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Exactly Soluble Quantum Model for Repeated Harmonic Perturbation

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ABSTRACT

We consider an exactly soluble dynamical system with inelastic repeated harmonic perturbation. Hamiltonian dynamics is quasi-free and it leads in the large-time limit to relaxation of initial states and to the entropy production. To study correlations we consider time evolution of subsystems. We prove a universality of dynamics driven by repeated harmonic perturbation in a short-time interaction limit.

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1 The Model

We consider an exactly soluble model of quantum system proposed in [TZ]. It is a harmonic system (one-mode quantum oscillator $\mathcal{S}$) successively perturbed by time-dependent stationary repeated harmonic interactions. This sequence of perturbation is switched on at the moment $t = 0$ and it acts successively on the interval $0 \leq t < \infty$. It is a common fashion to present this sequence as repeated interactions of the system $\mathcal{S}$ with an *infinite* time-equidistant *chain*: $\mathcal{C} = \mathcal{S}_1 + \mathcal{S}_2 + \dots$, of subsystems $\{\mathcal{S}_k\}_{k \geq 1}$ [BJM].

Note that there is a physical interpretation [NVZ], [BJM], behind of this mathematical setting. For the model [TZ], the system $\mathcal{C}_N$ is identified with a chain of N quantum particles ("atoms") with infinitely many *harmonic* internal degrees of freedom. They interact *one-by-one* with a one-mode quantum resonator (cavity) $\mathcal{S}$. This is a caricature of the one-atom maser system. In contract to [NVZ], but similar to the two-level Jaynes-Cummings atoms [BJM], the interaction with harmonic atoms is *inelastic*. This yields a drastic difference between evolution of the model [TZ] and the model [NVZ] with completely elastic interaction.

Recall that experimental study of interaction of a single atom in a cavity is expected to be drastically modified as compared with its behaviour in a free space. First, the spontaneous emission is enhanced in a resonant high- Q (i.e. non-leaky) cavity and it is suppressed if the cavity is off the resonance, see e.g.[M]. Another important difference is related to the nature of interaction of Ridberg's atoms and the cavity radiance. In [NVZ] an exactly soluble model in the limit of the *rigid* atoms shows that it corresponds to the regime of a "kick" cavity evolution [FJMa]. Whereas in the regime of the inelastic atom-cavity interaction the system may to relax to a steady state even for a non-leaky cavity [FJMb]. For example, this property manifests the models for the two-level Jaynes-Cummings atoms [BJM]. In the present paper we study a model for atoms with *infinitely* many levels, which imitates very soft Ridberg's atoms.

Below we suppose that the states of $\mathcal{S}$ and of every $\mathcal{S}_k$ are *normal*, i.e. defined by the *density matrices* ρ_0 and $\{\rho_k\}_{k=1}^\infty$ on the Hilbert spaces $\mathcal{H}_{\mathcal{S}}$ and $\{\mathcal{H}_{\mathcal{S}_k}\}_{k=1}^\infty$, respectively. The Hilbert space of the total system is then the tensor product $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{C}}$. Here the

infinite product $\mathcal{H}_C = \otimes_{k \geq 1} \mathcal{H}_{S_k}$ stays for the Hilbert space chain. Details of dynamics are presented in the next Section 2. Below we collect our hypothesis.

(H1) *Initial states.* For $t \leq 0$, all components of $\mathcal{S}$ and $\{\mathcal{S}_k\}_{k=1}^N$ are *independent*, i.e. the state of $\mathcal{S} + \mathcal{C}_N$ is described as a finite tensor product: $\omega_{\mathcal{S}+\mathcal{C}_N} := \omega_{\mathcal{S}} \otimes \bigotimes_{k=1}^N \omega_{\mathcal{S}_k}$. We suppose that each of the state in the product is *normal*.

(H2) *Tuned interaction.* We consider repeated perturbations in the tuned regime: for any moment $t \geq 0$ exactly *one* subsystem ("atom") $\mathcal{S}_n$ is interacting with the system $\mathcal{S}$ (quantum resonator) during a fixed time $\tau > 0$. Here $n = [t/\tau] + 1$, where $[x]$ denotes the integer part of $x \geq 0$.

Let $\mathcal{H}_0$ be the Hilbert space for the system $\mathcal{S}$ and $\mathcal{H}_k$ be the Hilbert space for the system $\mathcal{S}_k$ for $k = 1, \dots, N$. Then for $k = 0, 1, \dots, N$, the space $\mathcal{H}_k$ is a copy of the one-mode boson Fock space $\mathcal{F}$ with the vacuum vector $\Omega \in \mathcal{F}$ and with densely defined boson annihilation and (adjoint) creation operators: a and a^* , defined by $a\Omega = 0$. The total system $\mathcal{S} + \mathcal{C}_N$ lives in the Hilbert space

$$\mathcal{H}^{(N)} := \mathcal{H}_0 \otimes \bigotimes_{k=1}^N \mathcal{H}_k = \mathcal{F}^{\otimes(N+1)}. \quad (1.1)$$

Here $\mathbb{1}$ is the unit operator on $\mathcal{F}$. In the space (1.1) we define operators

$$b_k := \mathbb{1} \otimes \dots \otimes \mathbb{1} \otimes a \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}, \quad b_k^* := \mathbb{1} \otimes \dots \otimes \mathbb{1} \otimes a^* \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}, \quad (1.2)$$

where operator a , or a^* , is the $(k+1)$ th factor in (1.2). Operators (1.2) formally satisfy the Canonical Commutation Relations (CCR)

$$[b_k, b_{k'}^*] = \delta_{k,k'} \mathbb{1}, \quad [b_k, b_{k'}] = [b_k^*, b_{k'}^*] = 0, \quad k, k' = 0, 1, 2, \dots, N. \quad (1.3)$$

(H3) *Harmonic interaction.* The time-dependent repeated interaction described by (H2) is a piecewise *constant* operator in (1.1). It is the sum over $n \geq 1$ of the *bilinear* forms in operators (1.2) in the space $\mathcal{H}_0 \otimes \mathcal{H}_n$:

$$K_n(t) := \chi_{[(n-1)\tau, n\tau)}(t) \eta (b_0^* b_n + b_n^* b_0), \quad \eta > 0. \quad (1.4)$$

Here $\chi_{\mathcal{I}}(x)$ is the characteristic function of the set $\mathcal{I}$.

