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Exactly Soluble Quantum Model for Repeated Harmonic Perturbation

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ABSTRACT

We consider an exactly soluble dynamical system with inelastic repeated harmonic perturbation. Hamiltonian dynamics is quasi-free and it leads in the large-time limit to relaxation of initial states and to the entropy production. To study correlations we consider time evolution of subsystems. We prove a universality of dynamics driven by repeated harmonic perturbation in a short-time interaction limit.

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Contents

1 The Model

We consider an exactly soluble model of quantum system proposed in [TZ]. It is a harmonic system (one-mode quantum oscillator $\mathcal{S}$) successively perturbed by time-dependent stationary repeated harmonic interactions. This sequence of perturbation is switched on at the moment $t = 0$ and it acts successively on the interval $0 \leq t < \infty$. It is a common fashion to present this sequence as repeated interactions of the system $\mathcal{S}$ with an *infinite* time-equidistant *chain*: $\mathcal{C} = \mathcal{S}_1 + \mathcal{S}_2 + \ldots$, of subsystems $\{\mathcal{S}_k\}_{k\geq 1}$ [BJM].

Note that there is a physical interpretation [NVZ], [BJM], behind of this mathematical setting. For the model [TZ], the system $\mathcal{C}_N$ is identified with a chain of N quantum particles ("atoms") with infinitely many *harmonic* internal degrees of freedom. They interact *one-by-one* with a one-mode quantum resonator (cavity) $\mathcal{S}$. This is a caricature of the one-atom maser system. In contract to [NVZ], but similar to the two-level Jaynes-Cummings atoms [BJM], the interaction with harmonic atoms is *inelastic*. This yields a drastic difference between evolution of the model [TZ] and the model [NVZ] with completely elastic interaction.

Recall that experimental study of interaction of a single atom in a cavity is expected to be drastically modified as compared with its behaviour in a free space. First, the spontaneous emission is enhanced in a resonant high-Q (i.e. non-leaky) cavity and it is suppressed if the cavity is off the resonance, see e.g.[M]. Another important difference is related to the nature of interaction of Ridberg's atoms and the cavity radiance. In [NVZ] an exactly soluble model in the limit of the *rigid* atoms shows that it corresponds to the regime of a "kick" cavity evolution [FJMa]. Whereas in the regime of the inelastic atom-cavity interaction the system may to relax to a steady state even for a non-leaky cavity [FJMb]. For example, this property manifests the models for the two-level Jaynes-Cummings atoms [BJM]. In the present paper we study a model for atoms with *infinitely* many levels, which imitates very soft Ridberg's atoms.

Below we suppose that the states of $\mathcal{S}$ and of every $\mathcal{S}_k$ are *normal*, i.e. defined by the *density matrices* ρ_0 and $\{\rho_k\}_{k=1}^{\infty}$ on the Hilbert spaces $\mathscr{H}_{\mathcal{S}}$ and $\{\mathscr{H}_{\mathcal{S}_k}\}_{k=1}^{\infty}$, respectively. The Hilbert space of the total system is then the tensor product $\mathscr{H}_{\mathcal{S}} \otimes \mathscr{H}_{\mathcal{C}}$. Here the infinite product $\mathscr{H}_{\mathcal{C}} = \otimes_{k\geq 1}\mathscr{H}_{\mathcal{S}_k}$ stays for the Hilbert space chain. Details of dynamics are presented in the next Section 2. Below we collect our hypothesis.

(H1) *Initial states.* For $t \leq 0$, *all* components of $\mathcal{S}$ and $\{\mathcal{S}_k\}_{k=1}^N$ are *independent*, i.e. the state of $\mathcal{S} + \mathcal{C}_N$ is described as a finite tensor product: $\omega_{\mathcal{S}+\mathcal{C}_N} := \omega_{\mathcal{S}} \otimes \bigotimes_{k=1}^N \omega_{\mathcal{S}_k}$. We suppose that each of the state in the product is *normal*.

(H2) *Tuned interaction.* We consider repeated perturbations in the tuned regime: for any moment $t \geq 0$ exactly *one* subsystem ("atom") $\mathcal{S}_n$ is interacting with the system $\mathcal{S}$ (quantum resonator) during a fixed time $\tau > 0$. Here $n = [t/\tau] + 1$, where $[x]$ denotes the *integer* part of $x \geq 0$.

Let $\mathscr{H}_0$ be the Hilbert space for the system $\mathcal{S}$ and $\mathscr{H}_k$ be the Hilbert space for the the system $\mathcal{S}_k$ for $k = 1, \cdots, N$. Then for $k = 0, 1, \cdots, N$, the space $\mathscr{H}_k$ is a copy of the one-mode boson Fock space $\mathscr{F}$ with the vacuum vector $\Omega \in \mathscr{F}$ and with densely defined boson annihilation and (adjoint) creation operators: a and a^*, defined by $a\,\Omega = 0$. The total system $\mathcal{S} + \mathcal{C}_N$ lives in the Hilbert space

$$\mathscr{H}^{(N)} := \mathscr{H}_0 \otimes \bigotimes_{k=1}^{N} \mathscr{H}_k = \mathscr{F}^{\otimes(N+1)} \ . \tag{1.1}$$

Here $\mathbb{1}$ is the unit operator on $\mathscr{F}$. In the space (1.1) we define operators

$$b_k := \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes a \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \ , \ b_k^* := \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes a^* \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \ , \tag{1.2}$$

where operator a, or a^*, is the $(k+1)$th factor in (1.2). Operators (1.2) formally satisfy the Canonical Commutation Relations (CCR)

$$[b_k, b_{k'}^*] = \delta_{k,k'}\mathbb{1}, \quad [b_k, b_{k'}] = [b_k^*, b_{k'}^*] = 0 \ , \ k, k' = 0, 1, 2, \cdots, N. \tag{1.3}$$

(H3) *Harmonic interaction.* The time-dependent repeated interaction described by (H2) is a piecewise *constant* operator in (1.1). It is the sum over $n \geq 1$ of the *bilinear* forms in operators (1.2) in the space $\mathscr{H}_0 \otimes \mathscr{H}_n$:

$$K_n(t) := \chi_{[(n-1)\tau, n\tau)}(t)\, \eta\ (b_0^* b_n + \ b_n^* b_0) \ , \qquad \eta > 0. \tag{1.4}$$

Here $\chi_{\mathcal{I}}(x)$ is the characteristic function of the set $\mathcal{I}$.

For any $N \geq 1$ and $t < N\tau$, the self-adjoint Hamiltonian $H_N(t)$ of the non-autonomous system $\mathcal{S} + \mathcal{C}_N$ is defined in the space (1.1) as the sum of Hamiltonians corresponding the systems $\mathcal{S}$, $\mathcal{S}_k$ and interaction (1.4) [TZ]:

$$\begin{aligned} H_N(t) &:= H_{\mathcal{S}} + \sum_{k=1}^{N} (H_{\mathcal{S}_k} + K_k(t)) \qquad\qquad (1.5) \\ &= E b_0^* b_0 + \epsilon \sum_{k=1}^{N} b_k^* b_k + \eta \sum_{k=1}^{N} \chi_{[(k-1)\tau, k\tau)}(t)\ (b_0^* b_k + \ b_k^* b_0) \, , \end{aligned}$$

(H4) *Semi-boundedness.* To keep the self-adjoint Hamiltonian (1.5) semi-bounded from *below* we suppose that $E, \epsilon > 0$ and we impose the condition $\eta^2 \leq E\,\epsilon$.

By virtue of (1.4), (1.5) only $\mathcal{S}_n$ interacts with $\mathcal{S}$ for $t \in [(n-1)\tau, n\tau)$, $n \geq 1$, i.e. the system $\mathcal{S} + \mathcal{C}_N$ is *autonomous* on this time-interval with self-adjoint Hamiltonian

$$H_n := E\, b_0^* b_0 + \epsilon \sum_{k=1}^{N} b_k^* b_k + \eta\, (b_0^* b_n + \; b_n^* b_0) \ , \ n \leq N \ . \tag{1.6}$$

The key for the exact solution lemma follows from the harmonic structure of (1.6).

Lemma 1.1 *For $j = 0, 1, 2, \ldots, N$ and $n = 1, 2, \ldots, N$, one gets*

$$e^{itH_n} b_j e^{-itH_n} = \sum_{k=0}^{N} (U_n^*(t))_{jk} b_k, \quad e^{itH_n} b_j^* e^{-itH_n} = \sum_{k=0}^{N} \overline{(U_n^*(t))_{jk}} b_k^*, \tag{1.7}$$

$$e^{-itH_n} b_j e^{itH_n} = \sum_{k=0}^{N} (U_n(t))_{jk} b_k, \quad e^{-itH_n} b_j^* e^{itH_n} = \sum_{k=0}^{N} \overline{(U_n(t))_{jk}} b_k^* \ , \tag{1.8}$$

for $t \geq 0$. Here $U_n(t)$ and $V_n(t)$ are $(N+1)\times(N+1)$ matrices related by $U_n(t) := e^{it\epsilon} V_n(t)$, where

$$(V_n(t))_{jk} := \begin{cases} g(t)z(t)\, \delta_{k0} + g(t)w(t)\, \delta_{kn} & (j = 0) \\ g(t)w(t)\, \delta_{k0} + g(t)z(-t)\, \delta_{kn} & (j = n) \\ \qquad \delta_{jk} & (\textit{otherwise}) \ , \end{cases} \tag{1.9}$$

and

$$g(t) := e^{it(E-\epsilon)/2} \ , \ w(t) := \frac{2i\eta}{\sqrt{(E-\epsilon)^2 + 4\eta^2}} \sin t \sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2} \ , \tag{1.10}$$

$$z(t) := \cos t \sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2} + \frac{i(E-\epsilon)}{\sqrt{(E-\epsilon)^2 + 4\eta^2}} \sin t \sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2} \ . \tag{1.11}$$

Remark 1.2 *Note that by definitions (1.10) and (1.11), we get $|z(t)|^2 + |w(t)|^2 = 1$, $z(-t) = \overline{z(t)}$ and $w(t) = -\overline{w(t)}$. Therefore, the matrix*

$$M(t) := \begin{pmatrix} z(t) & w(t) \\ \\ w(t) & z(-t) \end{pmatrix}$$

is unitary. For $N = 1$, one gets $M(t) = \overline{g(t)} V_1(t)$, see (1.9). Moreover, (1.7) and (1.8) imply that $\{V_n(t)\}_{t\in\mathbb{R}}$ and $\{U_n(t)\}_{t\in\mathbb{R}}$ are in fact one-parameter groups of $(N+1)\times(N+1)$ unitary matrices.

