le M e^{\omega T} e^{L_g T} \left\{ \frac{\|F_p(\tau_0)\|_X}{\underline{g}_T} + \int_0^t \|f(\cdot,\sigma)\|_{L^\infty(0,s_\dagger;X)}\, d\sigma \right\} \tag{37}$$

and for a.e. $(s,t) \in S_T$ satisfying $s \in (z_0(t), s_\dagger)$,

$$\|p(s,t)\|_X \le M e^{\omega T} e^{L_g T} \left\{ \|p_0\|_{L^\infty(0,s_\dagger;X)} + \int_0^t \|f(\cdot,\sigma)\|_{L^\infty(0,s_\dagger;X)}\, d\sigma \right\}. \quad (38)$$

From (15), it is easy to see that

$$\|F_p(\tau_0)\|_X \le \|C\|_{L^\infty(0,T;X)} + \|\mathcal{B}\|_{L^\infty(S_T;\mathcal{L}(X))} \|p(\cdot,\tau_0)\|_{L^1(0,s_\dagger;X)}, \quad (39)$$

Note that by the estimate (31) in Theorem 4.1 and by the hypotheses of this proposition, we obtain

$$\begin{aligned} &\|p(\cdot,t)\|_{L^1(0,s_\dagger;X)} \\ &\le \widetilde{C}'_T \left\{ \|p_0\|_{L^\infty(0,s_\dagger;X)} + \|C\|_{L^\infty(0,T;X)} + \int_0^t \|f(\cdot,\sigma)\|_{L^\infty(0,s_\dagger;X)} \right\}, \end{aligned} \quad (40)$$

where $\widetilde{C}'_T$ is a constant depending on $\|\mathcal{B}\|_{L^\infty(S_T;\mathcal{L}(X))}$, $s_\dagger$ and $T > 0$. Then combining (37), (38) with (39) and (40), we see the estimate (35) holds and consequently, we find $p \in L^\infty(S_T; X)$.

In Case 3 or Case 4, since $z_0(t) \equiv 0$, (36) is nothing but (38). □

As mentioned above, we denote by $x \le x'$ the order relation defined by $x' - x \in X_+$ for $x,\ x' \in X$. Under the hypotheses of Theorem 4.1, the following comparison result holds.

Theorem 4.3. *Assume that* (H1)–(H6) *hold. Let $\mathcal{B}_i$, $\mathcal{M}_i$, f_i, C_i, and p_{0i} satisfy* (H7)–(H8) *for $i = 1, 2$. Suppose that*

$$\left.\begin{array}{ll} \mathcal{B}_1(s,t)x \le \mathcal{B}_2(s,t)x, & \text{a.e. } (s,t) \in S_T,\ \forall x \in X_+, \\ \mathcal{M}_1(s,t)x \ge \mathcal{M}_2(s,t)x, & \text{a.e. } (s,t) \in S_T,\ \forall x \in X_+, \\ f_1(s,t) \le f_2(s,t), & \text{a.e. } (s,t) \in S_T, \\ C_1(t) \le C_2(t), & \text{a.e. } t \in (0,T), \\ p_{01}(s) \le p_{02}(s), & \text{a.e. } s \in (0,s_\dagger). \end{array}\right\} \quad (41)$$

Let p_1, $p_2 \in L^\infty(0,T; L^1(0,s_\dagger;X))$ be the corresponding mild solutions to (3)*. Then we have $p_1(s,t) \le p_2(s,t)$ for a.e. $(s,t) \in S_T$.*

Proof. Let $w(s,t) = p_2(s,t) - p_1(s,t)$. Then from definition of the mild solution, it is easily seen that w is a mild solution to (3) corresponding to

$$
\begin{aligned}
\mathcal{B}(s,t) &:= \mathcal{B}_2(s,t), \\
\mathcal{M}(s,t) &:= \mathcal{M}_2(s,t), \\
f(s,t) &:= \{\mathcal{M}_1(s,t)p_1(s,t) - \mathcal{M}_2(s,t)p_1(s,t)\} + f_2(s,t) - f_1(s,t), \\
C(t) &:= \int_0^{s_\dagger} \{\mathcal{B}_2(s,t)p_1(s,t) - \mathcal{B}_1(s,t)p_1(s,t)\}\, ds + C_2(t) - C_1(t), \\
p_0(s) &:= p_{02}(s) - p_{01}(s).
\end{aligned}
$$

By (41), the positivity of the mild solution in Theorem 4.1, w satisfies $w(s,t) \in X_+$, and hence the assertion holds. $\square$

Combining Theorem 4.3 with Proposition 4.2, we obtain the following boundedness property of mild solutions.