For any $N \geq 1$ and $t < N\tau$, the self-adjoint Hamiltonian $H_N(t)$ of the non-autonomous system $\mathcal{S} + \mathcal{C}_N$ is defined in the space (1.1) as the sum of Hamiltonians corresponding the systems $\mathcal{S}$, $\mathcal{S}_k$ and interaction (1.4) [TZ]:

$$\begin{aligned} H_N(t) &:= H_{\mathcal{S}} + \sum_{k=1}^N (H_{\mathcal{S}_k} + K_k(t)) \\ &= E b_0^* b_0 + \epsilon \sum_{k=1}^N b_k^* b_k + \eta \sum_{k=1}^N \chi_{[(k-1)\tau, k\tau)}(t) (b_0^* b_k + b_k^* b_0), \end{aligned} \quad (1.5)$$

(H4) *Semi-boundedness.* To keep the self-adjoint Hamiltonian (1.5) semi-bounded from below we suppose that $E, \epsilon > 0$ and we impose the condition $\eta^2 \leq E\epsilon$.

By virtue of (1.4), (1.5) only $\mathcal{S}_n$ interacts with $\mathcal{S}$ for $t \in [(n-1)\tau, n\tau)$, $n \geq 1$, i.e. the system $\mathcal{S} + \mathcal{C}_N$ is *autonomous* on this time-interval with self-adjoint Hamiltonian

$$H_n := E b_0^* b_0 + \epsilon \sum_{k=1}^N b_k^* b_k + \eta (b_0^* b_n + b_n^* b_0), \quad n \leq N. \quad (1.6)$$

The key for the exact solution lemma follows from the harmonic structure of (1.6).

Lemma 1.1 *For $j = 0, 1, 2, \dots, N$ and $n = 1, 2, \dots, N$, one gets*

$$e^{itH_n} b_j e^{-itH_n} = \sum_{k=0}^N (U_n^*(t))_{jk} b_k, \quad e^{itH_n} b_j^* e^{-itH_n} = \sum_{k=0}^N \overline{(U_n^*(t))_{jk}} b_k^*, \quad (1.7)$$

$$e^{-itH_n} b_j e^{itH_n} = \sum_{k=0}^N (U_n(t))_{jk} b_k, \quad e^{-itH_n} b_j^* e^{itH_n} = \sum_{k=0}^N \overline{(U_n(t))_{jk}} b_k^*, \quad (1.8)$$

for $t \geq 0$. Here $U_n(t)$ and $V_n(t)$ are $(N+1) \times (N+1)$ matrices related by $U_n(t) := e^{it\epsilon} V_n(t)$, where

$$(V_n(t))_{jk} := \begin{cases} g(t)z(t) \delta_{k0} + g(t)w(t) \delta_{kn} & (j=0) \\ g(t)w(t) \delta_{k0} + g(t)z(-t) \delta_{kn} & (j=n) \\ \delta_{jk} & (\text{otherwise}) \end{cases}, \quad (1.9)$$

and

$$g(t) := e^{it(E-\epsilon)/2}, \quad w(t) := \frac{2i\eta}{\sqrt{(E-\epsilon)^2 + 4\eta^2}} \sin t \sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2}, \quad (1.10)$$

$$z(t) := \cos t \sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2} + \frac{i(E-\epsilon)}{\sqrt{(E-\epsilon)^2 + 4\eta^2}} \sin t \sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2}. \quad (1.11)$$

Remark 1.2 *Note that by definitions (1.10) and (1.11), we get $|z(t)|^2 + |w(t)|^2 = 1$, $z(-t) = \overline{z(t)}$ and $w(t) = -\overline{w(-t)}$. Therefore, the matrix*

$$M(t) := \begin{pmatrix} z(t) & w(t) \\ w(t) & z(-t) \end{pmatrix}$$

is unitary. For $N = 1$, one gets $M(t) = \overline{g(t)} V_1(t)$, see (1.9). Moreover, (1.7) and (1.8) imply that $\{V_n(t)\}_{t \in \mathbb{R}}$ and $\{U_n(t)\}_{t \in \mathbb{R}}$ are in fact one-parameter groups of $(N+1) \times (N+1)$ unitary matrices.

Proof (of Lemma 1.1): Let $\{J_n\}_{n=1}^N$ and $\{X_n\}_{n=1}^N$ be $(N+1) \times (N+1)$ Hermitian matrices given by

$$(J_n)_{jk} := \begin{cases} 1 & (j=k=0 \text{ or } j=k=n) \\ 0 & \text{otherwise} \end{cases}, \quad (1.12)$$

$$(X_n)_{jk} := \begin{cases} (E - \epsilon)/2 & (j, k) = (0, 0) \\ -(E - \epsilon)/2 & (j, k) = (n, n) \\ \eta & (j, k) = (0, n) \\ \eta & (j, k) = (n, 0) \\ 0 & \text{otherwise} \end{cases} . \quad (1.13)$$