Proof (of Lemma 1.1): Let $\{J_n\}_{n=1}^N$ and $\{X_n\}_{n=1}^N$ be $(N+1)\times(N+1)$ Hermitian matrices given by

$$(J_n)_{jk} := \begin{cases} 1 & (j = k = 0 \ \text{ or } \ j = k = n) \\ 0 & \text{otherwise} \end{cases} , \tag{1.12}$$

$$(X_n)_{jk} := \begin{cases} (E-\epsilon)/2 & (j,k)=(0,0) \\ -(E-\epsilon)/2 & (j,k)=(n,n) \\ \eta & (j,k)=(0,n) \\ \eta & (j,k)=(n,0) \\ 0 & \text{otherwise} \end{cases} . \tag{1.13}$$

We define the matrices

$$Y_n := \epsilon I + \frac{E-\epsilon}{2} J_n + X_n \quad (n=1,\ldots,N) , \tag{1.14}$$

where I is the $(N+1)\times(N+1)$ identity matrix. Then Hamiltonian (1.6) takes the form

$$H_n = \sum_{j,k=0}^{N} (Y_n)_{jk} b_j^* b_k . \tag{1.15}$$

Since Y_n is Hermitian, there exists a *diagonal* matrix Λ and unitary mapping $\mathcal{U}_n : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$, such that $Y_n = \mathcal{U}_n^* \Lambda \mathcal{U}_n$ holds. After canonical transformation $\mathcal{U}_n$ the matrix $\Lambda := \{\Lambda_{ij}\}_{i,j=0}^N = \{\delta_{ij}\,\varepsilon_j\}_{i,j=0}^N$ is universal and independent of n. The new operators:

$$c_j = \sum_{k=0}^{N} (\mathcal{U}_n)_{jk}\; b_k, \quad c_j^* = \sum_{k=0}^{N} \overline{(\mathcal{U}_n)_{jk}}\; b_k^* \quad (j=0,1,\ldots,N) , \tag{1.16}$$

satisfy CCR in the space $\mathscr{H}^{(N)}$ (1.1) and diagonalise (1.15): $\widetilde{H}_n = \sum_{j=0}^N \Lambda_{jj} c_j^* c_j$, where $\Lambda_{jj} = \varepsilon_j$. Therefore, the set of all eigenvectors of $\widetilde{H}_n$ is

$$\left\{ \prod_{j=0}^{N} \frac{(c_j^*)^{n_j}}{\sqrt{n_j!}}\, \Omega \otimes \ldots \otimes \Omega \,\middle|\, n_j \in \mathbb{Z}_+ \quad (j=0,1,\ldots,N) \right\} . \tag{1.17}$$

Note that it forms a complete orthonormal basis in $\mathscr{H}^{(N)}$. The linear envelope $\mathscr{H}_0^{(N)}$ of the set (1.17) is invariant subspace for transformations $e^{it\widetilde{H}_n}$ and its norm-closure coincides with $\mathscr{H}^{(N)}$. Then by (1.16) one gets on vectors (1.17):

$$e^{it\widetilde{H}_n} c_j e^{-it\widetilde{H}_n} = e^{-it\Lambda_{jj}} c_j, \quad e^{it\widetilde{H}_n} c_j^* e^{-it\widetilde{H}_n} = e^{it\Lambda_{jj}} c_j^* .$$

Now taking into account canonical transformation (1.16), we obtain

$$e^{itH_n} b_j e^{-itH_n} = \sum_{k=0}^{N} (\mathcal{U}_n^*)_{jk}\, e^{it\widetilde{H}_n} c_k e^{-it\widetilde{H}_n}$$

$$= \sum_{k,l=0}^{N} (\mathcal{U}_n^*)_{jk} e^{-it\Lambda_{kk}} (\mathcal{U}_n)_{kl} b_l = \sum_{l=0}^{N} \left(e^{-it\mathcal{U}_n^*\Lambda\mathcal{U}_n}\right)_{jl} b_l = \sum_{l=0}^{N} \left(e^{-itY_n}\right)_{jl} b_l . \tag{1.18}$$

Similarly we obtain $e^{itH_n} b_j^* e^{-itH_n} = \sum_{l=0}^N \overline{\left(e^{-itY_n}\right)_{jl}} b_l^*$.

Note that by virtue of (1.12), (1.13), one has identities

$$X_n^2 = \Big(\frac{(E-\epsilon)^2}{4} + \eta^2\Big) J_n \quad \text{and} \quad J_n X_n = X_n \ .$$

Together with definition (1.14) and (1.9), they yield

$$e^{itY_n} = e^{it\epsilon}\Bigg(I - J_n + e^{it(E-\epsilon)/2}\Big\{J_n \cos t\sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2} \tag{1.19}$$

$$+iX_n\left[\frac{(E-\epsilon)^2}{4} + \eta^2\right]^{-1/2} \sin t\sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2}\ \Big\}\Bigg) = e^{it\epsilon}V_n(t) = U_n(t).$$

Inserting now (1.19) into (1.18), we prove (1.7). Since $U_n(t)^* = U_n(-t)$, one can similarly establish (1.8). $\square$

Remark 1.3 *Hereafter, we are going to use the short-hand notations:*

$$g := g(\tau),\ w := w(\tau),\ z := z(\tau) \ \text{ and } \ V_n := V_n(\tau),\ U_n := U_n(\tau) \ . \tag{1.20}$$

In Section 2, we give explicit description of the Hamiltonian dynamics for the non-autonomous system $\mathcal{S} + \mathcal{C}$ driven by harmonic repeated interactions (H3). We show that our model of bosons (1.5) is a quasi-free W^*-dynamical system. In Section 3 we recall formulae for the entropy of the CCR quasi-free states. We use them in Section 4 for calculations of the entropy production. Section 5 is dedicated to analysis of reduced dynamics of subsystems, of their correlations and of convergence to equilibrium. We prove a universality of the short-time interaction limit of this dynamics for the subsystem $\mathcal{S}$.

2 Hamiltonian Dynamics

A well-known way to avoid the problem of evolution of unbounded creation-annihilation operators is to construct dynamics of the subsystem $\mathcal{S}$ on the unital Weyl CCR C^*-algebra $\mathscr{A}(\mathscr{F})$, see e.g. [AJP1] (Lectures 4 and 5), [BR2]. Here $\mathscr{A}(\mathscr{F})$ is generated on the Fock space $\mathscr{F}$ as the operator-norm closure of the linear span $\mathscr{A}_w$ of the Weyl operator system:

$$\{\widehat{w}(\alpha) = e^{i\Phi(\alpha)/\sqrt{2}}\}_{\alpha\in\mathbb{C}} \ . \tag{2.1}$$

Here $\Phi(\alpha) := \bar{\alpha}a + \alpha a^*$ is a self-adjoint operator with domain in $\mathscr{F}$ and the CCR take then the Weyl form:

$$\widehat{w}(\alpha_1)\widehat{w}(\alpha_2) = e^{-i\,\mathrm{Im}(\bar{\alpha}_1\alpha_2)/2}\ \widehat{w}(\alpha_1+\alpha_2) \ , \quad \alpha_1, \alpha_2 \in \mathbb{C} \ . \tag{2.2}$$

Note that $\mathscr{A}(\mathscr{F})$ is a minimal C^*-algebra, which contains the linear span $\mathscr{A}_w$ of the Weyl operator system (2.1). Algebra $\mathscr{A}(\mathscr{F})$ is contained in the unital C^*-algebra $\mathcal{L}(\mathscr{F})$ of all bounded operators on $\mathscr{F}$.

Similarly we define the Weyl C^*-algebra $\mathscr{A}(\mathscr{H}) \subset \mathcal{L}(\mathscr{H})$ over $\mathscr{H} := \mathscr{H}^{(N)}$ (1.1). It is appropriate for description the system $\mathcal{S} + \mathcal{C}$. This algebra is generated by operators

$$W(\zeta) = \bigotimes_{k=0}^{N} \widehat{w}(\zeta_k), \qquad \zeta = \{\zeta_k\}_{k=0}^{N} \in \mathbb{C}^{N+1}\ ,\ N \geq 1. \tag{2.3}$$

Using definitions of the boson operators $\{b_k, b_k^*\}_{k=1}^{N}$ and of the sesquilinear forms

$$\langle \zeta, b \rangle := \sum_{j=0}^{N} \bar{\zeta}_j b_j, \qquad \langle b, \zeta \rangle := \sum_{j=0}^{N} \zeta_j b_j^*\ , \tag{2.4}$$

the Weyl operators (2.3) can be rewritten as

$$W(\zeta) = \exp[i\big(\langle \zeta, b \rangle + \langle b, \zeta \rangle\big)/\sqrt{2}]\ . \tag{2.5}$$

We denote by $\mathfrak{C}_1(\mathscr{F}) \subset \mathcal{L}(\mathscr{F})$, the set of all trace-class operators on $\mathscr{F}$. A self-adjoint, non-negative operator $\rho \in \mathfrak{C}_1(\mathscr{F})$ with *unit* trace is called *density matrix*. The state $\omega_\rho(\cdot)$ generated on the C^*-algebra of bounded operators $\mathcal{L}(\mathscr{F})$ by ρ :

$$\omega_\rho(A) := \mathrm{Tr}_{\mathscr{F}}(\rho\, A)\ , \quad A \in \mathcal{L}(\mathscr{F})\ , \tag{2.6}$$

is a *normal* state. Let $\{\rho_k\}_{k=0}^{N}$ be density matrices on $\mathscr{F}$. Then the normal *product-state* on the C^*-algebra $\mathscr{A}(\mathscr{H})$ (isometrically isomorphic to the tensor product $\otimes_{k=0}^{N}\mathscr{A}(\mathscr{F})$) is

$$\omega_{\rho^\otimes}(\cdot) := \mathrm{Tr}_{\mathscr{H}}(\rho^\otimes\ \cdot\ )\ , \quad \rho^\otimes := \otimes_{k=0}^{N}\rho_k\ . \tag{2.7}$$

If we put $C_k(\alpha) := \mathrm{Tr}_{\mathscr{F}}[\rho_k\, \widehat{w}(\alpha)]$, $\alpha \in \mathbb{C}$, then by (2.3) one obtains for $\rho^\otimes$ (2.7) the representation:

$$\omega_{\rho^\otimes}(W(\zeta)) := \mathrm{Tr}_{\mathscr{H}}[\rho^\otimes\, W(\zeta)] = \prod_{k=0}^{N} C_k(\zeta_k)\ . \tag{2.8}$$

Let $\varrho \in \mathfrak{C}_1(\mathscr{H})$ be a density matrix on $\mathscr{H}$. Then for the system $\mathcal{S}+\mathcal{C}$, the Hamiltonian evolution $T_t : \varrho \mapsto \varrho(t)$ of initial density matrix $\varrho(0) := \varrho$ is defined as a solution of the Cauchy problem for the *non-autonomous* Liouville equation

$$\partial_t \varrho(t) = L(t)(\varrho(t))\ ,\ \varrho(t)\big|_{t=0} = \varrho\ . \tag{2.9}$$

By virtue of (1.6) the equation (2.9) is *autonomous* for each of the interval $[(n-1)\tau, n\tau)$:

$$L(t)(\cdot) = L_n(\cdot) = -i[H_n\, ,\, \cdot\,]\ , \quad t \in [(n-1)\tau, n\tau)\ ,\ n \geqslant 1\ . \tag{2.10}$$

Since any $t \geqslant 0$ has the representation:

$$t := n(t)\tau + \nu(t)\ ,\ n(t) := [t/\tau] \ \text{ and }\ \nu(t) \in [0, \tau)\ , \tag{2.11}$$

by the Markovian independence of generators (2.10), the trace-norm ($\|\cdot\|_1$)-continuous solution of the Cauchy problem (2.9) takes the iterative form:

$$\begin{aligned} &\varrho(t) = T_t(\varrho) := T_{\nu(t),n}(T_{\tau,n-1}(\ldots T_{\tau,1}(\varrho)\ldots)) = \\ &e^{-i\nu(t)H_n} e^{-i\tau H_{n-1}} \ldots e^{-i\tau H_1}\varrho\, e^{i\tau H_1} \ldots e^{i\tau H_{n-1}} e^{i\nu(t)H_n}\ . \end{aligned} \tag{2.12}$$

Here $t \in [(n-1)\tau, n\tau)$, $n = n(t) < N$. By the $\|\cdot\|_1$-continuity we obtain from (2.12) that

$$\varrho(N\tau - 0) = \varrho(N\tau) = T_{N\tau}(\varrho) = e^{-i\tau H_N} \dots e^{-i\tau H_1} \varrho\, e^{i\tau H_1} \dots e^{i\tau H_N} . \tag{2.13}$$

Note that equivalent and often more convenient description of evolution of the systems $\mathcal{S} + \mathcal{C}$ is the *dual* dynamics $T_t^* : \mathfrak{L}(\mathscr{H}) \to \mathfrak{L}(\mathscr{H})$:

$$\omega_{\varrho(t)}(A) = \mathrm{Tr}_{\mathscr{H}}(T_t(\varrho)\, A) =: \mathrm{Tr}_{\mathscr{H}}(\varrho\, T_t^*(A)), \text{ for } (\varrho, A) \in \mathfrak{C}_1(\mathscr{F}) \times \mathfrak{L}(\mathscr{H}). \tag{2.14}$$

Remark 2.1 *Below we show that T_t^* maps $\mathscr{A}(\mathscr{H})$ into itself, and that the action of T_t^* on Weyl operators can be calculated in the explicit form. Since $\mathscr{A}(\mathscr{H})$ is weak*-dense in $\mathfrak{L}(\mathscr{H})$, these allow to deduce properties of evolution $\rho(t)$, see [AJP1] (Lectures 2 and 4).*

Using (2.13) and dual representation (2.14), we obtain the main result of this section.