Theorem 4.4. *Let X be a Banach lattice with positive cone X_+. Assume that $f \in L^\infty(S_T; X_+)$, $C \in L^\infty(0,T; X_+)$, and $p_0 \in L^\infty(0, s_\dagger; X_+)$. Then any mild solution $p \in L^\infty(0,T; L^1(0, s_\dagger; X_+))$ belongs to $L^\infty(S_T; X_+)$ and satisfies the same estimate as in* (35) *in* Case 1 *or* Case 2*, and* (36) *in* Case 3 *or* Case 4.

Proof. Let $\overline{p} \in L^\infty(0,T; L^1(0, s_\dagger; X_+))$ be a mild solution corresponding to

$$
\begin{aligned}
\mathcal{B}(s,t) &:= \|\mathcal{B}\|_{L^\infty(S_T;\mathcal{L}(X))} I, \\
\mathcal{M}(s,t) &:= \mathbf{0}, \\
f(s,t) &:= \|f(\cdot,t)\|_{L^\infty(0,s_\dagger;X)} \mathbf{1}, \\
C(t) &:= \|C\|_{L^\infty(0,T;X)} \mathbf{1}, \\
p_0(s) &:= \|p_0\|_{L^\infty(0,s_\dagger;X)} \mathbf{1},
\end{aligned}
$$

where I is the identity operator in $\mathcal{L}(X)$, $\mathbf{0}$ is the zero operator in $\mathcal{L}(X)$ and $\mathbf{1}$ is the unit vector in X. Then Theorem 4.3 implies $0 \le p(s,t) \le \overline{p}(s,t)$ for a.e. $(s,t) \in S_T$. Since X is a Banach lattice, we have $\|p(s,t)\|_X \le \|\overline{p}(s,t)\|_X$ for a.e. $(s,t) \in S_T$. By Proposition 4.2, we know $\overline{p} \in L^\infty(S_T; X)$ and $\overline{p}$ satisfies the estimate (35) in Case 1 or Case 2, and (36) in Case 3 or Case 4. Hence so does p. $\square$

5. Dual problems

Let X^* be the dual space of X. The bracket $\langle p, \xi \rangle$ means the dual pair for $p \in X$ and $\xi \in X^*$. Let $\mathcal{A}^*$ be the adjoint operator of $\mathcal{A}$. Then the following holds:

$$\langle \mathcal{A}p, \xi \rangle = \langle p, \mathcal{A}^* \xi \rangle \quad \text{for } p \in D(\mathcal{A}) \text{ and } \xi \in D(\mathcal{A}^*),$$

where

$$D(\mathcal{A}^*) = \{\xi \in X^* \mid \exists \eta \in X^* \text{ such that } \langle \mathcal{A}p, \xi \rangle = \langle p, \eta \rangle \text{ for all } p \in D(\mathcal{A})\}.$$

Recall that if X is reflexive, then $D(\mathcal{A}^*)$ is dense in X^* and $\mathcal{A}^*$ is the generator of the adjoint semigroup $\{\mathcal{T}^*(t) \mid t \geq 0\}$ in X^*. See e.g. [13]. Let $\mathcal{M}^*(s,t)$ and $\mathcal{B}^*(s,t)$ be the adjoint operators for $\mathcal{M}(s,t)$ and $\mathcal{B}(s,t)$ for a.e. $(s,t) \in S_T$, respectively. Note that $\mathcal{B}^*$, $\mathcal{M}^* \in L^\infty(S_T; \mathcal{L}(X^*))$ since $\|\mathcal{B}^*(s,t)\|_{\mathcal{L}(X^*)} = \|\mathcal{B}(s,t)\|_{\mathcal{L}(X)}$ and $\|\mathcal{M}^*(s,t)\|_{\mathcal{L}(X^*)} = \|\mathcal{M}(s,t)\|_{\mathcal{L}(X)}$. We consider the following problem:

(In Case 1 or Case 3) Given $f^* \in L^\infty(S_T; X^*)$, find $\xi \in L^\infty(S_T; D(\mathcal{A}^*)) \cap C^1_\varphi(S_T; X^*)$ which satisfies

$$\left.\begin{aligned}
&D_\varphi \xi(s,t) + \mathcal{A}^*\xi(s,t) - \mathcal{M}^*(s,t)\xi(s,t) + \mathcal{B}^*(s,t)\xi(0,t) = f^*(s,t),\\
&\qquad a.e.\ (s,t) \in S_T,\\
&\xi(s_\dagger, t) := \lim_{h \to +0} \xi(\varphi(t-h; t, s_\dagger), t-h) = 0, \quad a.e.\ t \in (0,T),\\
&\xi(s, T) := \lim_{h \to +0} \xi(\varphi(T-h; T, s), T-h) = 0, \quad a.e.\ s \in (0, s_\dagger).
\end{aligned}\right\} \tag{42}$$