We define the matrices

$$Y_n := \epsilon I + \frac{E - \epsilon}{2} J_n + X_n \quad (n = 1, \dots, N) , \quad (1.14)$$

where I is the $(N + 1) \times (N + 1)$ identity matrix. Then Hamiltonian (1.6) takes the form

$$H_n = \sum_{j,k=0}^N (Y_n)_{jk} b_j^* b_k . \quad (1.15)$$

Since Y_n is Hermitian, there exists a *diagonal* matrix Λ and unitary mapping $\mathcal{U}_n : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{N+1}$, such that $Y_n = \mathcal{U}_n^* \Lambda \mathcal{U}_n$ holds. After canonical transformation $\mathcal{U}_n$ the matrix $\Lambda := \{\Lambda_{ij}\}_{i,j=0}^N = \{\delta_{ij} \varepsilon_j\}_{i,j=0}^N$ is universal and independent of n . The new operators:

$$c_j = \sum_{k=0}^N (\mathcal{U}_n)_{jk} b_k, \quad c_j^* = \sum_{k=0}^N \overline{(\mathcal{U}_n)_{jk}} b_k^* \quad (j = 0, 1, \dots, N) , \quad (1.16)$$

satisfy CCR in the space $\mathcal{H}^{(N)}$ (1.1) and diagonalise (1.15): $\tilde{H}_n = \sum_{j=0}^N \Lambda_{jj} c_j^* c_j$, where $\Lambda_{jj} = \varepsilon_j$. Therefore, the set of all eigenvectors of $\tilde{H}_n$ is

$$\left\{ \prod_{j=0}^N \frac{(c_j^*)^{n_j}}{\sqrt{n_j!}} \Omega \otimes \dots \otimes \Omega \mid n_j \in \mathbb{Z}_+ \quad (j = 0, 1, \dots, N) \right\} . \quad (1.17)$$

Note that it forms a complete orthonormal basis in $\mathcal{H}^{(N)}$. The linear envelope $\mathcal{H}_0^{(N)}$ of the set (1.17) is invariant subspace for transformations $e^{it\tilde{H}_n}$ and its norm-closure coincides with $\mathcal{H}^{(N)}$. Then by (1.16) one gets on vectors (1.17):

$$e^{it\tilde{H}_n} c_j e^{-it\tilde{H}_n} = e^{-it\Lambda_{jj}} c_j, \quad e^{it\tilde{H}_n} c_j^* e^{-it\tilde{H}_n} = e^{it\Lambda_{jj}} c_j^* .$$

Now taking into account canonical transformation (1.16), we obtain

$$\begin{aligned} e^{itH_n} b_j e^{-itH_n} &= \sum_{k=0}^N (\mathcal{U}_n^*)_{jk} e^{it\tilde{H}_n} c_k e^{-it\tilde{H}_n} \\ &= \sum_{k,l=0}^N (\mathcal{U}_n^*)_{jk} e^{-it\Lambda_{kk}} (\mathcal{U}_n)_{kl} b_l = \sum_{l=0}^N (e^{-it\mathcal{U}_n^* \Lambda \mathcal{U}_n})_{jl} b_l = \sum_{l=0}^N (e^{-itY_n})_{jl} b_l . \end{aligned} \quad (1.18)$$

Similarly we obtain $e^{itH_n} b_j^* e^{-itH_n} = \sum_{l=0}^N \overline{(e^{-itY_n})_{jl}} b_l^* .$

Note that by virtue of (1.12), (1.13), one has identities

$$X_n^2 = \left(\frac{(E - \epsilon)^2}{4} + \eta^2 \right) J_n \quad \text{and} \quad J_n X_n = X_n .$$

Together with definition (1.14) and (1.9), they yield

$$\begin{aligned} e^{itY_n} &= e^{it\epsilon} \left(I - J_n + e^{it(E-\epsilon)/2} \left\{ J_n \cos t \sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2} \right. \right. \\ &\quad \left. \left. + iX_n \left[\frac{(E-\epsilon)^2}{4} + \eta^2 \right]^{-1/2} \sin t \sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2} \right\} \right) = e^{it\epsilon} V_n(t) = U_n(t). \end{aligned} \quad (1.19)$$

Inserting now (1.19) into (1.18), we prove (1.7). Since $U_n(t)^* = U_n(-t)$, one can similarly establish (1.8). $\square$

Remark 1.3 *Hereafter, we are going to use the short-hand notations:*

$$g := g(\tau), \quad w := w(\tau), \quad z := z(\tau) \quad \text{and} \quad V_n := V_n(\tau), \quad U_n := U_n(\tau) . \quad (1.20)$$

In Section 2, we give explicit description of the Hamiltonian dynamics for the non-autonomous system $\mathcal{S} + \mathcal{C}$ driven by harmonic repeated interactions (H3). We show that our model of bosons (1.5) is a quasi-free W^* -dynamical system. In Section 3 we recall formulae for the entropy of the CCR quasi-free states. We use them in Section 4 for calculations of the entropy production. Section 5 is dedicated to analysis of reduced dynamics of subsystems, of their correlations and of convergence to equilibrium. We prove a universality of the short-time interaction limit of this dynamics for the subsystem $\mathcal{S}$.

2 Hamiltonian Dynamics

A well-known way to avoid the problem of evolution of unbounded creation-annihilation operators is to construct dynamics of the subsystem $\mathcal{S}$ on the unital Weyl CCR C^* -algebra $\mathcal{A}(\mathcal{F})$, see e.g. [AJP1] (Lectures 4 and 5), [BR2]. Here $\mathcal{A}(\mathcal{F})$ is generated on the Fock space $\mathcal{F}$ as the operator-norm closure of the linear span $\mathcal{A}_w$ of the Weyl operator system:

$$\{\widehat{w}(\alpha) = e^{i\Phi(\alpha)/\sqrt{2}}\}_{\alpha \in \mathbb{C}} . \quad (2.1)$$

Here $\Phi(\alpha) := \bar{\alpha}a + \alpha a^*$ is a self-adjoint operator with domain in $\mathcal{F}$ and the CCR take then the Weyl form:

$$\widehat{w}(\alpha_1)\widehat{w}(\alpha_2) = e^{-i\text{Im}(\bar{\alpha}_1\alpha_2)/2} \widehat{w}(\alpha_1 + \alpha_2) , \quad \alpha_1, \alpha_2 \in \mathbb{C} . \quad (2.2)$$

Note that $\mathcal{A}(\mathcal{F})$ is a minimal C^* -algebra, which contains the linear span $\mathcal{A}_w$ of the Weyl operator system (2.1). Algebra $\mathcal{A}(\mathcal{F})$ is contained in the unital C^* -algebra $\mathcal{L}(\mathcal{F})$ of all bounded operators on $\mathcal{F}$.