Proposition 2.2 *For $t = N\tau$, the expectation (2.8) of the Weyl operator (2.5) with respect to the evolved state has the form*

$$\omega_{\rho(N\tau)}(W(\zeta)) = \omega_\rho(W(U_1 \dots U_N\ \zeta)) = \prod_{k=0}^{N} C_k((U_1 \dots U_N\ \zeta)_k) . \tag{2.15}$$

Here

$$(U_1 \dots U_N\ \zeta)_0 = e^{iN\tau\epsilon}\big((gz)^N \zeta_0 + \sum_{j=1}^{N} gw(gz)^{j-1}\ \zeta_j\big) , \tag{2.16}$$

whereas

$$(U_1 \dots U_N\ \zeta)_k = e^{iN\tau\epsilon}\big(gw(gz)^{N-k}\zeta_0 + g\bar{z}\zeta_k + \sum_{j=k+1}^{N} g^2 w^2 (gz)^{j-k-1}\zeta_j\big) , \tag{2.17}$$

for $0 < k < N$, and

$$(U_1 \dots U_N\ \zeta)_N = e^{iN\tau\epsilon}\big(gw\zeta_0 + g\bar{z}\zeta_N\big) , \tag{2.18}$$

see definitions (1.10) and (1.11).

Proof : Note that (2.8), (2.13) and duality (2.14) yield

$$\omega_{\rho(N\tau)}(W(\zeta)) = \mathrm{Tr}_{\mathscr{H}}[\rho\ T_{N\tau}^*(W(\zeta))] = \mathrm{Tr}_{\mathscr{H}}[\rho\ e^{i\tau H_1} \dots e^{i\tau H_N} W(\zeta) e^{-i\tau H_N} \dots e^{-i\tau H_1}]$$

$$= \mathrm{Tr}_{\mathscr{H}}[\rho\ W(U_1 \dots U_N\ \zeta)] = \prod_{k=0}^{N} C_k((U_1 \dots U_N\ \zeta)_k). \tag{2.19}$$

To generate the mapping $\zeta \mapsto U_1 \dots U_N\, \zeta$ in (2.19), we use Lemma 1.1 and sesquilinear forms (2.4) to obtain

$$e^{i\tau H_1} \dots e^{i\tau H_N} \langle \zeta, b \rangle e^{-i\tau H_N} \dots e^{-i\tau H_1} = \langle \zeta, U_N^* \dots U_1^* b \rangle = \langle U_1 \dots U_N\ \zeta, b \rangle\ , \tag{2.20}$$

and the similar expression for its conjugate, which we then insert into (2.5).

Moreover, by the same Lemma 1.1, we get that $U_1 \ldots U_N\, \zeta = e^{iN\tau\epsilon} V_1 \ldots V_N\, \zeta$, where

$$(V_1 \ldots V_N)_{0j} = \begin{cases} (V_1)_{00} \ldots (V_N)_{00} = (gz)^N & (j = 0) \\ (V_1)_{00} \ldots (V_{j-1})_{00} (V_j)_{0j} (V_{j+1})_{jj} \ldots (V_N)_{jj} = (gz)^{j-1} gw & (0 < j \leqslant N), \end{cases}$$

and for $0 < k \leqslant N$:

$$(V_1 \ldots V_N)_{kj} = \begin{cases} (V_1 \ldots V_{k-1})_{kk} (V_k)_{k0} (V_{k+1} \ldots V_N)_{00} = gw(gz)^{N-k} & (j = 0) \\ 0 & (0 < j < k) \\ (V_1 \ldots V_{k-1})_{kk} (V_k)_{kk} (V_{k+1} \ldots V_N)_{kk} = g\bar{z} & (j = k) \\ (V_1 \ldots V_{k-1})_{kk} (V_k)_{k0} (V_{k+1} \ldots V_{j-1})_{00} (V_j)_{0j} (V_{j+1} \ldots V_N)_{jj} & \\ \qquad = gw(gz)^{j-k-1} gw & (k < j \leqslant N). \end{cases}$$

Collecting these formulae, one obtains explicit expressions for components (2.16) and (2.17) of the vector $U_1 \ldots U_N\, \zeta$. □

Remark 2.3 *Note that for a fixed N and for any $t = m\tau$, $1 \leq m \leq N$, the arguments of Lemma 2.2 give a general formula*

$$\omega_{\rho(m\tau)}(W(\zeta)) = \omega_\rho(T^*_{m\tau}(W(\zeta))) = \omega_\rho(W(U_1 \ldots U_m\ \zeta)) = \prod_{k=0}^{N} C_k((U_1 \ldots U_m\ \zeta)_k)\ . \quad (2.21)$$

Following the same line of reasoning as for (2.17) one obtains explicit formulae for the components $\{(U_1 \ldots U_m\ \zeta)_k\}_{k=0}^N$:

$$(U_1 \ldots U_m\ \zeta)_k =$$
$$\begin{cases} e^{im\tau\epsilon}\big((gz)^m \zeta_0 + \sum_{j=1}^m gw(gz)^{j-1}\ \zeta_j\big) & (k = 0) \\ e^{im\tau\epsilon}\big(gw(gz)^{m-k}\zeta_0 + g\bar{z}\zeta_k + \sum_{j=k+1}^m g^2 w^2 (gz)^{j-k-1}\zeta_j\big) & (1 \leqslant k < m) \\ e^{im\tau\epsilon}\big(gw\zeta_0 + g\bar{z}\zeta_m\big) & (k = m) \\ e^{im\tau\epsilon}\ \zeta_k & (m < k \leqslant N) \end{cases}$$

Note that for $m = N$, these formulae coincide with (2.16)-(2.18), except the last line, which is void in this case.

Recall that unity preserving $*$-dynamics $t \mapsto T^*_t$ on the von Neumann algebra $\mathfrak{M}(\mathscr{H})$ generated by $\{W(\zeta)\}_{\zeta \in \mathbb{C}}$ (2.5) is *quasi-free*, if there exist a mapping $U_t : \zeta \mapsto U_t \zeta$ and a complex-valued function $\Omega_t : \zeta \mapsto \Omega_t(\zeta)$, such that

$$T^*_t(W(\zeta)) = \Omega_t(\zeta) W(U_t \zeta)\ ,\ \Omega_0 = 1\ ,\ U_0 = I\ , \quad (2.22)$$

see e.g. [DVV], [AJP1] (Lecture 4) or [BR2]. Then by Remark 2.3, the stepwise dynamics

$$T^*_{m\tau}(W(\zeta)) = W(U_1 \ldots U_m\ \zeta)\ , \qquad m = 0, 1, \ldots, N$$

is quasi-free, with $\Omega_t(\zeta) = 1$ and the matrices $\{U_j\}_{j=1}^N$ on $\mathbb{C}^{N+1}$ defined by Lemma 1.1.

3 Entropy of Quasi-Free States on CCR C^*-Algebras

In this section, we establish some useful formulae relating expectations of the Weyl operators (Weyl characteristic function) and the entropy of boson quasi-free states. For the reader convenience we formulate them in a way which is restricted but sufficient for our purposes. For general settings one can consult [Fa], [AJP1], [BR2], [Ve] and references therein.

Definition 3.1 *A state ω on the CCR C^*-algebra $\mathscr{A}(\mathscr{F})$ (2.1) is called quasi-free, if its characteristic function has the form*

$$\omega(\widehat{w}(\alpha)) := e^{-\frac{1}{4}|\alpha|^2-\frac{1}{2}h(\alpha)} \ , \ \alpha \in \mathbb{C} \ , \tag{3.1}$$

where $h : \alpha \mapsto \widehat{h}(\alpha, \alpha)$ is a (closable) non-negative sesquilinear form on $\mathbb{C}\times\mathbb{C}$. A quasi-free state ω is gauge-invariant if $\omega(\widehat{w}(\alpha)) = \omega(\widehat{w}(e^{i\varphi}\alpha))$ for $\varphi \in [0, 2\pi)$.

Let ω_β denote the Gibbs state with parameter β (dimensionless inverse temperature) given by the density matrix $\rho(\beta) = e^{-\beta a^*a}/Z(\beta)$, where $Z(\beta) = (1 - e^{-\beta})^{-1}$. Since

$$\omega_\beta(\widehat{w}(\alpha)) = e^{-\frac{1}{4}|\alpha|^2-\frac{1}{2}h_\beta(\alpha)} \ , \quad h_\beta(\alpha) = \frac{|\alpha|^2}{e^\beta - 1} \ , \ \alpha \in \mathbb{C} \ , \tag{3.2}$$

this state is quasi-free and gauge-invariant. Note that the entropy of ω_β is given by

$$s(\beta) := -\mathrm{Tr}_{\mathscr{F}}[\rho(\beta)\ln\rho(\beta)] = \beta\omega_\beta(a^*a) - \ln(1 - e^{-\beta}) \ \text{ and } \ \omega_\beta(a^*a) = \frac{1}{e^\beta - 1} \ . \tag{3.3}$$

In terms of the variable $x := (1 + e^{-\beta})/(1 - e^{-\beta})$ the entropy (3.3) is

$$s(\beta) = \sigma(x) := \frac{x+1}{2}\ln\frac{x+1}{2} - \frac{x-1}{2}\ln\frac{x-1}{2} \ . \tag{3.4}$$

Here $\sigma : (1, \infty) \to (0, \infty)$ and $\sigma'(x) > 0$.

To extend (3.4) to the space (1.1) we note that a general *gauge-invariant* quasi-free states on the CCR C^*-algebra $\mathscr{A}(\mathscr{H})$ are defined by density matrices of the form [Ve]:

$$\rho_L = \frac{1}{Z_L}\, e^{-\langle b, Lb\rangle} \ , \ Z_L = \det\big[I - e^{-L}\big]^{-1} \ . \tag{3.5}$$

Here sesquilinear operator-valued forms $\langle b, Lb\rangle = \sum_{n,m=0}^N \ell_{nm} b_n^* b_m$ are parameterised by $(N + 1) \times (N + 1)$ *positive-definite* Hermitian matrix $L = \{\ell_{nm}\}_{0\leqslant n,m\leqslant N}$. Note that the $*$-automorphism G_φ on $\mathscr{A}(\mathscr{H})$ (the gauge transformation) :

$$G_\varphi : b_n^* \mapsto b_n^* e^{i\varphi}, \ b_m \mapsto b_m e^{-i\varphi} \qquad (\varphi \in \mathbb{R} \ , \ n, m = 0, 1, \ldots N) \ , \tag{3.6}$$

leaves the state (3.5) invariant. Then characteristic function of the Weyl operators $W(\zeta)$ takes the form

$$\omega_{\rho_L}(W(\zeta)) = \mathrm{Tr}_{\mathscr{H}}[\rho_L W(\zeta)] = \exp\big[-\frac{1}{4}\langle\zeta,\zeta\rangle - \frac{1}{2}\langle\zeta, \frac{I}{e^L - I}\,\zeta\rangle\big] \ . \tag{3.7}$$

Here $\zeta = (\zeta^{\rm tr})^{\rm tr}$, where *transposition* of this vector is equal to $\zeta^{\rm tr} := (\zeta_0, \zeta_1, \dots \zeta_N) \in \mathbb{C}^{N+1}$. Note that the entropy of the state ω_{ρ_L} is given by

$$S(\rho_L) = -{\rm Tr}_{\mathscr{H}}[\rho_L \ln \rho_L] = {\rm tr}_{\mathbb{C}^{N+1}}[L(e^L - I)^{-1} - \ln(I - e^{-L})] \ . \tag{3.8}$$

If we define the matrix $X := (I + e^{-L})(I - e^{-L})^{-1}$, then the characteristic function (3.7) takes the form:

$$\omega_{\rho_L}(W(\zeta)) = \exp\big[-\frac{1}{4}\langle \zeta, X\zeta\rangle\big] \ , \tag{3.9}$$

and for the entropy (3.8) we obtain

$$S(\rho_L) = {\rm tr}\Big[\frac{X+I}{2}\ln\frac{X+I}{2} - \frac{X-I}{2}\ln\frac{X-I}{2}\Big]. \tag{3.10}$$

Below we need a bit more specified set up than (3.9), (3.10). Let $\rho(\beta,\delta;\xi)$ be density matrix of a quasi-free state (3.5) corresponding to the operator-valued sesquilinear form

$$\langle b, L(\beta,\delta;\xi)b\rangle := \beta\sum_{n=0}^{N} b_n^* b_n + \delta\langle b,\xi\rangle\langle\xi, b\rangle. \tag{3.11}$$

on $\mathbb{C}^{N+1}\times\mathbb{C}^{N+1}$. Here $\beta > 0,\ \delta > -\beta$, and the vector $\xi^{\rm tr} = (\xi_0, \xi_1, \dots, \xi_N) \in \mathbb{C}^{N+1}$.