(In Case 2 or Case 4) Given $f^* \in L^\infty(S_T; X^*)$, find $\xi \in L^\infty(S_T; D(\mathcal{A}^*)) \cap C^1_\varphi(S_T; X^*)$ which satisfies

$$\left.\begin{aligned}
&D_\varphi \xi(s,t) + \mathcal{A}^*\xi(s,t) - \mathcal{M}^*(s,t)\xi(s,t) + \mathcal{B}^*(s,t)\xi(0,t) = f^*(s,t),\\
&\qquad a.e.\ (s,t) \in S_T,\\
&\xi(s, T) := \lim_{h \to +0} \xi(\varphi(T-h; T, s), T-h) = 0, \quad a.e.\ s \in (0, s_\dagger).
\end{aligned}\right\} \tag{43}$$

Let $\overline{g}(s,t) := g(s_\dagger - s, T - t)$ for $(s,t) \in \overline{S}_T$. Then $\overline{g}$ satisfies (H1). Define the characteristic curve $\overline{\varphi}(t; t_0, s_0)$ through (s_0, t_0) for the function $\overline{g}$ by the solution $s(t)$ of

$$s'(t) = \overline{g}(s(t), t), \ t \in [0,T], \quad s(t_0) = s_0.$$

Note that the following relation holds:

$$\overline{\varphi}(t;t_0,s_0) = s_\dagger - \varphi(T-t;t_0,s_0).$$

Note also that if $g(s,t)$ satisfies Case 1 or Case 4, then $\overline{g}(s,t)$ satisfies Case 1 or Case 4, respectively, and if $g(s,t)$ satisfies Case 2 or Case 3, then $\overline{g}(s,t)$ satisfies Case 3 or Case 2, respectively.

Let $\overline{\xi}(s,t) := \xi(s_\dagger - s, T-t)$. Then we have

$$\begin{aligned} D_{\overline{\varphi}}\overline{\xi}(s,t) &= \lim_{h\to 0}\frac{1}{h}[\overline{\xi}(\overline{\varphi}(t+h;t,s),t+h) - \overline{\xi}(s,t)] \\ &= \lim_{h\to 0}\frac{1}{h}[\xi(s_\dagger - \overline{\varphi}(t+h;t,s),T-t-h) - \xi(s_\dagger - s, T-t)] \\ &= \lim_{h\to 0}\frac{1}{h}[\xi(\varphi(T-t-h;t,s),T-t-h) - \xi(s_\dagger - s, T-t)] \\ &= -D_\varphi \xi(s_\dagger - s, T-t). \end{aligned}$$

Hence, in Case 1 or 3, the problem (42) is reduced to

$$\left.\begin{aligned} D_{\overline{\varphi}}\overline{\xi}(s,t) &= \mathcal{A}^*\overline{\xi}(s,t) - \overline{\mathcal{M}}^*(s,t)\overline{\xi}(s,t) + \overline{\mathcal{B}}^*(s,t)\overline{\xi}(s_\dagger,t) - \overline{f}^*(s,t) \\ &\qquad a.e.\ (s,t)\in S_T, \\ \overline{\xi}(0,t) &= 0 \quad a.e.\ t\in(0,T) \\ \overline{\xi}(s,0) &= 0 \quad a.e.\ s\in(0,s_\dagger) \end{aligned}\right\} \tag{44}$$

and in Case 2 or Case 4, the problem (43) is reduced to

$$\begin{aligned} D_{\overline{\varphi}}\overline{\xi}(s,t) &= \mathcal{A}^*\overline{\xi}(s,t) - \overline{\mathcal{M}}^*(s,t)\overline{\xi}(s,t) + \overline{\mathcal{B}}^*(s,t)\overline{\xi}(s_\dagger,t) - \overline{f}^*(s,t) \\ &\qquad a.e.\ (s,t)\in S_T, \\ \overline{\xi}(s,0) &= 0 \quad a.e.\ s\in(0,s_\dagger) \end{aligned}$$

where

$$\begin{aligned} \overline{\mathcal{M}}^*(s,t) &= \mathcal{M}^*(s_\dagger - s, T-t), \\ \overline{\mathcal{B}}^*(s,t) &= \mathcal{B}^*(s_\dagger - s, T-t), \\ \overline{f}^*(s,t) &= f^*(s_\dagger - s, T-t). \end{aligned}$$