Similarly we define the Weyl C^* -algebra $\mathcal{A}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ over $\mathcal{H} := \mathcal{H}^{(N)}$ (1.1). It is appropriate for description the system $\mathcal{S} + \mathcal{C}$. This algebra is generated by operators

$$W(\zeta) = \bigotimes_{k=0}^N \widehat{w}(\zeta_k), \quad \zeta = \{\zeta_k\}_{k=0}^N \in \mathbb{C}^{N+1}, \quad N \geq 1. \quad (2.3)$$

Using definitions of the boson operators $\{b_k, b_k^*\}_{k=1}^N$ and of the sesquilinear forms

$$\langle \zeta, b \rangle := \sum_{j=0}^N \bar{\zeta}_j b_j, \quad \langle b, \zeta \rangle := \sum_{j=0}^N \zeta_j b_j^*, \quad (2.4)$$

the Weyl operators (2.3) can be rewritten as

$$W(\zeta) = \exp[i(\langle \zeta, b \rangle + \langle b, \zeta \rangle)/\sqrt{2}]. \quad (2.5)$$

We denote by $\mathfrak{C}_1(\mathcal{F}) \subset \mathcal{L}(\mathcal{F})$, the set of all trace-class operators on $\mathcal{F}$. A self-adjoint, non-negative operator $\rho \in \mathfrak{C}_1(\mathcal{F})$ with *unit* trace is called *density matrix*. The state $\omega_\rho(\cdot)$ generated on the C^* -algebra of bounded operators $\mathcal{L}(\mathcal{F})$ by ρ :

$$\omega_\rho(A) := \text{Tr}_{\mathcal{F}}(\rho A), \quad A \in \mathcal{L}(\mathcal{F}), \quad (2.6)$$

is a *normal* state. Let $\{\rho_k\}_{k=0}^N$ be density matrices on $\mathcal{F}$. Then the normal *product-state* on the C^* -algebra $\mathcal{A}(\mathcal{H})$ (isometrically isomorphic to the tensor product $\bigotimes_{k=0}^N \mathcal{A}(\mathcal{F})$) is

$$\omega_{\rho^\otimes}(\cdot) := \text{Tr}_{\mathcal{H}}(\rho^\otimes \cdot), \quad \rho^\otimes := \bigotimes_{k=0}^N \rho_k. \quad (2.7)$$

If we put $C_k(\alpha) := \text{Tr}_{\mathcal{F}}[\rho_k \widehat{w}(\alpha)]$, $\alpha \in \mathbb{C}$, then by (2.3) one obtains for $\rho^\otimes$ (2.7) the representation:

$$\omega_{\rho^\otimes}(W(\zeta)) := \text{Tr}_{\mathcal{H}}[\rho^\otimes W(\zeta)] = \prod_{k=0}^N C_k(\zeta_k). \quad (2.8)$$

Let $\varrho \in \mathfrak{C}_1(\mathcal{H})$ be a density matrix on $\mathcal{H}$. Then for the system $\mathcal{S} + \mathcal{C}$, the Hamiltonian evolution $T_t : \varrho \mapsto \varrho(t)$ of initial density matrix $\varrho(0) := \varrho$ is defined as a solution of the Cauchy problem for the *non-autonomous* Liouville equation

$$\partial_t \varrho(t) = L(t)(\varrho(t)), \quad \varrho(t)|_{t=0} = \varrho. \quad (2.9)$$

By virtue of (1.6) the equation (2.9) is *autonomous* for each of the interval $[(n-1)\tau, n\tau]$:

$$L(t)(\cdot) = L_n(\cdot) = -i[H_n, \cdot], \quad t \in [(n-1)\tau, n\tau], \quad n \geq 1. \quad (2.10)$$

Since any $t \geq 0$ has the representation:

$$t := n(t)\tau + \nu(t), \quad n(t) := [t/\tau] \quad \text{and} \quad \nu(t) \in [0, \tau), \quad (2.11)$$

by the Markovian independence of generators (2.10), the trace-norm ($\|\cdot\|_1$)-continuous solution of the Cauchy problem (2.9) takes the iterative form:

$$\begin{aligned} \varrho(t) = T_t(\varrho) &:= T_{\nu(t), n}(T_{\tau, n-1}(\dots T_{\tau, 1}(\varrho) \dots)) = \\ &e^{-i\nu(t)H_n} e^{-i\tau H_{n-1}} \dots e^{-i\tau H_1} \varrho e^{i\tau H_1} \dots e^{i\tau H_{n-1}} e^{i\nu(t)H_n}. \end{aligned} \quad (2.12)$$

Here $t \in [(n-1)\tau, n\tau)$, $n = n(t) < N$. By the $\|\cdot\|_1$ -continuity we obtain from (2.12) that

$$\varrho(N\tau - 0) = \varrho(N\tau) = T_{N\tau}(\varrho) = e^{-i\tau H_N} \dots e^{-i\tau H_1} \varrho e^{i\tau H_1} \dots e^{i\tau H_N}. \quad (2.13)$$

Note that equivalent and often more convenient description of evolution of the systems $\mathcal{S} + \mathcal{C}$ is the *dual* dynamics $T_t^* : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$:

$$\omega_{\varrho(t)}(A) = \text{Tr}_{\mathcal{H}}(T_t(\varrho) A) =: \text{Tr}_{\mathcal{H}}(\varrho T_t^*(A)), \text{ for } (\varrho, A) \in \mathfrak{C}_1(\mathcal{F}) \times \mathcal{L}(\mathcal{H}). \quad (2.14)$$

Remark 2.1 Below we show that T_t^* maps $\mathcal{A}(\mathcal{H})$ into itself, and that the action of T_t^* on Weyl operators can be calculated in the explicit form. Since $\mathcal{A}(\mathcal{H})$ is weak*-dense in $\mathcal{L}(\mathcal{H})$, these allow to deduce properties of evolution $\rho(t)$, see [AJP1] (Lectures 2 and 4).