Lemma 3.2 *The partition function of the state*

$$\rho(\beta,\delta;\xi) = \frac{1}{Z(\beta,\delta;\xi)}\exp\big[-\langle b, L(\beta,\delta;\xi)b\rangle\big] \ ,$$

is given by

$$Z(\beta,\delta;\xi) = {\rm Tr}_{\mathscr{H}}[e^{-\langle b, L(\beta,\delta;\xi)b\rangle}] = (1-e^{-\beta})^{-N}(1-e^{-(\beta+\delta\langle\xi,\xi\rangle)})^{-1} \ . \tag{3.12}$$

The characteristic function and the entropy of this state are respectively:

$${\rm Tr}_{\mathscr{H}}[\rho(\beta,\delta;\xi)W(\zeta)] = \exp\big[-\frac{1}{4}\,\frac{1+e^{-\beta}}{1-e^{-\beta}}\,\langle\zeta,\zeta\rangle\big]$$

$$\times\exp\big[-\frac{1}{4}\Big(\frac{1+e^{-\beta-\delta\langle\xi,\xi\rangle}}{1-e^{-\beta-\delta\langle\xi,\xi\rangle}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big)|\langle\xi,\zeta\rangle|^2/\langle\xi,\xi\rangle\big] \ , \tag{3.13}$$

and

$$S(\rho(\beta,\delta;\xi)) = -{\rm Tr}_{\mathscr{H}}[\rho(\beta,\delta;\xi)\ln\rho(\beta,\delta;\xi)] = N s(\beta) + \ s(\beta+\delta\langle\xi,\xi\rangle) \ . \tag{3.14}$$

Proof : Proof of (3.12) follows from (3.5) and (3.11). Indeed, since by (3.5) any orthogonal transformation $\mathcal{O}$ on $\mathbb{C}^{N+1}$ leaves the partition function invariant: $Z_{\mathcal{O}^T L\mathcal{O}} = Z_L$, one can calculate it with $O\xi$ (instead of ξ), where $\mathcal{O}\xi$ has only one non-zero component equals to the vector norm $\langle\xi,\xi\rangle^{1/2}$. Then the right-hand side of (3.12) follows straightforwardly from the calculation of the left-hand side for this choice of $\mathcal{O}\xi$.

Since this transformation $\mathcal{O}$ also diagonalise the matrix $L := L(\beta, \delta; \xi)$, one uses it to simplify (3.9) and then to return back to ξ at the last step. To this aim we note that

$$\omega_{\rho_L}(W(\zeta)) = \exp\big[-\frac{1}{4}\langle\mathcal{O}\zeta, \mathcal{O}X\mathcal{O}^*\mathcal{O}\zeta\rangle\big] = \tag{3.15}$$
$$\exp\big[-\frac{1}{4}\,\frac{1+e^{-\beta}}{1-e^{-\beta}}\,\langle\mathcal{O}\zeta, \mathcal{O}\zeta\rangle'\big]\exp\big[-\frac{1}{4}\,\frac{1+e^{-\beta-\delta\langle\xi,\xi\rangle}}{1-e^{-\beta-\delta\langle\xi,\xi\rangle}}|(\mathcal{O}\zeta)_0|^2\big]\ .$$

Here $\langle\mathcal{O}\zeta, \mathcal{O}\zeta\rangle' := \sum_{k=1}^{N}|(\mathcal{O}\zeta)_k|^2$ and we choose transformation $\mathcal{O}$ in such a way that $(\mathcal{O}\xi)_j = \delta_{0,j}\|\xi\|$. Since

$$|(\mathcal{O}\zeta)_0|^2 = \frac{1}{\langle\xi,\xi\rangle}\langle\mathcal{O}\zeta, \mathcal{O}\xi\rangle\langle\mathcal{O}\xi, \mathcal{O}\zeta\rangle\ , \tag{3.16}$$

the identities (3.15) prove (3.13). The same method is valid for entropy (3.8). Calculation of the trace in diagonal representation for $L = L(\beta, \delta; \xi)$ gives formula (3.14). $\square$

Recall that the state ω on the CCR C^*-algebra $\mathscr{A}(\mathscr{H})$ is *regular*, if the map $s \mapsto \omega(W(s\,\zeta))$ is a continuous function of $s \in \mathbb{R}$ for any $\zeta \in \mathbb{C}^{N+1}$. This property follows from the explicit expression (3.13). Since by the Araki-Segal theorem, see e.g. [AJP1](Lecture 5), a regular state is completely defined by its characteristic function, (3.13) and (3.14) yield the following statement.

Proposition 3.3 *The entropy $S(\rho)$ of the quasi-free state ω_ρ on the CCR C^*-algebra $\mathscr{A}(\mathscr{H})$ with characteristic function*

$$\omega_\rho(W(\zeta)) = \exp\Big[-\frac{1}{4}\big(x\langle\zeta,\zeta\rangle + x_0|\langle\xi,\zeta\rangle|^2\big)\Big] \tag{3.17}$$

is uniquely determined by the parameters (ξ, x, x_0), where $\xi \in \mathbb{C}^{N+1}$,
$x > 1,\ x_0 > 1 - x$ and it has the form

$$S(\rho) = N\sigma(x) + \sigma(x + x_0\langle\xi,\xi\rangle)\ , \tag{3.18}$$

where $\sigma(\cdot)$ is defined by (3.4).

Proof : The proof follows directly from definition (3.4), if one puts

$$x_0\,\langle\xi,\xi\rangle = \frac{1+e^{-\beta-\delta\langle\xi,\xi\rangle}}{1-e^{-\beta-\delta\langle\xi,\xi\rangle}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\ ,$$

in (3.13) and uses (3.4) in (3.14). $\square$

4 Repeated Perturbations and Entropy Production

We consider evolution (2.12) of the system $\mathcal{S}+\mathcal{C}$, when initial density matrix (2.7) corresponds to the product of gauge-invariant Gibbs quasi-free states with parameter $\beta_0 \geq 0$ for $\mathcal{S}$ and with parameter $\beta \geq 0$ for $\mathcal{C}$:

$$\rho = \rho_0 \otimes \bigotimes_{k=1}^{N}\rho_k\ , \quad \rho_0 = e^{-\beta_0 a^* a}/Z(\beta_0)\ , \quad \rho_k = e^{-\beta a^* a}/Z(\beta)\ ,\ k = 1, 2, \ldots, N\ . \tag{4.1}$$

This case corresponds to ρ_L in (3.5) with diagonal matrix $L = \mathrm{diag}(\beta_0, \beta, \ldots, \beta)$ and to $\rho(\beta, \delta; \xi)$ in representation (3.11) with $(\beta, \delta; \xi) = (\beta, \beta_0 - \beta; e)$, i.e.,

$$\rho = \rho(\beta, \beta_0 - \beta; e) = \exp\big[-\beta_0 b_0^* b_0 - \beta \sum_{j=1}^{N} b_j^* b_j \big] / Z(\beta, \beta_0 - \beta) \ . \tag{4.2}$$

Here $e^{\mathrm{tr}} = (1, 0, \ldots, 0) \in \mathbb{C}^{N+1}$ and

$$Z(\beta, \beta_0 - \beta) = Z(\beta_0) Z(\beta)^N = \frac{1}{(1 - e^{-\beta_0})(1 - e^{-\beta})^N}.$$

A straightforward application of formulae (3.13), (3.14) and Lemma 3.2 for $\xi = e$ (i.e. for $\langle \xi, \xi \rangle = 1$, $\langle \xi, \zeta \rangle = \zeta_0$) to the state (4.1) (or (4.2)), yields the following statement:

Lemma 4.1 *The characteristic function of (4.1) (or (4.2)) is*

$$\begin{aligned} &\omega_\rho(W(\zeta)) = \mathrm{Tr}_{\mathscr{H}}[\rho \ W(\zeta)] = \\ &\exp\Big[- \frac{|\zeta_0|^2}{4} \Big(\frac{1 + e^{-\beta_0}}{1 - e^{-\beta_0}} - \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \Big) - \frac{\langle \zeta, \zeta \rangle}{4} \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \Big] \ , \end{aligned} \tag{4.3}$$

and the entropy is equal to

$$S(\rho) = N s(\beta) + s(\beta_0) \, . \tag{4.4}$$

Lemma 4.2 *Characteristic function of the state with density matrix $\rho(N\tau)$ is equal to*

$$\omega_{\rho(N\tau)}(W(\zeta)) = \exp\Big[- \frac{|(U_1 \ldots U_N \zeta)_0|^2}{4} \Big(\frac{1 + e^{-\beta_0}}{1 - e^{-\beta_0}} - \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \Big) - \frac{\langle \zeta, \zeta \rangle}{4} \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \Big] \ , \tag{4.5}$$

whereas the total entropy rests invariant:

$$S(\rho(N\tau)) = S(\rho) = N s(\beta) + s(\beta_0) \ .$$

Here the mapping $U_1 \ldots U_N : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ is given by (2.16) and (2.17).

Proof : From (2.15), one gets $\omega_{\rho(N\tau)}(W(\zeta)) = \omega_\rho(W(U_1 \ldots U_N \ \zeta))$. Since the mappings $U_j : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$, $j = 1, \ldots, N$ are unitary (Lemma 2.2), (4.3) yields (4.5). Finally, we obtain that the mapping (2.12) leaves the total entropy (4.4) invariant, see (3.3). □

Let ω and ω_0 be two normal states on the Weyl CCR algebra $\mathscr{A}(\mathscr{H})$ with density matrices ϱ and ϱ_0. Following Araki [Ar1] (see also [AJP3], Lectures 1 and 3) we introduce the *relative entropy* of the state ω with respect to ω_0:

$$\mathrm{Ent}(\varrho | \varrho_0) := \mathrm{Tr}_{\mathscr{H}}[\varrho (\ln \varrho - \ln \varrho_0)] \geq 0 \ . \tag{4.6}$$

Proposition 4.3 *The relative entropy of $\omega_{\rho(N\tau)}$ with respect to ω_ρ is*

$$\mathrm{Ent}(\rho(N\tau) | \rho) \ = \ \frac{(\beta_0 - \beta)(e^{\beta_0} - e^\beta)}{(e^{\beta_0} - 1)(e^\beta - 1)} (1 - |z|^{2N}) \ , \tag{4.7}$$

where $z := z(\tau)$ is defined by (1.11) and (1.20).