Let X be a Banach lattice with positive cone X_+. Then its dual X^* is a Banach lattice with positive cone $X_+^* := \{\xi\in X^* \mid \langle p,\xi\rangle \geq 0,\ \forall p\in X_+\}$. Furthermore, $L^1(S_T;X)$ is a Banach lattice with positive cone $L^1(S_T;X_+)$ and its dual space $L^\infty(S_T;X^*)$ is a Banach lattice with positive cone $L^\infty(S_T;X_+^*)$.

For the dual problem we have the following:

Theorem 5.1. *Assume that X is a reflexive Banach lattice and $\mathcal{A}$ is the generator of an analytic semigroup in X. Then for any $f^* \in L^\infty(S_T; X^*)$ satisfying $-f^*(s,t) \in X^*_+$, there exists a unique $\xi \in L^\infty(S_T; D(\mathcal{A}^*) \cap X^*_+) \cap C^1_\varphi(S_T; X^*)$ which satisfies* (42) *in* Case 1 *or* Case 3*, and* (43) *in* Case 2 *or* Case 4*, respectively.*

Proof. We first consider the cases $g(s,t)$ satisfies Case 1 or Case 3. In these cases, note that $\overline{g}(s,t)$ satisfies Case 1 or Case 2, respectively. Let $E^*_+ = L^\infty(S_T; D(\mathcal{A}^*) \cap X^*_+) \cap C^1_\varphi(S_T; X^*)$. For given $\zeta \in E^*_+$, we consider the following problem:

$$\left.\begin{aligned} D_{\overline{\varphi}}\overline{\xi}(s,t) &= \mathcal{A}^*\overline{\xi}(s,t) - \overline{\mathcal{M}}^*(s,t)\overline{\xi}(s,t) + \overline{\mathcal{B}}^*(s,t)\zeta(s_\dagger,t) - \overline{f}^*(s,t) \\ &\qquad a.e.\ (s,t) \in S_T, \\ \overline{\xi}(0,t) &= 0 \quad a.e.\ t \in (0,T), \\ \overline{\xi}(s,0) &= 0 \quad a.e.\ s \in (0,s_\dagger), \end{aligned}\right\} \tag{45}$$

where $\zeta(s_\dagger,t) := \lim_{h\to+0}\zeta(\varphi(t-h;t,s_\dagger),t-h)$. Recall that the ajoint semigroup $\{\mathcal{T}^*(t) \mid t \geq 0\}$ becomes an analytic semigroup (e.g. [2, Prop. 1.2.3]), and it is easily seen that $\{\mathcal{T}^*(t) \mid t \geq 0\}$ is a positive semigroup. Note that (45) has the same structure as (20) considering

$$\begin{aligned} \gamma(s,t) &:= \partial_s\overline{g}(s,t), \\ \mathcal{M}(s,t) &:= \mathcal{M}^*(s,t) - \partial_s\overline{g}(s,t), \\ f(s,t) &:= \overline{\mathcal{B}}^*(s,t)\zeta(s_\dagger,t) - \overline{f}^*(s,t), \\ C(t) &:= 0,\ \mathcal{B}(s,t) := 0,\ p_0(s) := 0. \end{aligned}$$

Then Theorems 31 and 4.4 combined with Theorem 3.1 as well as Remark 4.1 imply that the problem (45) admits a unique solution $\overline{\xi} \in E^*_+$ and by (35), the following estimate holds:

$$\|\overline{\xi}(s,t)\|_{X^*} \leq \widetilde{C}^{*,\infty}_T \int_0^t \|\overline{\mathcal{B}}^*(s,t)\zeta(s_\dagger,t) - \overline{f}^*(s,t)\|_{L^\infty(0,s_\dagger;X^*)}, \tag{46}$$

where $\widetilde{C}^{*,\infty}_T > 0$ is a constant depending on $L_{\overline{g}} > 0$, $\min_{t\in[0,T]}\overline{g}(0,t) > 0$, $s_\dagger$ and $T > 0$. Put $[Z\zeta](s,t) = \overline{\xi}(s,t)$. Then Z maps E^*_+ into itself. Let ζ_1, $\zeta_2 \in E^*_+$ and let $\overline{\xi}_1$ and $\overline{\xi}_2$ be the corresponding solutions to (45). Then $\widetilde{\xi} := \overline{\xi}_1 - \overline{\xi}_2$ is a solution to (45) with $\widetilde{\zeta}(s_\dagger,t) := \zeta_1(s_\dagger,t) - \zeta_2(s_\dagger,t)$ and