Using (2.13) and dual representation (2.14), we obtain the main result of this section.

Proposition 2.2 For $t = N\tau$, the expectation (2.8) of the Weyl operator (2.5) with respect to the evolved state has the form

$$\omega_{\rho(N\tau)}(W(\zeta)) = \omega_{\rho}(W(U_1 \dots U_N \zeta)) = \prod_{k=0}^N C_k((U_1 \dots U_N \zeta)_k). \quad (2.15)$$

Here

$$(U_1 \dots U_N \zeta)_0 = e^{iN\tau\epsilon}((gz)^N \zeta_0 + \sum_{j=1}^N gw(gz)^{j-1} \zeta_j), \quad (2.16)$$

whereas

$$(U_1 \dots U_N \zeta)_k = e^{iN\tau\epsilon}(gw(gz)^{N-k} \zeta_0 + g\bar{z}\zeta_k + \sum_{j=k+1}^N g^2 w^2(gz)^{j-k-1} \zeta_j), \quad (2.17)$$

for $0 < k < N$, and

$$(U_1 \dots U_N \zeta)_N = e^{iN\tau\epsilon}(gw\zeta_0 + g\bar{z}\zeta_N), \quad (2.18)$$

see definitions (1.10) and (1.11).

Proof : Note that (2.8), (2.13) and duality (2.14) yield

$$\begin{aligned} \omega_{\rho(N\tau)}(W(\zeta)) &= \text{Tr}_{\mathcal{H}}[\rho T_{N\tau}^*(W(\zeta))] = \text{Tr}_{\mathcal{H}}[\rho e^{i\tau H_1} \dots e^{i\tau H_N} W(\zeta) e^{-i\tau H_N} \dots e^{-i\tau H_1}] \\ &= \text{Tr}_{\mathcal{H}}[\rho W(U_1 \dots U_N \zeta)] = \prod_{k=0}^N C_k((U_1 \dots U_N \zeta)_k). \end{aligned} \quad (2.19)$$

To generate the mapping $\zeta \mapsto U_1 \dots U_N \zeta$ in (2.19), we use Lemma 1.1 and sesquilinear forms (2.4) to obtain

$$e^{i\tau H_1} \dots e^{i\tau H_N} \langle \zeta, b \rangle e^{-i\tau H_N} \dots e^{-i\tau H_1} = \langle \zeta, U_N^* \dots U_1^* b \rangle = \langle U_1 \dots U_N \zeta, b \rangle, \quad (2.20)$$

and the similar expression for its conjugate, which we then insert into (2.5).

Moreover, by the same Lemma 1.1, we get that $U_1 \dots U_N \zeta = e^{iN\tau\epsilon} V_1 \dots V_N \zeta$, where

$$(V_1 \dots V_N)_{0j} = \begin{cases} (V_1)_{00} \dots (V_N)_{00} = (gz)^N & (j = 0) \\ (V_1)_{00} \dots (V_{j-1})_{00} (V_j)_{0j} (V_{j+1})_{jj} \dots (V_N)_{jj} = (gz)^{j-1} gw & (0 < j \leq N), \end{cases}$$

and for $0 < k \leq N$:

$$(V_1 \dots V_N)_{kj} = \begin{cases} (V_1 \dots V_{k-1})_{kk} (V_k)_{k0} (V_{k+1} \dots V_N)_{00} = gw(gz)^{N-k} & (j = 0) \\ 0 & (0 < j < k) \\ (V_1 \dots V_{k-1})_{kk} (V_k)_{kk} (V_{k+1} \dots V_N)_{kk} = g\bar{z} & (j = k) \\ (V_1 \dots V_{k-1})_{kk} (V_k)_{k0} (V_{k+1} \dots V_{j-1})_{00} (V_j)_{0j} (V_{j+1} \dots V_N)_{jj} \\ = gw(gz)^{j-k-1} gw & (k < j \leq N). \end{cases}$$

Collecting these formulae, one obtains explicit expressions for components (2.16) and (2.17) of the vector $U_1 \dots U_N \zeta$. $\square$

Remark 2.3 Note that for a fixed N and for any $t = m\tau$, $1 \leq m \leq N$, the arguments of Lemma 2.2 give a general formula

$$\omega_{\rho(m\tau)}(W(\zeta)) = \omega_{\rho}(T_{m\tau}^*(W(\zeta))) = \omega_{\rho}(W(U_1 \dots U_m \zeta)) = \prod_{k=0}^N C_k((U_1 \dots U_m \zeta)_k). \quad (2.21)$$

Following the same line of reasoning as for (2.17) one obtains explicit formulae for the components $\{(U_1 \dots U_m \zeta)_k\}_{k=0}^N$:

$$(U_1 \dots U_m \zeta)_k = \begin{cases} e^{im\tau\epsilon} ((gz)^m \zeta_0 + \sum_{j=1}^m gw(gz)^{j-1} \zeta_j) & (k = 0) \\ e^{im\tau\epsilon} (gw(gz)^{m-k} \zeta_0 + g\bar{z}\zeta_k + \sum_{j=k+1}^m g^2 w^2 (gz)^{j-k-1} \zeta_j) & (1 \leq k < m) \\ e^{im\tau\epsilon} (gw\zeta_0 + g\bar{z}\zeta_m) & (k = m) \\ e^{im\tau\epsilon} \zeta_k & (m < k \leq N) \end{cases}$$

Note that for $m = N$, these formulae coincide with (2.16)-(2.18), except the last line, which is void in this case.