Proof : The trace cyclicity yields

$$\mathrm{Ent}(\rho(N\tau)|\rho) = \mathrm{Tr}_{\mathscr{H}}[\rho(N\tau)(\ln\rho(N\tau) - \ln\rho)] \tag{4.8}$$

$$= \mathrm{Tr}_{\mathscr{H}}[\rho(\ln\rho - e^{i\tau H_1}\dots e^{i\tau H_N}\ln\rho e^{-i\tau H_N}\dots e^{-i\tau H_1})]$$

$$= \frac{\beta-\beta_0}{Z(\beta,\beta_0-\beta)}\mathrm{Tr}_{\mathscr{H}}\left[e^{-\beta_0 b_0^* b_0 - \beta\sum_{j=1}^N b_j^* b_j}\left(b_0^* b_0 - e^{i\tau H_1}\dots e^{i\tau H_N} b_0^* b_0 e^{-i\tau H_N}\dots e^{-i\tau H_1}\right)\right] .$$

Note that one gets $b_0^* b_0 = \langle b, e\rangle\langle e, b\rangle$ by (2.4). Hence, (2.20) implies

$$e^{i\tau H_1}\cdots e^{i\tau H_N} b_0^* b_0 e^{-i\tau H_N}\cdots e^{-i\tau H_1} = \sum_{k=0}^{N}(U_1\dots U_N\ e)_k b_k^* \ \sum_{k'=0}^{N}\overline{(U_1\dots U_N\ e)}_{k'}\, b_{k'} . \tag{4.9}$$

Note also that the gauge invariance of the state ρ implies the selection rule:

$$\frac{1}{Z(\beta,\beta_0-\beta)}\mathrm{Tr}_{\mathscr{H}}\left[e^{-\beta_0 b_0^* b_0 - \beta\sum_{j=1}^N b_j^* b_j}\ b_k^* b_{k'}\ \right] = 0 \ \text{ for } \ k\neq k' \ . \tag{4.10}$$

By this rule after injection of (4.9) into (4.8) only diagonal terms with $k = k'$ survive in the expectation:

$$\mathrm{Ent}(\rho(N\tau)|\rho) =$$
$$\frac{\beta-\beta_0}{Z(\beta,\beta_0-\beta)}\mathrm{Tr}_{\mathscr{H}}\left[e^{-\beta_0 b_0^* b_0 - \beta\sum_{j=1}^N b_j^* b_j}\left(b_0^* b_0 - \sum_{k=0}^{N}|(U_1\dots U_N e)_k|^2 b_k^* b_k\right)\right].$$

Finally, by Lemma 2.2, (2.16), (2.17), and by (3.3), we obtain

$$\mathrm{Ent}(\rho(N\tau)|\rho) =$$
$$\frac{\beta-\beta_0}{Z(\beta,\beta_0-\beta)}\mathrm{Tr}_{\mathscr{H}}\left[e^{-\beta_0 b_0^* b_0 - \beta\sum_{j=1}^N b_j^* b_j}\left((1-|z|^{2N})b_0^* b_0 - \sum_{k=1}^{N}|w|^2|z|^{2N-2k} b_k^* b_k\right)\right]$$
$$= \frac{(\beta_0-\beta)(e^{\beta_0}-e^{\beta})}{(e^{\beta_0}-1)(e^{\beta}-1)}(1-|z|^{2N}) ,$$

that proves (4.7). □

The relative entropy defined by (4.6) is non-negative. In contrast to *invariant* total entropy (Lemma 4.2), the relative entropy (4.7) is a monotonously increasing function of time $t = N\tau$, for $|z| < 1$ (see Lemma 1.1, Remark 1.2). It converges to the limit:

$$\lim_{N\to\infty}\mathrm{Ent}(\rho(N\tau)|\rho) \ = \ (\beta-\beta_0)\left[\frac{1}{e^{\beta_0}-1} - \frac{1}{e^{\beta}-1}\right] \geq 0 , \tag{4.11}$$

which is positive for $\beta_0 \neq \beta$. The limit (4.11) gives asymptotic amount of the entropy production, when one starts with the initial product state corresponding to (4.1) and then consider $N\tau \to \infty$, see [BJM].

5 Evolution of Subsystems

Subsystem $\mathcal{S}$. We start with the simplest subsystem $\mathcal{S}$. Let the initial state of the total system $\mathcal{S}+\mathcal{C}$ in (1.1) be a tensor-product of the corresponding density matrices $\rho = \rho_S \otimes \rho_C$, see (H1). Then for $t \geq 0$ the state $\omega^t_{\mathcal{S}}(\cdot)$ of the subsystem $\mathcal{S}$ is given on the Weyl C^*-algebra $\mathscr{A}(\mathscr{H}_0)$ by

$$\omega^t_{\mathcal{S}}(\cdot) := \omega_{\rho(t)}(\cdot \otimes \mathbb{1}) \ . \tag{5.1}$$

For $\zeta^{\mathrm{tr}} = (\alpha, 0, \ldots, 0) \in \mathbb{C}^{N+1}$, we consider the Weyl operator $W(\zeta) = \widehat{w}(\alpha) \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}$ (2.3). By virtue of (2.8), (2.21) and (5.1), we obtain for $t = m\tau$ ($1 \leq m \leq N$):

$$\omega^{m\tau}_{\mathcal{S}}(\widehat{w}(\alpha)) = \omega_{\rho(m\tau)}(W(\zeta)) = \omega_\rho(W(U_1 \ldots U_m \ \zeta)) \ . \tag{5.2}$$

Then for components $\{(U_1 \ldots U_m \ \zeta)_k\}_{k=0}^N$ of the vector $U_1 \ldots U_m \zeta$ in (5.2), one obtains the expression:

$$(U_1 \ldots U_m \ \zeta)_k = \begin{cases} e^{im\tau\epsilon}(gz)^m \alpha & (k=0) \\ e^{im\tau\epsilon} gw(gz)^{m-k}\alpha & (1 \leqslant k < m) \\ e^{im\tau\epsilon} gw\alpha & (k=m) \\ 0 & (m < k \leqslant N) \ , \end{cases} \tag{5.3}$$

which follows from Remark 2.3.

If the initial density matrices: $\rho = \rho_S \otimes \rho_C$ corresponds to the product of Gibbs quasi-free states for different temperatures as in (4.1), then (5.2) and Lemma 4.1 yield

$$\omega^{m\tau}_{\mathcal{S}}(\widehat{w}(\alpha)) = \exp\Big[-\frac{|\alpha|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}} - \frac{|z^m\alpha|^2}{4}\Big(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big)\Big] \tag{5.4}$$

Note that for any moment $t = m\tau$ the state $\omega^{m\tau}_{\mathcal{S}}(\cdot)$ is a quasi-free Gibbs equilibrium state with parameter $\beta^*(m\tau)$ which satisfies the equation

$$\frac{1+e^{-\beta^*(m\tau)}}{1-e^{-\beta^*(m\tau)}} = |z|^{2m}\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} + (1-|z|^{2m})\frac{1+e^{-\beta}}{1-e^{-\beta}} \ . \tag{5.5}$$

This equation yields that either $\beta \leq \beta^*(m\tau) \leq \beta_0$, or $\beta_0 \leq \beta^*(m\tau) \leq \beta$.

For $m \to \infty$ ($N \to \infty$) the Weyl characteristic function (5.4) has the limit

$$\lim_{m\to\infty} \omega^{m\tau}_{\mathcal{S}}(\widehat{w}(\alpha)) = \exp\Big[-\frac{|\alpha|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] \ . \tag{5.6}$$

Hence, in the limit $t \to \infty$ the subsystem $\mathcal{S}$ evolves from the Gibbs equilibrium state with parameter β_0 to another equilibrium state with parameter β *imposed* by the chain $\mathcal{C}$.

Subsystem $\mathcal{S}_1$. The initial state $\omega^0_{\mathcal{S}_1}(\cdot) = \omega^t_{\mathcal{S}_1}(\cdot)|_{t=0}$ of this subsystem corresponds to a one-point reduced density matrix or to the partial trace on the CCR Weyl algebra $\mathscr{A}(\mathscr{H}_1)$:

$$\omega^0_{\mathcal{S}_1}(\widehat{w}(\alpha)) = \omega_\rho(\mathbb{1} \otimes \widehat{w}(\alpha) \otimes \bigotimes_{k=2}^N \mathbb{1}) = \exp\Big[-\frac{|\alpha|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] \ . \tag{5.7}$$

Now we choose vector $(\zeta^1)^{\mathrm{tr}} := (0, \alpha, 0, \ldots, 0) \in \mathbb{C}^{N+1}$. Then

$$\omega^{m\tau}_{\mathcal{S}_1}(\widehat{w}(\alpha)) = \omega_{\rho(m\tau)}(W(\zeta^{(1)})) = \omega_{\rho_S \otimes \rho_C}(W(U_1 \ldots U_m\ \zeta^{(1)})) \tag{5.8}$$

for $1 < m \leq N$. By Remark 2.3, the components $\{(U_1 \ldots U_m\ \zeta^{(1)})_k\}_{k=0}^N$ are:

$$(U_1 \ldots U_m\ \zeta)_k = \begin{cases} e^{im\tau\epsilon} gw\,\alpha & (k=0) \\ e^{im\tau\epsilon}\,\delta_{k,1}\,g\bar{z}\,\alpha & (1 \leqslant k < m) \\ 0 & (m \leqslant k \leqslant N). \end{cases} \tag{5.9}$$

Then, we obtain

$$\omega^{m\tau}_{\mathcal{S}_1}(\widehat{w}(\alpha)) = \exp\Big[-\frac{|\alpha|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}} - \frac{|w\alpha|^2}{4}\left(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\right)\Big] \tag{5.10}$$

for any $1 < m \leq N$. Therefore, the initial state (5.7) changes to (5.10) after the first act of interaction on the interval $[0, \tau)$ and there is no further evolution of this state for $t > \tau$.

Note that (5.10) is characteristic function of a quasi-free Gibbs equilibrium state with parameter β^*, which satisfies the equation

$$\frac{1+e^{-\beta^*}}{1-e^{-\beta^*}} = |w|^2 \frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} + (1-|w|^2)\frac{1+e^{-\beta}}{1-e^{-\beta}}\ .$$

Again, this equation implies that either $\beta \leq \beta^* \leq \beta_0$, or $\beta_0 \leq \beta^* \leq \beta$.

Evolution of $\mathcal{S}_1$ has a transparent physical interpretation: after the one act of interaction during the time $t \in [0, \tau)$, subsystem $\mathcal{S}_1$ relaxes to an *intermediate* equilibrium with the subsystem $\mathcal{S}$. This manifests in a shift of initial parameter β to β^*, which rests unchangeable since there is no perturbations of subsystem $\mathcal{S}_1$ for $t > \tau$.

Subsystem $\mathcal{S}_m$. For $1 < m \leq N$ the initial state $\omega^0_{\mathcal{S}_m}(\cdot) = \omega^t_{\mathcal{S}_m}(\cdot)|_{t=0}$ of this subsystem is defined on the CCR Weyl algebra $\mathscr{A}(\mathscr{H}_m)$ by the partial trace :

$$\omega^0_{\mathcal{S}_m}(\widehat{w}(\alpha)) = \omega_\rho(\bigotimes_{k=0}^{m-1} \mathbb{1} \otimes \widehat{w}(\alpha) \otimes \bigotimes_{k=m+1}^{N} \mathbb{1}) = \exp\Big[-\frac{|\alpha|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big]\ . \tag{5.11}$$

Now we choose vector $(\zeta^{(m)})^{\mathrm{tr}} := (0, \ldots, 0, \alpha, 0, \ldots, 0) \in \mathbb{C}^{N+1}$, where α occupies the $m+1$ position. Consequently

$$\omega^{m\tau}_{\mathcal{S}_m}(\widehat{w}(\alpha)) = \omega_{\rho(m\tau)}(W(\zeta^{(m)})) = \omega_{\rho_S \otimes \rho_C}(W(U_1 \ldots U_m\ \zeta^{(m)}))\ . \tag{5.12}$$

The components $\{(U_1 \ldots U_m\ \zeta^{(m)})_k\}_{k=0}^N$ are:

$$(U_1 \ldots U_m\ \zeta^{(m)})_k = \begin{cases} e^{im\tau\epsilon}\, gw(gz)^{m-1}\,\alpha & (k=0) \\ e^{im\tau\epsilon}\, g^2 w^2\,(gz)^{m-k-1}\,\alpha & (1 \leqslant k < m) \\ e^{im\tau\epsilon} g\bar{z}\,\alpha\ , & (k=m) \\ 0 & (m < k \leqslant N). \end{cases} \tag{5.13}$$

which again follows from Remark 2.3. Then evolution of the state of subsystem $\mathcal{S}_m$ is:

$$\omega^{m\tau}_{\mathcal{S}_m}(\widehat{w}(\alpha)) = \exp\Big[-\frac{|\alpha|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}} - \frac{|w\alpha|^2}{4}|z|^{2(m-1)}\left(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\right)\Big]. \quad (5.14)$$

Note that interaction for $t \in [(m-1)\tau, m\tau)$ push out the subsystem $\mathcal{S}_m$ from the Gibbs equilibrium state (5.11), but its effect attenuates for large m:

$$\lim_{m\to\infty} \omega^{m\tau}_{\mathcal{S}_m}(\widehat{w}(\alpha)) = \exp\Big[-\frac{|\alpha|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] . \quad (5.15)$$

Again, this is evolution of a quasi-free Gibbs equilibrium state with time-dependent inverse temperature parameter $\beta^{**}(m\tau)$, which satisfies the equation

$$\frac{1+e^{-\beta^{**}(m\tau)}}{1-e^{-\beta^{**}(m\tau)}} = |w|^2|z|^{2(m-1)}\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} + (1-|w|^2|z|^{2(m-1)})\frac{1+e^{-\beta}}{1-e^{-\beta}} . \quad (5.16)$$

As above, the value of the parameter $\beta^{**}(m\tau)$ is always between β_0 and β.