$\overline{f}^* := 0$. Then by the estimate of (46), we obtain

$$
\begin{aligned}
\|\widetilde{\xi}(s,t)\|_{X^*} &\le \widetilde{C}_T^{*,\infty} \int_0^t \|\overline{\mathcal{B}}^*(\cdot,\sigma)\widetilde{\zeta}(s_\dagger,\sigma)\|_{L^\infty(0,s_\dagger;X^*)}\, d\sigma \\
&\le \widetilde{C}_T^{*,\infty} \|\overline{\mathcal{B}}^*\|_{L^\infty(S_T;\mathcal{L}(X^*))} \int_0^t \|\widetilde{\zeta}(\cdot,\sigma)\|_{L^\infty(0,s_\dagger;X^*)}\, d\sigma,
\end{aligned}
\tag{47}
$$

As usual, we introduce the norm on $L^\infty(S_T;X^*)$ by

$$
\|\zeta\|_{\infty,\lambda} := \operatorname{ess\,sup}_{t\in(0,T)} e^{-\lambda t}\|\zeta(\cdot,t)\|_{L^\infty(0,s_\dagger;X^*)}
$$

for $\lambda > 0$, which is equivalent to the original norm. Then, it follows from (47) that

$$
\|Z\zeta\|_{\infty,\lambda} = \|\widetilde{\xi}\|_{\infty,\lambda} \le \frac{\widetilde{C}_T^{*,\infty}\|\overline{\mathcal{B}}^*\|_{L^\infty(S_T;\mathcal{L}(X^*))}}{\lambda}\|\widetilde{\zeta}\|_{\infty,\lambda}
$$

Taking $\lambda > 0$ large enough, Z is shown to be a contraction mapping from E_+^* into itself and (44) admits a unique solution $\overline{\xi} \in E_+^*$. Hence the dual problem (42) has a unique solution $\xi \in L^\infty(S_T; D(\mathcal{A}^*) \cap X_+^*) \cap C_\varphi^1(S_T;X^*)$.

Next, if $g(s,t)$ satisfies Case 2 or Case 4, then $\overline{g}(s,t)$ satisfies Case 3 or Case 4, respectively. In these cases, we just consider (45) without the second equation. The rest of the proof is the same. □

Corollary 5.2. *Assume that X is a reflexive Banach lattice and $\mathcal{A}$ is the generator of an analytic semigroup in X. Then for $f^* \in L^\infty(S_T;X^*)$, there exists $\xi \in L^\infty(S_T;D(\mathcal{A}^*)) \cap C_\varphi^1(S_T;X^*)$ which satisfies* (42) *in* Case 1 *or* Case 3*, and* (43) *in* Case 2 *or* Case 4*, respectively.*

Proof. For $f^* \in L^\infty(S_T;X^*)$, write $f^* = f_+^* - f_-^*$ with the positive part f_+^* and the negative part f_-^*. Put $f_1^* := -f_+^*$ and $f_2^* := -f_-^*$. For f_i^* $(i = 1,2)$, by Theorem 5.1, there exists $\xi_i \in L^\infty(S_T; D(\mathcal{A}^*) \cap X_+^*) \cap C_\varphi^1(S_T;X^*)$ which satisfies (42) for f_i^* instead of f^* in Case 1 or Case 3, and (43) for f_i^* instead of f^* in Case 2 or Case 4, respectively. Hence by linearity, $\xi := -(\xi_1 - \xi_2) = \xi_2 - \xi_1$ becomes the desired function. □

6. Weak solutions

In order to define a weak solution to (3), we set two classes of test functions corresponding with Case 1–Case4.

(In Case 1 or Case 3) Let Ξ denote the set of all $\xi \in L^\infty(S_T; D(\mathcal{A}^*)) \cap C^1_\varphi(S_T; X^*)$ satisfying

$$\begin{aligned}
&D_\varphi \xi + \mathcal{A}^*\xi - \mathcal{M}^*(\cdot,\cdot)\xi + \mathcal{B}^*(\cdot,\cdot)\xi(0,\cdot) \in L^\infty(S_T; X^*),\\
&\xi(s_\dagger, t) := \lim_{\eta \to t} \xi(\varphi(\eta; t, s_\dagger), \eta) = 0 \quad a.e.\ t \in (0,T),\\
&\xi(s, T) := \lim_{h \to +0} \xi(\varphi(T-h; T, s), T-h) = 0 \quad a.e.\ s \in (0, s_\dagger).
\end{aligned}$$