Recall that unity preserving $*$ -dynamics $t \mapsto T_t^*$ on the von Neumann algebra $\mathfrak{M}(\mathcal{H})$ generated by $\{W(\zeta)\}_{\zeta \in \mathbb{C}}$ (2.5) is *quasi-free*, if there exist a mapping $U_t : \zeta \mapsto U_t \zeta$ and a complex-valued function $\Omega_t : \zeta \mapsto \Omega_t(\zeta)$, such that

$$T_t^*(W(\zeta)) = \Omega_t(\zeta) W(U_t \zeta), \quad \Omega_0 = 1, \quad U_0 = I, \quad (2.22)$$

see e.g. [DVV], [AJP1] (Lecture 4) or [BR2]. Then by Remark 2.3, the stepwise dynamics

$$T_{m\tau}^*(W(\zeta)) = W(U_1 \dots U_m \zeta), \quad m = 0, 1, \dots, N$$

is quasi-free, with $\Omega_t(\zeta) = 1$ and the matrices $\{U_j\}_{j=1}^N$ on $\mathbb{C}^{N+1}$ defined by Lemma 1.1.

3 Entropy of Quasi-Free States on CCR C^* -Algebras

In this section, we establish some useful formulae relating expectations of the Weyl operators (Weyl characteristic function) and the entropy of boson quasi-free states. For the reader convenience we formulate them in a way which is restricted but sufficient for our purposes. For general settings one can consult [Fa], [AJP1], [BR2], [Ve] and references therein.

Definition 3.1 *A state ω on the CCR C^* -algebra $\mathcal{A}(\mathcal{F})$ (2.1) is called quasi-free, if its characteristic function has the form*

$$\omega(\widehat{w}(\alpha)) := e^{-\frac{1}{4}|\alpha|^2 - \frac{1}{2}h(\alpha)} \quad , \quad \alpha \in \mathbb{C} \quad , \quad (3.1)$$

where $h : \alpha \mapsto \widehat{h}(\alpha, \alpha)$ is a (closable) non-negative sesquilinear form on $\mathbb{C} \times \mathbb{C}$. A quasi-free state ω is gauge-invariant if $\omega(\widehat{w}(\alpha)) = \omega(\widehat{w}(e^{i\varphi}\alpha))$ for $\varphi \in [0, 2\pi)$.

Let ω_β denote the Gibbs state with parameter β (dimensionless inverse temperature) given by the density matrix $\rho(\beta) = e^{-\beta a^* a} / Z(\beta)$, where $Z(\beta) = (1 - e^{-\beta})^{-1}$. Since

$$\omega_\beta(\widehat{w}(\alpha)) = e^{-\frac{1}{4}|\alpha|^2 - \frac{1}{2}h_\beta(\alpha)} \quad , \quad h_\beta(\alpha) = \frac{|\alpha|^2}{e^\beta - 1} \quad , \quad \alpha \in \mathbb{C} \quad , \quad (3.2)$$

this state is quasi-free and gauge-invariant. Note that the entropy of ω_β is given by

$$s(\beta) := -\text{Tr}_{\mathcal{F}}[\rho(\beta) \ln \rho(\beta)] = \beta \omega_\beta(a^* a) - \ln(1 - e^{-\beta}) \quad \text{and} \quad \omega_\beta(a^* a) = \frac{1}{e^\beta - 1} \quad . \quad (3.3)$$

In terms of the variable $x := (1 + e^{-\beta}) / (1 - e^{-\beta})$ the entropy (3.3) is

$$s(\beta) = \sigma(x) := \frac{x+1}{2} \ln \frac{x+1}{2} - \frac{x-1}{2} \ln \frac{x-1}{2} \quad . \quad (3.4)$$

Here $\sigma : (1, \infty) \rightarrow (0, \infty)$ and $\sigma'(x) > 0$.

To extend (3.4) to the space (1.1) we note that a general *gauge-invariant* quasi-free states on the CCR C^* -algebra $\mathcal{A}(\mathcal{H})$ are defined by density matrices of the form [Ve]:

$$\rho_L = \frac{1}{Z_L} e^{-\langle b, Lb \rangle} \quad , \quad Z_L = \det[I - e^{-L}]^{-1} \quad . \quad (3.5)$$

Here sesquilinear operator-valued forms $\langle b, Lb \rangle = \sum_{n,m=0}^N \ell_{nm} b_n^* b_m$ are parameterised by $(N+1) \times (N+1)$ *positive-definite* Hermitian matrix $L = \{\ell_{nm}\}_{0 \leq n,m \leq N}$. Note that the $*$ -automorphism G_φ on $\mathcal{A}(\mathcal{H})$ (the gauge transformation) :

$$G_\varphi : b_n^* \mapsto b_n^* e^{i\varphi} \quad , \quad b_m \mapsto b_m e^{-i\varphi} \quad (\varphi \in \mathbb{R} \quad , \quad n, m = 0, 1, \dots, N) \quad , \quad (3.6)$$

leaves the state (3.5) invariant. Then characteristic function of the Weyl operators $W(\zeta)$ takes the form

$$\omega_{\rho_L}(W(\zeta)) = \text{Tr}_{\mathcal{H}}[\rho_L W(\zeta)] = \exp \left[-\frac{1}{4} \langle \zeta, \zeta \rangle - \frac{1}{2} \left\langle \zeta, \frac{I}{e^L - I} \zeta \right\rangle \right] \quad . \quad (3.7)$$

Here $\zeta = (\zeta^{\text{tr}})^{\text{tr}}$, where *transposition* of this vector is equal to $\zeta^{\text{tr}} := (\zeta_0, \zeta_1, \dots, \zeta_N) \in \mathbb{C}^{N+1}$. Note that the entropy of the state ω_{ρ_L} is given by

$$S(\rho_L) = -\text{Tr}_{\mathcal{H}}[\rho_L \ln \rho_L] = \text{tr}_{\mathbb{C}^{N+1}}[L(e^L - I)^{-1} - \ln(I - e^{-L})] . \quad (3.8)$$