To interpret the evolution of $\mathcal{S}_m$ and the coincidence between (5.15) and (5.6) note that the state of the subsystem $\mathcal{S}$ relaxes to that of initial state of the chain $\mathcal{C}$, see (5.6). Therefore, after interaction of the subsystem $\mathcal{S}_m$, i.e. at the moment $t = m\tau$, its parameter $\beta^{**}(m\tau)$ has a value between β and $\beta^*((m-1)\tau)$ since (5.5) and (5.16) yield

$$\frac{1+e^{-\beta^{**}(m\tau)}}{1-e^{-\beta^{**}(m\tau)}} = |w|^2\frac{1+e^{-\beta^*((m-1)\tau)}}{1-e^{-\beta^*((m-1)\tau)}} + (1-|w|^2)\frac{1+e^{-\beta}}{1-e^{-\beta}} .$$

As in the case $m = 1$, there is no further evolution: $\omega^{n\tau}_{\mathcal{S}_m} = \omega^{m\tau}_{\mathcal{S}_m}$ for $n \geqslant m$.

Next, we consider the composed subsystems $\mathcal{S} + \mathcal{S}_m$ and $\mathcal{S}_{m-n} + \mathcal{S}_m$. Our aim is to study the indirect *correlations* imposed by repeated interaction via $\mathcal{S}$.

Subsystem $\mathcal{S} + \mathcal{S}_m$. For $1 < m \leqslant N$ the initial state $\omega^0_{\mathcal{S}+\mathcal{S}_m}(\cdot) = \omega^t_{\mathcal{S}+\mathcal{S}_m}(\cdot)|_{t=0}$ of this *composed* subsystem is defined by the partial trace on the Weyl C^*-algebra $\mathscr{A}(\mathscr{H}_0\otimes\mathscr{H}_m) \approx \mathscr{A}(\mathscr{H}_0)\otimes\mathscr{A}(\mathscr{H}_m)$ by:

$$\begin{aligned}\omega^0_{\mathcal{S}+\mathcal{S}_m}(\widehat{w}(\alpha_0)\otimes\widehat{w}(\alpha_1)) &:= \omega_\rho(\widehat{w}(\alpha_0)\otimes\bigotimes_{k=1}^{m-1}\mathbb{1}\otimes\widehat{w}(\alpha_1)\otimes\bigotimes_{k=m+1}^{N}\mathbb{1}) \\ &= \exp\Big[-\frac{|\alpha_0|^2}{4}\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}}\Big] \exp\Big[-\frac{|\alpha_1|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] . \quad (5.17)\end{aligned}$$

This is the characteristic function of the product state corresponding to two isolated systems with different temperatures. If one defines vector $(\zeta^{(0,m)})^{\mathrm{tr}} := (\alpha_0, 0, \ldots, 0, \alpha_1, 0, \ldots, 0) \in \mathbb{C}^{N+1}$, where α_1 occupies the $m+1$ position, then

$$\omega^{m\tau}_{\mathcal{S}+\mathcal{S}_m}(\widehat{w}(\alpha_0)\otimes\widehat{w}(\alpha_1)) = \omega_{\rho(m\tau)}(W(\zeta^{(0,m)})) = \omega_{\rho_S\otimes\rho_C}(W(U_1\ldots U_m\,\zeta^{(0,m)})) . \quad (5.18)$$

The components $\{(U_1\ldots U_m\,\zeta^{(0,m)})_k\}_{k=0}^N$ are deduced from Remark 2.3:

$$(U_1\ldots U_m\,\zeta^{(0,m)})_k = \begin{cases} e^{im\tau\epsilon}\,(gz)^{m-1}\,[gz\,\alpha_0 + gw\,\alpha_1], & (k=0) \\ e^{im\tau\epsilon}(gz)^{m-k-1}g^2[wz\,\alpha_0 + w^2\,\alpha_1], & (1\leqslant k<m) \\ e^{im\tau\epsilon}\,[gw\,\alpha_0 + g\overline{z}\,\alpha_1], & (k=m) \\ 0 & (m<k\leqslant N). \end{cases} \quad (5.19)$$

Together with (2.8), one gets for $m \to \infty$:

$$\begin{aligned}
&\omega^{m\tau}_{\mathcal{S}+\mathcal{S}_m}(\widehat{w}(\alpha_0) \otimes \widehat{w}(\alpha_1)) = \exp\Big[-\frac{1}{4}|z\alpha_0 + w\alpha_1|^2 |z|^{2(m-1)} \frac{1+e^{-\beta_0}}{1-e^{-\beta_0}}\Big] \qquad (5.20)\\
&\times \exp\Big[-\frac{1}{4}|z\alpha_0 + w\alpha_1|^2 (1-|z|^{2(m-1)}) \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] \exp\Big[-\frac{1}{4}|w\alpha_0 + \overline{z}\alpha_1|^2 \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big]\\
&\longrightarrow \exp\Big[-\frac{1}{4}(|\alpha_0|^2 + |\alpha_1|^2)\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] \ .
\end{aligned}$$

Hence, in this limit the composed subsystem $\mathcal{S} + \mathcal{S}_m$ evolves from the product of two quasi-free equilibrium states (5.17) with different parameters β_0 and β to the product of quasi-free equilibrium states for the same parameter β imposed by repeated interaction with the chain $\mathcal{C}$, when $m \to \infty$. Interpretation is similar to the case *Subsystem* $\mathcal{S}_m$.

Subsystem $\mathcal{S}_{m-n} + \mathcal{S}_m$. We suppose that $1 < (m-n) < m \leqslant N$. Then the initial state $\omega^t_{\mathcal{S}_{m-n}+\mathcal{S}_m}(\cdot)|_{t=0}$ of this *composed* subsystem is the partial trace over the Weyl C^*-algebra $\mathscr{A}(\mathscr{H}_{m-n} \otimes \mathscr{H}_m) \approx \mathscr{A}(\mathscr{H}_{m-n}) \otimes \mathscr{A}(\mathscr{H}_m)$:

$$\begin{aligned}
&\omega^0_{\mathcal{S}_{m-n}+\mathcal{S}_m}(\widehat{w}(\alpha_1) \otimes \widehat{w}(\alpha_2)) := \qquad (5.21)\\
&\omega_\rho\Big(\bigotimes_{k=0}^{m-n-1} \mathbb{1} \otimes \widehat{w}(\alpha_1) \otimes \bigotimes_{k=m-n+1}^{m-1} \mathbb{1} \otimes \widehat{w}(\alpha_2) \otimes \bigotimes_{k=m+1}^{N} \mathbb{1}\Big) =\\
&= \exp\Big[-\frac{|\alpha_1|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] \exp\Big[-\frac{|\alpha_2|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] \ .
\end{aligned}$$

This is the characteristic function of the product state corresponding to two isolated systems with the same temperatures.

We define the vector $(\zeta^{(m-n,m)})^{\mathrm{tr}} := (0, 0, \ldots, 0, \alpha_1, 0, \ldots, 0, \alpha_2, 0, \ldots, 0) \in \mathbb{C}^{N+1}$. Here α_1 and α_2 occupy respectively the $(m-n+1)$th and the $(m+1)$th positions, then

$$\begin{aligned}
&\omega^{m\tau}_{\mathcal{S}_{m-n}+\mathcal{S}_m}(\widehat{w}(\alpha_1) \otimes \widehat{w}(\alpha_2)) = \qquad (5.22)\\
&\omega_{\rho(m\tau)}(W(\zeta^{(m-n,m)})) = \omega_{\rho_S \otimes \rho_C}(W(U_1 \ldots U_m \ \zeta^{(m-n,m)})) \ .
\end{aligned}$$

By Remark 2.3 we obtain for the values of components $\{(U_1 \ldots U_m \ \zeta^{(m-n,m)})_k\}_{k=0}^N$:

$$(U_1 \ldots U_m \ \zeta^{(m-n,m)})_k = \qquad (5.23)$$

$$= \begin{cases}
e^{im\tau\epsilon} \ (gz)^{m-n-1} \ gw[\alpha_1 + (gz)^n \alpha_2] & (k=0)\\
e^{im\tau\epsilon} \ [g^2 w^2 (gz)^{m-n-k-1} \, \alpha_1 + g^2 w^2 (gz)^{m-k-1} \, \alpha_2] & (1 \leqslant k < m-n)\\
e^{im\tau\epsilon} \ [g\overline{z} \, \alpha_1 + g^2 w^2 \, (gz)^{m-k-1} \, \alpha_2] & (k = m-n)\\
e^{im\tau\epsilon} \ g^2 w^2 \, (gz)^{m-k-1} \, \alpha_2 & (m-n < k < m)\\
e^{im\tau\epsilon} \ g\overline{z} \, \alpha_2 & (k=m)\\
0 & (m < k \leqslant N)
\end{cases} .$$

When $m \to \infty$, then for any fixed n we obtain for (5.22):

$$\omega^{m\tau}_{\mathcal{S}_{m-n}+\mathcal{S}_m}(\widehat{w}(\alpha_1) \otimes \widehat{w}(\alpha_2)) = \exp\Big[-\frac{1}{4}|w|^2|\alpha_1 + (gz)^{n+1}\alpha_2|^2|z|^{2(m-n-1)}\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}}\Big]$$
$$\times \exp\Big[-\frac{1}{4}(\{|w|^2(1-|z|^{2(m-n-1)}) + |z|^2\}|\alpha_1|^2 + (1-|w|^2|z|^{2(m-1)})|\alpha_2|^2)\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big]$$
$$\longrightarrow \exp\Big[-\frac{1}{4}(|\alpha_1|^2 + |\alpha_2|^2)\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] \ . \tag{5.24}$$

Therefore, in this limit, the composed subsystem $\mathcal{S}_{m-n} + \mathcal{S}_m$ evolves from the initial product of two quasi-free equilibrium states (5.21) to the *same* final state, although for a finite m the evolution (5.24) is *nontrivial.* This again easily understandable taking into account our analysis of *Subsystem* $\mathcal{S}_m$ and *Subsystem* $\mathcal{S} + \mathcal{S}_m$.

Consider now the case of a *fixed* $s := m - n \geqslant 1$. Then the limit in (5.24) is

$$\lim_{m\to\infty} \omega^{m\tau}_{\mathcal{S}_s+\mathcal{S}_m}(\widehat{w}(\alpha_1) \otimes \widehat{w}(\alpha_2)) = \tag{5.25}$$
$$\exp\Big[-\frac{1}{4}|w|^2|z|^{2(s-1)}|\alpha_1|^2\left\{\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\right\}\Big] \exp\Big[-\frac{1}{4}(|\alpha_1|^2 + |\alpha_2|^2)\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big]$$
$$= \exp\Big[-\frac{1}{4}|\alpha_1|^2\frac{1+e^{-\beta^{**}(s\tau)}}{1-e^{-\beta^{**}(s\tau)}}\Big] \exp\Big[-\frac{1}{4}|\alpha_2|^2\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] \ ,$$

where $\beta^{**}(s\tau)$ verifies equation (5.16). Hence, in this case the limit state (5.25) is the product of quasi-free Gibbs states with *different* parameters $\beta^{**}(s\tau)$ and β. This means that subsystem $\mathcal{S}_s$ keeps a *memory* about perturbation at the moment $t = s\tau$, when the parameter $\beta^{*}(s\tau)$ (5.5) of subsystem $\mathcal{S}$ was still different from β.

Note that (5.25) coincides with the product state (5.21) when $s \to \infty$.