(In Case 2 or Case 4) Let Ξ_0 denote the set of all $\xi \in L^\infty(S_T; D(\mathcal{A}^*)) \cap C^1_\varphi(S_T; X^*)$ satisfying

$$\begin{aligned}
&D_\varphi \xi + \mathcal{A}^*\xi - \mathcal{M}^*(\cdot,\cdot)\xi + \mathcal{B}^*(\cdot,\cdot)\xi(0,\cdot) \in L^\infty(S_T; X^*),\\
&\xi(s, T) := \lim_{\eta \to T} \xi(\varphi(\eta; T, s), \eta) = 0 \quad a.e.\ s \in (0, s_\dagger).
\end{aligned}$$

Definition 6.1. (In Case 1 or Case 2) A function $p \in L^1(S_T; X)$ is said to be a *weak solution* to (3) if p satisfies

$$\begin{aligned}
&\int_{S_T} \langle p(s,t), -D_\varphi \xi(s,t) - \mathcal{A}^*\xi(s,t) + \mathcal{M}^*(s,t)\xi(s,t) - \mathcal{B}^*(s,t)\xi(0,t)\rangle \, dsdt\\
&= \int_0^{s_\dagger} \langle p_0(s), \xi(s,0)\rangle \, ds + \int_0^T \langle C(t), \xi(0,t)\rangle \; dt + \int_{S_T} \langle f(s,t), \xi(s,t)\rangle \, dsdt
\end{aligned} \tag{48}$$

for any $\xi \in \Xi$ in Case 1 ($\xi \in \Xi_0$ in Case 2, respectively);
(In Case 3 or Case 4) A function $p \in L^1(S_T; X)$ is said to be a *weak solution* to (3) if p satisfies

$$\begin{aligned}
&\int_{S_T} \langle p(s,t), -D_\varphi \xi(s,t) - \mathcal{A}^*\xi(s,t) + \mathcal{M}^*(s,t)\xi(s,t)\rangle \, dsdt\\
&= \int_0^{s_\dagger} \langle p_0(s), \xi(s,0)\rangle \, ds + \int_{S_T} \langle f(s,t), \xi(s,t)\rangle \, dsdt
\end{aligned}$$

for any $\xi \in \Xi$ in Case 3 ($\xi \in \Xi_0$ in Case 4, respectively).

Proposition 6.1. *If $p \in L^\infty(0,T; L^1(0, s_\dagger; X))$ is a mild solution of* (3)*, then p is a weak solution.*

Proof. Let $p \in L^\infty(0,T; L^1(0,s_\dagger; X))$ be a mild solution of (3). Consider first the Case 1 or Case 2. Let $\xi \in \Xi$ in Case 1 and $\xi \in \Xi_0$ in Case 2, respectively. By Theorem 3.1, we have

$$
\begin{aligned}
I(h) &:= \int_0^{T-h}\int_0^{s_\dagger} \left\langle \frac{1}{h}[p(\varphi(t+h;t,s),t+h)-p(s,t)], \xi(s,t)\right\rangle dsdt \\
&\to \int_0^T\int_0^{s_\dagger} \langle D_\varphi p(s,t), \xi(s,t)\rangle \, dsdt \\
&= \int_0^T\int_0^{s_\dagger} \langle \mathcal{A}p(s,t) - \gamma(s,t)p(s,t) - \mathcal{M}(s,t)p(s,t), \xi(s,t)\rangle \, dsdt \\
&\quad + \int_0^T\int_0^{s_\dagger} \langle f(s,t), \xi(s,t)\rangle \, dsdt \\
&= \int_0^T\int_0^{s_\dagger} \langle p(s,t), \mathcal{A}^*\xi(s,t) - \gamma(s,t)\xi(s,t) - \mathcal{M}^*(s,t)\xi(s,t)\rangle \, dsdt \\
&\quad + \int_0^T\int_0^{s_\dagger} \langle f(s,t), \xi(s,t)\rangle \, dsdt
\end{aligned} \tag{49}
$$

as $h \to +0$. On the other hand, by changing variable from (s,t) to (y,σ) by the relation $(s,t) = (\varphi(\sigma-h;\sigma,y), \sigma-h)$, we have