If we define the matrix $X := (I + e^{-L})(I - e^{-L})^{-1}$, then the characteristic function (3.7) takes the form:

$$\omega_{\rho_L}(W(\zeta)) = \exp \left[-\frac{1}{4} \langle \zeta, X \zeta \rangle \right] , \quad (3.9)$$

and for the entropy (3.8) we obtain

$$S(\rho_L) = \text{tr} \left[\frac{X + I}{2} \ln \frac{X + I}{2} - \frac{X - I}{2} \ln \frac{X - I}{2} \right] . \quad (3.10)$$

Below we need a bit more specified set up than (3.9), (3.10). Let $\rho(\beta, \delta; \xi)$ be density matrix of a quasi-free state (3.5) corresponding to the operator-valued sesquilinear form

$$\langle b, L(\beta, \delta; \xi) b \rangle := \beta \sum_{n=0}^N b_n^* b_n + \delta \langle b, \xi \rangle \langle \xi, b \rangle . \quad (3.11)$$

on $\mathbb{C}^{N+1} \times \mathbb{C}^{N+1}$. Here $\beta > 0$, $\delta > -\beta$, and the vector $\xi^{\text{tr}} = (\xi_0, \xi_1, \dots, \xi_N) \in \mathbb{C}^{N+1}$.

Lemma 3.2 *The partition function of the state*

$$\rho(\beta, \delta; \xi) = \frac{1}{Z(\beta, \delta; \xi)} \exp \left[-\langle b, L(\beta, \delta; \xi) b \rangle \right] ,$$

is given by

$$Z(\beta, \delta; \xi) = \text{Tr}_{\mathcal{H}}[e^{-\langle b, L(\beta, \delta; \xi) b \rangle}] = (1 - e^{-\beta})^{-N} (1 - e^{-(\beta + \delta \langle \xi, \xi \rangle)})^{-1} . \quad (3.12)$$

The characteristic function and the entropy of this state are respectively:

$$\begin{aligned} \text{Tr}_{\mathcal{H}}[\rho(\beta, \delta; \xi) W(\zeta)] &= \exp \left[-\frac{1}{4} \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \langle \zeta, \zeta \rangle \right] \\ &\times \exp \left[-\frac{1}{4} \left(\frac{1 + e^{-\beta - \delta \langle \xi, \xi \rangle}}{1 - e^{-\beta - \delta \langle \xi, \xi \rangle}} - \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \right) |\langle \xi, \zeta \rangle|^2 / \langle \xi, \xi \rangle \right] , \end{aligned} \quad (3.13)$$

and

$$S(\rho(\beta, \delta; \xi)) = -\text{Tr}_{\mathcal{H}}[\rho(\beta, \delta; \xi) \ln \rho(\beta, \delta; \xi)] = N s(\beta) + s(\beta + \delta \langle \xi, \xi \rangle) . \quad (3.14)$$

Proof: Proof of (3.12) follows from (3.5) and (3.11). Indeed, since by (3.5) any orthogonal transformation $\mathcal{O}$ on $\mathbb{C}^{N+1}$ leaves the partition function invariant: $Z_{\mathcal{O}^T L \mathcal{O}} = Z_L$, one can calculate it with $\mathcal{O}\xi$ (instead of ξ), where $\mathcal{O}\xi$ has only one non-zero component equals to the vector norm $\langle \xi, \xi \rangle^{1/2}$. Then the right-hand side of (3.12) follows straightforwardly from the calculation of the left-hand side for this choice of $\mathcal{O}\xi$.

Since this transformation $\mathcal{O}$ also diagonalise the matrix $L := L(\beta, \delta; \xi)$, one uses it to simplify (3.9) and then to return back to ξ at the last step. To this aim we note that

$$\begin{aligned} \omega_{\rho_L}(W(\zeta)) &= \exp \left[-\frac{1}{4} \langle \mathcal{O}\zeta, \mathcal{O}X\mathcal{O}^*\mathcal{O}\zeta \rangle \right] = \\ &\exp \left[-\frac{1}{4} \frac{1+e^{-\beta}}{1-e^{-\beta}} \langle \mathcal{O}\zeta, \mathcal{O}\zeta \rangle' \right] \exp \left[-\frac{1}{4} \frac{1+e^{-\beta-\delta\langle \xi, \xi \rangle}}{1-e^{-\beta-\delta\langle \xi, \xi \rangle}} |(\mathcal{O}\zeta)_0|^2 \right]. \end{aligned} \quad (3.15)$$

Here $\langle \mathcal{O}\zeta, \mathcal{O}\zeta \rangle' := \sum_{k=1}^N |(\mathcal{O}\zeta)_k|^2$ and we choose transformation $\mathcal{O}$ in such a way that $(\mathcal{O}\xi)_j = \delta_{0,j} \|\xi\|$. Since

$$|(\mathcal{O}\zeta)_0|^2 = \frac{1}{\langle \xi, \xi \rangle} \langle \mathcal{O}\zeta, \mathcal{O}\xi \rangle \langle \mathcal{O}\xi, \mathcal{O}\zeta \rangle, \quad (3.16)$$

the identities (3.15) prove (3.13). The same method is valid for entropy (3.8). Calculation of the trace in diagonal representation for $L = L(\beta, \delta; \xi)$ gives formula (3.14). $\square$

Recall that the state ω on the CCR C^* -algebra $\mathcal{A}(\mathcal{H})$ is *regular*, if the map $s \mapsto \omega(W(s\zeta))$ is a continuous function of $s \in \mathbb{R}$ for any $\zeta \in \mathbb{C}^{N+1}$. This property follows from the explicit expression (3.13). Since by the Araki-Segal theorem, see e.g. [AJP1](Lecture 5), a regular state is completely defined by its characteristic function, (3.13) and (3.14) yield the following statement.