Subsystem $\mathcal{S}_{\sim n}$. To define $\mathcal{S}_{\sim n}$ for $0 \leqslant n \leqslant k \leqslant N$, we divide the total system at the moment $t = k\tau$ into two subsystems: $\mathcal{S}_{n,k} + \mathcal{C}_{n,k}$. Here

$$\mathcal{S}_{n,k} := \mathcal{S} + \mathcal{S}_k + \mathcal{S}_{k-1} + \cdots + \mathcal{S}_{k-n+1} \ , \ (\mathcal{S}_{0,k} := \mathcal{S}) \ , \tag{5.26}$$

whereas

$$\mathcal{C}_{n,k} := \mathcal{S}_N + \cdots + \mathcal{S}_{k+1} + \mathcal{S}_{k-n} + \cdots + \mathcal{S}_1 \ . \tag{5.27}$$

We interpret $\mathcal{S}_{\sim n}$ is an entire "object" whose entity is $\mathcal{S}_{n,k}$ at the moment $t = k\tau$ ($k = n, n+1, \cdots, N$). As time is running, the elementary subsystems $\mathcal{S}_k$ in $\mathcal{S}_{\sim n}$ are replacing. We study the behaviour of $\mathcal{S}_{\sim n}$ for large $t = k\tau$, i.e., we analyse the k-dependence of the "state" of $\mathcal{S}_{n,k}$ at $t = k\tau$.

For any fixed $t = k\tau$ we can decompose the Hilbert space $\mathscr{H}$ into tensor product $\mathscr{H} = \mathscr{H}_s \otimes \mathscr{H}_c$. Here $\mathscr{H}_s$ is the Hilbert space of subsystem (5.26) and $\mathscr{H}_c$ corresponds to subsystem (5.27):

$$\mathscr{H}_s := \mathscr{H}_0 \otimes \bigotimes_{j=1}^{n} \mathscr{H}_{k-j+1}, \qquad \mathscr{H}_c := \bigotimes_{j=1}^{k-n} \mathscr{H}_j \otimes \bigotimes_{j=k+1}^{N} \mathscr{H}_j \ . \tag{5.28}$$

For a density matrix ϱ on $\mathscr{H}$, we introduce the *reduced* density matrix ϱ_s on $\mathscr{H}_s$ as the partial trace over $\mathscr{H}_c$:

$$\varrho_s := \mathrm{Tr}_{\mathscr{H}_c}\varrho \ . \tag{5.29}$$

To avoid a possible confusion causing by the fact that all $\mathscr{H}_j, j = 0, 1, \ldots$ are identical to $\mathscr{F}$ and by the change of components with time, we treat the Weyl algebra on the subsystem and the corresponding reduced density matrix of $\rho \in \mathfrak{C}_1(\mathscr{H})$ in the following way. For $n \leqslant N$ on the Fock space $\mathcal{F}^{\otimes(n+1)}$ we consider the Weyl operator

$$W_n(\zeta) = \exp\Big[i\frac{\langle\zeta, \tilde{b},\rangle + \langle\tilde{b}, \zeta\rangle}{\sqrt{2}}\Big], \tag{5.30}$$

where $\zeta \in \mathbb{C}^{n+1}$, $\tilde{b}_0, \cdots, \tilde{b}_n$ and $\tilde{b}_0^*, \cdots, \tilde{b}_n^*$ are the annihilation and the creation operators in $\mathcal{F}^{\otimes(n+1)}$ satisfying the corresponding CCR, and

$$\langle\zeta, \tilde{b}\rangle = \sum_{j=0}^{n} \bar{\zeta}_j \tilde{b}_j, \qquad \langle\tilde{b}, \zeta\rangle = \sum_{j=0}^{n} \zeta_j \tilde{b}_j^*.$$

By $\mathscr{A}(\mathscr{F}^{\otimes(n+1)})$, we denote the C^*-algebra generated by the Weyl operators (5.30). For any subset $J \subset \{1, 2, \cdots, N\}$, we define the operation of taking the partial trace

$$R^J : \mathfrak{C}_1(\mathscr{F}^{\otimes(N+1)}) \ni \rho \longmapsto R^J\rho \in \mathfrak{C}_1(\mathscr{F}^{\otimes(N+1-|J|)})$$

by

$$\omega_{R^J\rho}\big(W_{N-|J|}(\zeta)\big) = \omega_\rho\big(W_N(r_J\zeta)\big)\ .$$

Here the mapping

$$r_J : \mathbb{C}^{N+1-|J|} \ni \zeta \longmapsto r_J\zeta \in \mathbb{C}^{N+1}$$

is defined by

$$(r_J\zeta)_j := \begin{cases} \zeta_0 & (j = 0) \\ 0 & (j \in J) \\ \zeta_{j-|\{i\in J\,|\,i<j\}|} & (\text{otherwise}) \end{cases},$$

where $|A|$ denotes the cardinality of the set A.

Since all $\mathscr{H}_1, \mathscr{H}_2, \cdots$ are identical to $\mathscr{F}$, we do not care to distinguish the spaces

$$\bigotimes_{j\in\{0,1,\cdots,N\}\setminus J} \mathscr{H}_j \qquad \text{and} \qquad \bigotimes_{j\in\{0,1,\cdots,N\}\setminus J'} \mathscr{H}_j$$

when $J \neq J'$, but $|J| = |J'|$, and consider them as the same space $\mathscr{F}^{\otimes(N+1-|J|)}$. Instead, we pay attention to distinguishing projections

$$\bigotimes_{j=0}^{N} \mathscr{H}_j \longrightarrow \bigotimes_{j\in\{0,1,\cdots,N\}\setminus J} \mathscr{H}_j$$

for different subsets $J \subset \{1, 2, \cdots, N\}$ with same $|J|$.

Since we treat $\mathcal{S}_{n,k}$ at time $t = k\tau$ for $k = n, n + 1, \cdots$ as the result of the time evolution of a *single* subsystem $\mathcal{S}_{\sim n}$, we define its state at the moment $t = k\tau$ by the reduced density matrix $\{\rho_s(k\tau)\}_{k\geqslant n}$ of this subsystem as follows:

$$\rho_s(k\tau) := R^{\{1,\cdots,k-n,k+1,\cdots,N\}}\big(\rho(k\tau)\big) = R^{\{1,\cdots,k-n,k+1,\cdots,N\}}T_{k\tau}(\rho), \tag{5.31}$$

see (2.12). Taking into account Lemma 4.2 and identity $\langle r_J\zeta, r_J\zeta\rangle_{\mathbb{C}^{N+1}} = \langle\zeta, \zeta\rangle_{\mathbb{C}^{N+1-|J|}}$, one readily obtains the following result.

Lemma 5.1 *For the initial density matrix (4.1),*

$$\omega_{\rho_s(k\tau)}(W_n(\zeta)) = \omega_{R^{J_{n,k}}\rho(k\tau)}(W_n(\zeta))$$

$$= \exp\Big[-\frac{|(U_1 \dots U_k\, r_{J_{n,k}}\zeta)_0|^2}{4}\Big(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big) - \frac{\langle\zeta,\zeta\rangle}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big]$$

holds, where $J_{n,k} = \{1, 2, \cdots, k-n, k+1, \cdots, N\}$.

To study the limit $k \to \infty$ and $N \to \infty$ ($k \leqslant N$) for a fixed n, we note that $(U_1 \dots U_k\, r_{J_{n,k}}\zeta)_0 \to 0$ follows from (2.16) and $|z| < 1$. Lemma 5.1 implies that

$$\lim_{k\to\infty} \omega_{\rho_s(k\tau)}(W_n(\zeta)) = \exp\Big[-\frac{\langle\zeta,\zeta\rangle}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] = \omega_{\rho_n^{(\beta)}}(W_n(\zeta)) \ , \qquad (5.32)$$

where by the Araki-Segal theorem and irreducibility of the CCR algebra $\mathscr{A}(\mathscr{F}^{\otimes(n+1)})$

$$\rho_n^{(\beta)} = \exp\big[-\beta\sum_{j=0}^{n}\tilde{b}_j^*\tilde{b}_j\big]/Z(\beta)^{n+1} \ , \ Z(\beta) = (1-e^{-\beta})^{-1} \ . \qquad (5.33)$$

Therefore, we proved the following statement:

Theorem 5.2 *Let the initial state of the total system $\mathcal{S}+\mathcal{C}$ is defined by the density matrix (4.2): $\rho = \rho(\beta, \beta_0 - \beta; e)$. Then for any fixed n, the state $\omega_{\rho_s(k\tau)}(\cdot)$ of subsystem $\mathcal{S}_{n,k}$ converges to the equilibrium Gibbs state $\omega_{\rho_n^{(\beta)}}(\cdot)$ as $k \to \infty$ in the weak*-topology for the states on $\mathscr{A}(\mathscr{F}^{\otimes(n+1)})$.*

Theorem 5.3 *Under the same conditions as in Theorem 5.2, we obtain*

$$\lim_{k\to\infty} S(\rho_s(k\tau)) = S(\rho_n^{(\beta)}) \ .$$

Proof : Let the vector $\xi_{n,k} \in \mathbb{C}^{n+1}$ be defined by $(U_1 \dots U_k r_{J_{n,k}}\zeta)_0 =: \langle\xi_{n,k}, \zeta\rangle$. Then $k \to \infty$, for a fixed n, implies $\langle\xi_{n,k}, \xi_{n,k}\rangle \to 0$. By Proposition 3.3 and Lemma 5.1 we obtain that in this limit

$$S(\rho_s(k\tau)) = n\sigma\Big(\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big) + \sigma\Big(\frac{1+e^{-\beta}}{1-e^{-\beta}} + \langle\xi_{n,k}, \xi_{n,k}\rangle\Big(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big)\Big)$$

$$\longrightarrow (n+1)\sigma\Big(\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big) = S(\rho_n^{(\beta)}) \ .$$

□

Remark 5.4 *The local entropy decreases or increases with $k\tau$ according to $\beta > \beta_0$ or $\beta < \beta_0$, respectively.*

6 A Short-Time Limit for Repeated Perturbation

The results in the Section 5 are essentially due explicit knowledge of the initial density matrix (4.1) of the total system $\mathcal{S}+\mathcal{C}$. In this section, we show that the lack of this information is not decisive for certain results concerning the convergence to equilibrium if one considers the repeated perturbation in a *short-time* limit.

We study this limit for the subsystem $\mathcal{S}$. We keep to consider the initial state of the system $\mathcal{S}+\mathcal{C}$ to be a product state with the density matrix

$$\rho = \rho_0 \otimes \bigotimes_{k=1}^{N} \rho_k \in \mathfrak{C}_1(\mathscr{H}) \ , \tag{6.1}$$

see (2.7), but we essentially relax the conditions on ρ_0 and on $\{\rho_k\}_{k=1}^N$ (cf.(4.1)):

$$\begin{array}{ll} \text{(h1)} & \rho_1 = \rho_2 = \cdots = \rho_N \in \mathfrak{C}_1(\mathscr{F}) \ ; \\ \text{(h2)} & \mathrm{Tr}_{\mathscr{F}}(\rho_1 a) = \mathrm{Tr}_{\mathscr{F}}(\rho_1 a^2) = \mathrm{Tr}_{\mathscr{F}}(\rho_1 a^*) = \mathrm{Tr}_{\mathscr{F}}(\rho_1 a^{*2}) = 0 \ ; \\ \text{(h3)} & \mathrm{Tr}_{\mathscr{F}}[\rho_1 (a^*a)^2] < \infty \ . \end{array}$$

Remark 6.1 *Note that hypothesis (h1)-(h3) are satisfied when the density matrices $\{\rho_k\}_{k=0}^N$ correspond to the gauge-invariant quasi-free states with parameter β_0 for $k=0$ and β for $k=1,2,\ldots,N$, see (4.1). Then (h2) is due to the gauge invariance and one gets for (h3):*

$$\mathrm{Tr}_{\mathscr{F}}[\rho_k (a^*a)^2] = (2n_\beta^2 + n_\beta) \ , \tag{6.2}$$

where $n_\beta = \mathrm{Tr}_{\mathscr{F}}(\rho_k\, a^*a) = (e^\beta - 1)^{-1}$, $k=1,\ldots,N$.