$$
\begin{aligned}
I(h) &= \frac{1}{h}\Bigg[\int_h^T \int_{\varphi(\sigma;\sigma-h,0)}^{\varphi(\sigma;\sigma-h,s_\dagger)} \langle p(y,\sigma), \xi(\varphi(\sigma-h;\sigma,y),\sigma-h)\rangle \, J_h(y,\sigma)\, dyd\sigma \\
&\qquad - \int_0^{T-h}\int_0^{s_\dagger} \langle p(s,t), \xi(s,t)\rangle \, dsdt\Bigg] \\
&= \int_h^{T-h}\int_{\varphi(\sigma;\sigma-h,0)}^{\varphi(\sigma;\sigma-h,s_\dagger)} \Bigg\{\left\langle p(y,\sigma), \frac{1}{h}\left[\xi(\varphi(\sigma-h;\sigma,y),\sigma-h) - \xi(y,\sigma)\right]\right\rangle \\
&\qquad \times J_h(y,\sigma) + \left\langle p(y,\sigma), \frac{1}{h}(J_h(y,\sigma)-1)\xi(y,\sigma)\right\rangle\Bigg\} dyd\sigma \\
&\quad + \frac{1}{h}\int_{T-h}^T \int_{\varphi(\sigma;\sigma-h,0)}^{\varphi(\sigma;\sigma-h,s_\dagger)} \langle p(y,\sigma), \xi(\varphi(\sigma-h;\sigma,y),\sigma-h)\rangle \, J_h(y,\sigma)\, dyd\sigma \\
&\quad - \int_h^{T-h} \frac{1}{h}\int_0^{\varphi(\sigma;\sigma-h,0)} \langle p(y,\sigma), \xi(y,\sigma)\rangle \, dyd\sigma \\
&\quad - \frac{1}{h}\int_0^h\int_0^{s_\dagger} \langle p(y,\sigma), \xi(y,\sigma)\rangle \, dyd\sigma \\
&=: I_1(h) + I_2(h) - I_3(h) - I_4(h),
\end{aligned} \tag{50}
$$

where $J_h(y,\sigma) = \dfrac{\partial(s,t)}{\partial(y,\sigma)} = \Pi_{\partial_s g}(\sigma - h, \sigma; \sigma, y) > 0$ is the Jacobian. As $h \to +0$, it is shown that

$$I_1(h) \to \int_0^T \int_0^{s_\dagger} \langle p(y,\sigma), -D_\varphi \xi(y,\sigma) - \gamma(y,\sigma)\xi(y,\sigma)\rangle \, dy d\sigma, \tag{51}$$

$$I_2(h) \to \int_0^{s_\dagger} \langle p(y,T), \xi(y,T)\rangle \, dy = 0, \tag{52}$$

$$\begin{aligned} I_3(h) &= \int_h^{T-h} \frac{\varphi(\sigma;\sigma-h,0) - \varphi(\sigma;\sigma,0)}{h\varphi(\sigma;\sigma-h,0)} \int_0^{\varphi(\sigma;\sigma-h,0)} \langle p(y,\sigma), \xi(y,\sigma)\rangle \, dy d\sigma \\ &\to \int_0^T \langle g(0,\sigma)p(0,\sigma), \xi(0,\sigma)\rangle \, d\sigma \\ &= \int_0^T \left\langle C(\sigma) + \int_0^{s_\dagger} \mathcal{B}(s,\sigma)p(s,\sigma)\, ds, \xi(0,\sigma) \right\rangle d\sigma, \end{aligned} \tag{53}$$

$$I_4(h) \to \int_0^{s_\dagger} \langle p_0(y), \xi(y,0)\rangle \, dy. \tag{54}$$

It follows from (51), (52), (53), and (54) that

$$\begin{aligned} I(h) \to &\int_0^T \int_0^{s_\dagger} \langle p(s,t), -D_\varphi \xi(s,t) - \gamma(s,t)\xi(s,t)\rangle \, ds dt \\ &+ \int_0^T \langle C(t), \xi(0,t)\rangle \, dt + \int_0^T \int_0^{s_\dagger} \langle p(s,t), \mathcal{B}^*(s,t)\xi(0,t)\rangle \, ds dt \\ &+ \int_0^{s_\dagger} \langle p_0(s), \xi(s,0)\rangle \, ds \end{aligned} \tag{55}$$

as $h \to +0$. From (49) and (55), we find that (48) is satisfied and p is a weak solution.

Next, consider the Case 3 or Case 4. Let $\xi \in \Xi$ in Case 3 and $\xi \in \Xi_0$ in

Case 4. Since $\varphi(\sigma;\sigma-h,0)=0$, (50) reads as

$$
\begin{aligned}
I(h) = &\int_h^{T-h}\int_0^{\varphi(\sigma;\sigma-h,s_\dagger)}\Bigg\{\left\langle p(y,\sigma), \frac{1}{h}[\xi(\varphi(\sigma-h;\sigma,y),\sigma-h)-\xi(y,\sigma)]\right\rangle \\
&\qquad \times |J_h(y,\sigma)| + \left\langle p(y,\sigma), \frac{1}{h}[|J_h(y,\sigma)|-1]\xi(y,\sigma)\right\rangle\Bigg\}\, dyd\sigma \\
&+\frac{1}{h}\int_{T-h}^{T}\int_0^{\varphi(\sigma;\sigma-h,s_\dagger)} \langle p(y,\sigma), \xi(\varphi(\sigma-h;\sigma,y),\sigma-h)\rangle\, |J_h(y,\sigma)|\, dyd\sigma \\
&-\frac{1}{h}\int_0^h\int_0^{s_\dagger}\langle p(y,\sigma),\xi(y,\sigma)\rangle\; dyd\sigma.
\end{aligned}
$$

The rest of the arguments is same. □

It is known that $\mathcal{X} := L^1(S_T;X)$ is a Banach lattice with positive cone $\mathcal{X}_+ := L^1(S_T;X_+)$ and $\mathcal{X}$ is order complete if every order interval in X is weakly compact. See [4]. Thus if we assume X is reflexive, then $\mathcal{X} = L^1(S_T;X)$ is shown to be order complete. Then one can define the signum operator for $p\in\mathcal{X}$, denoted by $\operatorname{sign} p \in \mathcal{L}(\mathcal{X})$ which satisfies the following properties:

(i) $(\operatorname{sign} p)p = |p|$,
(ii) $|(\operatorname{sign} p)q| \le |q|$ for $q\in\mathcal{X}$,
(iii) $(\operatorname{sign} p)q = 0$ for $q\in\mathcal{X}$ orthogonal to p.

Here $|p|$ denotes the absolute value of $p\in\mathcal{X}$. See [12, C-I] for basic facts of Banach lattice and signum operators. The dual space $\mathcal{X}^* = L^\infty(S_T;X^*)$ is a Banach lattice with positive cone $\mathcal{X}^*_+ = L^\infty(S_T;X^*_+)$.

We are ready to show the uniqueness of weak solutions.

Theorem 6.2. *Assume that X is a reflexive Banach lattice and $\mathcal{A}$ is the generator of an analytic semigroup in X. Then there exists at most one weak solution to* (3).

Proof. Let us consider Case 1 or Case 2 first. Let p_1 and p_2 be two weak solutions to (3). Then the difference $p := p_1 - p_2$ satisfies $p\in L^1(S_T;X)$ and

$$
\int_{S_T}\langle p(s,t), D_\varphi\xi(s,t) + \mathcal{A}^*\xi(s,t) - \mathcal{M}^*(s,t)\xi(s,t) + \mathcal{B}^*(s,t)\xi(0,t)\rangle\; dsdt = 0 \tag{56}
$$

for any $\xi \in \Xi$ in Case 1 ($\xi \in \Xi_0$ in Case 2, respectively). For any $q^* \in \mathcal{X}^*$, let $f^* = (\operatorname{sign} p)^* q^* \in \mathcal{X}^* = L^\infty(S_T; X^*)$, where $(\operatorname{sign} p)^* \in \mathcal{L}(\mathcal{X}^*)$ is the adjoint of $\operatorname{sign} p \in \mathcal{L}(\mathcal{X})$. By Corollary 5.2, there exists $\xi \in \Xi$ in Case 1, and $\xi \in \Xi_0$ in Case 2, respectively such that

$$D_\varphi \xi(s,t) + \mathcal{A}^* \xi(s,t) - \mathcal{M}^*(s,t)\xi(s,t) + \mathcal{B}^*(s,t)\xi(0,t) = -f^*(s,t),$$
$$a.e.\ (s,t) \in S_T.$$

Plugging this ξ into (56) yields $\int_{S_T} \langle p(s,t), f^*(s,t)\rangle \, dsdt = 0$. This implies $\langle |p|, q^* \rangle_{\mathcal{X},\mathcal{X}^*} = \langle p, f^* \rangle_{\mathcal{X},\mathcal{X}^*} = 0$. Hence we have $|p| = 0$ in $\mathcal{X}$ and so, $\|p\|_{\mathcal{X}} = \||p|\|_{\mathcal{X}} = 0$. Consequently, we have $p_1(s,t) = p_2(s,t)$ holds for a.e. $(s,t) \in S_T$. $\square$

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