Below we denote by $|ya^* + \bar{y}a|$ the operator originated from the *polar decomposition* of the operator $ya^* + \bar{y}a = U\,|ya^* + \bar{y}a|$, where U is the partial isometry on $\mathscr{F}$.

Lemma 6.2 *Under hypothesis (h1)-(h3), the following bounds hold:*

$$\begin{array}{ll} \text{(i)} & \mathrm{Tr}_{\mathscr{F}}(\rho_k a^* a) < \infty, \\ \text{(ii)} & \mathrm{Tr}_{\mathscr{F}}(\rho_k |ya^* + \bar{y}a|^2) \leqslant C|y|^2, \\ \text{(iii)} & \mathrm{Tr}_{\mathscr{F}}(\rho_k |ya^* + \bar{y}a|^3) \leqslant C'|y|^3, \\ \text{(iv)} & \mathrm{Tr}_{\mathscr{F}}(\rho_k |ya^* + \bar{y}a|^4) \leqslant C''|y|^4, \end{array}$$

for all $k=1,\ldots,N$. Here C, C', C'' are positive constants, which depend only on $\mathrm{Tr}[\rho_1(a^*a)^2]$.

Proof : The first bound (i) is a consequence of the Cauchy-Schwarz inequality and (h3). Applying the inequalities

$$|A + A^*|^2 \leqslant |A + A^*|^2 + |A - A^*|^2 = 2(AA^* + A^*A),$$

$$\begin{aligned} |A + A^*|^4 &\leqslant |A + A^*|^4 + |A - A^*|^4 + |A + iA^*|^4 + |A - iA^*|^4 \\ &= 4(AA^* + A^*A)^2 + 4(A^2A^{*2} + A^{*2}A^2), \end{aligned}$$

to $A = \bar{y}a$, we obtain (ii) and (iv). Finally, a combination of (ii), (iv) with the Cauchy-Schwarz inequality yields (iii). □

Theorem 6.3 *Let $\tau \to 0$, $N \to \infty$ be short-time perturbation limit subjected to demands: $\tau^2 N \to \infty$ and $\tau^3 N \to 0$. Then for any initial condition (6.1) verifying (h1)-(h3), the characteristic function $\omega_{\mathcal{S}}^{N\tau}(\widehat{w}(\theta))$ of the state for subsystem $\mathcal{S}$ at $t = N\tau$, converges to*

$$\omega_{\mathcal{S}}(\widehat{w}(\theta)) := \lim_{\tau \to 0, N \to \infty} \omega_{\rho(N\tau)}(W(\zeta_\theta)) = e^{-|\theta|^2 \mathrm{Tr}_{\mathscr{F}}[\rho_1 (a^*a + aa^*)]/4}. \tag{6.3}$$

Here $\theta \in \mathbb{C}$ and the $(N+1)$-component vector is $(\zeta_\theta)^{\mathrm{tr}} := (\theta, 0, 0, \ldots, 0) \in \mathbb{C}^{N+1}$.

By (6.3) the state $\omega_{\mathcal{S}}^{N\tau}$ converges to $\omega_{\mathcal{S}}$ in the weak*-topology. From the right-hand side of (6.3) and Definition 3.1 we deduce that the limit state is gauge-invariant and quasi-free with $h(\theta) := |\theta|^2 \,\mathrm{Tr}_{\mathscr{F}}(\rho_1\, a^* a)$.

Remark 6.4 *Recall that the state ω over the Weyl algebra $\mathscr{A}(\mathscr{F}) = \overline{\mathscr{A}_w(\mathscr{F})}$ is regular, C^n-smooth or analytic, if the function (see (2.1))*

$$s \mapsto \omega(\widehat{w}(s\theta)) = \omega(e^{i\, s\Phi(\theta)/\sqrt{2}}) \tag{6.4}$$

is respectively continuous, C^n-smooth or analytic in the vicinity of $s = 0$. In the last case the characteristic function $\omega(\widehat{w}(s\theta))$ (and therefore the state) is completely determined by

$$\omega(\widehat{w}(s\theta)) = \exp\left\{\sum_{m=1}^{\infty} \frac{i^m s^m}{m!}\, 2^{-m/2}\, \omega^T(\Phi^m(\theta))\right\} . \tag{6.5}$$

Here $\{\omega^T(\Phi^m(\theta))\}_{m=0}^{\infty}$ are truncated correlation functions defined recursively by relations

$$\begin{aligned}
&\omega^T(\Phi(\theta)) := \omega(\Phi(\theta)) \ ,\\
&\omega^T(\Phi^2(\theta)) := \omega(\Phi^2(\theta)) - \omega(\Phi(\theta))^2 \ ,\\
&\omega^T(\Phi^3(\theta)) := \omega(\Phi^3(\theta)) - 3\omega(\Phi^2(\theta))\omega(\Phi(\theta)) + 2\omega(\Phi(\theta))^3 \ , \ \text{etc}
\end{aligned}$$

Lemma 6.2 implies that the states for density matrices $\rho_1 = \rho_2 = \ldots$ are C^4-smooth.

Proof (of Theorem 6.3): By (h2) and by Lemma 6.2 (i)-(iii) together with Remark 6.4, we obtain for the states $\omega(\cdot) = \omega_{\rho_k}(\cdot)$ the representation of (6.5) in the form:

$$C_k(\theta) = \omega_{\rho_k}(\widehat{w}(\theta)) = \exp[-\frac{1}{4}\, \omega_{\rho_k}^T(\Phi^2(\theta)) + R(\theta)] \ , \ k = 1, 2, \ldots, N, \tag{6.6}$$

where $R(\theta) = O(|\theta|^3)$ in the vicinity of $\theta = 0$. For the self-adjoint operator $\Phi(\theta) = \bar{\theta}a + \theta a^*$, the hypothesis (h2) and Lemma 6.2 (i) imply

$$\omega_{\rho_k}^T(\Phi^2(\theta)) = |\theta|^2 \ \mathrm{Tr}_{\mathscr{F}}[\rho_k\, (a^*a + aa^*)] \ . \tag{6.7}$$

Now, taking into account Lemma 2.2 for the vector ζ_θ, as well as (6.6) and (6.7), we obtain the representation:

$$\omega_{\mathcal{S}}^{N\tau}(\widehat{w}(\theta)) = \omega_{\rho(N\tau)}(W(\zeta_\theta)) = C_0(e^{i\epsilon\tau N}(gz)^N\theta) \prod_{k=1}^{N} C_k(e^{i\epsilon\tau N} gw\ (gz)^{N-k}\theta)$$

$$= C_0(e^{i\epsilon\tau N}(gz)^N\theta)\exp\Big(-\sum_{k=1}^{N}\frac{|\theta_k|^2}{4}\mathrm{Tr}_{\mathscr{F}}[(a^*a+aa^*)\rho_k]+\widehat{R}\Big)\ . \qquad (6.8)$$

Here by (2.17) and by (6.6) one has

$$\theta_k := e^{i\epsilon N\tau}gw\ (gz)^{N-k}\theta\ ,\ \sum_{k=1}^{N}|\theta_k|^2 = |\theta|^2|w|^2\frac{1-|z|^{2N}}{1-|z|^2}\ ,\ \widehat{R} = \sum_{k=1}^{N}O(|\theta_k|^3)\ .$$

By virtue of (1.10) and (1.11), we get $|g(\tau)| = 1$, $|w(\tau)|^2 + |z(\tau)|^2 = 1$ and also

$$w(\tau) = i\eta\tau + O(\tau^3)\ ,\ |z(\tau)| = 1 - \frac{|\eta|^2\tau^2}{2} + O(\tau^4)\ ,$$

for small τ. This yields for small $\tau > 0$ and large N, the estimates $|(gz)^N| \leq O(e^{-|\eta|^2\tau^2N/2})$, $|\theta_k| \leq O(\tau)$, and $\widehat{R} = O(\tau^3 N)$ by virtue of (h1). Then taking into account the conditions $\tau^2 N \to \infty$ and $\tau^3 N \to 0$, we get the limits:

$$\lim_{\tau\to 0, N\to\infty} C_0(e^{i\epsilon\tau N}(gz)^N\theta) = 1\ ,\ \lim_{\tau\to 0, N\to\infty}\sum_{k=1}^{N}|\theta_k|^2 = |\theta|^2\ ,\ \lim_{\tau\to 0, N\to\infty}\widehat{R} = 0\ .$$

C_0 is a continuous function since it is defined by a normal state with density matrix ρ_0. Inserting all these limits into (6.8), we obtain what is claimed as the limit (6.3). □

Corollary 6.5 *Suppose that density matrices $\{\rho_k\}_{k=1}^N$ correspond to the gauge-invariant quasi-free Gibbs state with parameter β (4.1). These states satisfy (h1)-(h3). The statement of Theorem 6.3 is valid with the limit*

$$\omega_{\mathcal{S}}(\widehat{w}(\theta)) = \lim_{\tau\to 0, N\to\infty}\omega_{\mathcal{S}}^{N\tau}(\widehat{w}(\theta)) = \exp\left\{-\frac{|\theta|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\right\}\ . \qquad (6.9)$$

It coincides with the result for equilibrium state (5.6) of the subsystem $\mathcal{S}$.

Hence, the short-time perturbation limit $\tau \to 0$, $N \to \infty$ subjected to $\tau^2 N \to \infty$ and $\tau^3 N \to 0$ gives a *universal* gauge-invariant quasi-free limiting state under hypothesis (h1)-(h3). The hypotheses (h2),(h3) control only first *two* moments of the initial states of the subsystem $\mathcal{C}$. Then stationarity and independence of repeated perturbation due to (h1), correspond to conditions for the non-commutative Central Limit Theorem [Ve]. Note also that the state ω_{ρ_0} of the subsystem $\mathcal{S}$ may be replaced by any *regular* state.

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References

[Ar1] H.Araki, Relative Entropy of States of von Neumann Algebras, *Publl. RIMS, Kyoto Univ.*, **11** (1976), 809-833.

[AJP1] *Open Quantum Systems I, The Hamiltonian Approach*, S. Attal, A. Joye, C.-A. Pillet (Eds.), Lecture Notes in Mathematics **1880**, Springer-Verlag, Berlin-Heidelberg 2006.

[AJPII] *Open Quantum Systems II, The Markovian Approach*, S. Attal, A. Joye, C.-A. Pillet (Eds.), Lecture Notes in Mathematics **1881**, Springer-Verlag, Berlin-Heidelberg 2006.

[AJP3] *Open Quantum Systems III, Recent Developements*, S. Attal, A. Joye, C.-A. Pillet (Eds.), Lecture Notes in Mathematics **1882**, Springer-Verlag, Berlin-Heidelberg 2006.

[BJM] L.Bruneau, A.Joye, and M.Merkli, Repeated interactions in open quantum systems, *J.Math.Phys.* **55** (2014), 075204.

[BR2] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*,vol.2, Springer-Verlag (2nd Edt), Berlin 1997.

[DVV] B.Demoen, P.Vanheuverzwijn, and A.Verbeure, Completely positive quasi-free maps of the CCR-algebra, *Rep.Math.Phys.*, **15** (1979) 27-39.

[Fa] M.Fannes, The entropy of quasi-free states for a continuous boson system, *Ann.de l'IHP*, section A, **28**(1978) 187-196.

[FJMa] P. Filipowicz, J. Javanainen, and P. Meystre, Quantum and semiclassical steady states of a kicked cavity mode. *J. Opt. Soc. Am.*, B**3** (1986) 906–910.

[FJMb] P. Filipowicz, J. Javanainen, and P. Meystre, Theory of a microscopic maser, *Phys. Rev.*, A**34** (1986) 3077–3087.

[M] P. Meystre, Theoretical developments in cavity quantum optics: a brief review, *Physics Reports*, **219**(3)(1992), 243–262.

[NVZ] B. Nachtergaele, A. Vershynina, and V. A. Zagrebnov, Non-Equilibrium States of a Photon Cavity Pumped by an Atomic Beam, *Annales Henri Poincaré*, **15** (2014), 213-262.

[TZ] H.Tamura and V. A. Zagrebnov, A Dynamics Driven by Repeaded Harmonic Perturbations, arXiv:1404.2998 (2014), 1-40.

[Ve] A.F. Verbeure, *Many-Body Boson Systems*, Springer-Verlag, Berlin 2011.