Proposition 3.3 *The entropy $S(\rho)$ of the quasi-free state ω_ρ on the CCR C^* -algebra $\mathcal{A}(\mathcal{H})$ with characteristic function*

$$\omega_\rho(W(\zeta)) = \exp \left[-\frac{1}{4} (x\langle \zeta, \zeta \rangle + x_0 |\langle \xi, \zeta \rangle|^2) \right] \quad (3.17)$$

is uniquely determined by the parameters (ξ, x, x_0) , where $\xi \in \mathbb{C}^{N+1}$, $x > 1$, $x_0 > 1 - x$ and it has the form

$$S(\rho) = N\sigma(x) + \sigma(x + x_0\langle \xi, \xi \rangle), \quad (3.18)$$

where $\sigma(\cdot)$ is defined by (3.4).

Proof : The proof follows directly from definition (3.4), if one puts

$$x_0 \langle \xi, \xi \rangle = \frac{1 + e^{-\beta-\delta\langle \xi, \xi \rangle}}{1 - e^{-\beta-\delta\langle \xi, \xi \rangle}} - \frac{1 + e^{-\beta}}{1 - e^{-\beta}},$$

in (3.13) and uses (3.4) in (3.14). $\square$

4 Repeated Perturbations and Entropy Production

We consider evolution (2.12) of the system $\mathcal{S} + \mathcal{C}$, when initial density matrix (2.7) corresponds to the product of gauge-invariant Gibbs quasi-free states with parameter $\beta_0 \geq 0$ for $\mathcal{S}$ and with parameter $\beta \geq 0$ for $\mathcal{C}$:

$$\rho = \rho_0 \otimes \bigotimes_{k=1}^N \rho_k, \quad \rho_0 = e^{-\beta_0 a^* a} / Z(\beta_0), \quad \rho_k = e^{-\beta a^* a} / Z(\beta), \quad k = 1, 2, \dots, N. \quad (4.1)$$

This case corresponds to ρ_L in (3.5) with diagonal matrix $L = \text{diag}(\beta_0, \beta, \dots, \beta)$ and to $\rho(\beta, \delta; \xi)$ in representation (3.11) with $(\beta, \delta; \xi) = (\beta, \beta_0 - \beta; e)$, i.e.,

$$\rho = \rho(\beta, \beta_0 - \beta; e) = \exp \left[-\beta_0 b_0^* b_0 - \beta \sum_{j=1}^N b_j^* b_j \right] / Z(\beta, \beta_0 - \beta) . \quad (4.2)$$

Here $e^{\text{tr}} = (1, 0, \dots, 0) \in \mathbb{C}^{N+1}$ and

$$Z(\beta, \beta_0 - \beta) = Z(\beta_0) Z(\beta)^N = \frac{1}{(1 - e^{-\beta_0})(1 - e^{-\beta})^N} .$$

A straightforward application of formulae (3.13), (3.14) and Lemma 3.2 for $\xi = e$ (i.e. for $\langle \xi, \xi \rangle = 1$, $\langle \xi, \zeta \rangle = \zeta_0$) to the state (4.1) (or (4.2)), yields the following statement:

Lemma 4.1 *The characteristic function of (4.1) (or (4.2)) is*

$$\begin{aligned} \omega_\rho(W(\zeta)) &= \text{Tr}_{\mathcal{H}}[\rho W(\zeta)] = \\ &\exp \left[-\frac{|\zeta_0|^2}{4} \left(\frac{1 + e^{-\beta_0}}{1 - e^{-\beta_0}} - \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \right) - \frac{\langle \zeta, \zeta \rangle}{4} \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \right] , \end{aligned} \quad (4.3)$$

and the entropy is equal to

$$S(\rho) = Ns(\beta) + s(\beta_0) . \quad (4.4)$$

Lemma 4.2 *Characteristic function of the state with density matrix $\rho(N\tau)$ is equal to*

$$\omega_{\rho(N\tau)}(W(\zeta)) = \exp \left[-\frac{|(U_1 \dots U_N \zeta)_0|^2}{4} \left(\frac{1 + e^{-\beta_0}}{1 - e^{-\beta_0}} - \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \right) - \frac{\langle \zeta, \zeta \rangle}{4} \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \right] , \quad (4.5)$$

whereas the total entropy rests invariant:

$$S(\rho(N\tau)) = S(\rho) = Ns(\beta) + s(\beta_0) .$$

Here the mapping $U_1 \dots U_N : \mathbb{C}^{N+1} \rightarrow \mathbb{C}^{N+1}$ is given by (2.16) and (2.17).

Proof : From (2.15), one gets $\omega_{\rho(N\tau)}(W(\zeta)) = \omega_\rho(W(U_1 \dots U_N \zeta))$. Since the mappings $U_j : \mathbb{C}^{N+1} \rightarrow \mathbb{C}^{N+1}$, $j = 1, \dots, N$ are unitary (Lemma 2.2), (4.3) yields (4.5). Finally, we obtain that the mapping (2.12) leaves the total entropy (4.4) invariant, see (3.3). $\square$

Let ω and ω_0 be two normal states on the Weyl CCR algebra $\mathcal{A}(\mathcal{H})$ with density matrices ϱ and ϱ_0 . Following Araki [Ar1] (see also [AJP3], Lectures 1 and 3) we introduce the *relative entropy* of the state ω with respect to ω_0 :

$$\text{Ent}(\varrho|\varrho_0) := \text{Tr}_{\mathcal{H}}[\varrho(\ln \varrho - \ln \varrho_0)] \geq 0 . \quad (4.6)$$

Proposition 4.3 *The relative entropy of $\omega_{\rho(N\tau)}$ with respect to ω_ρ is*

$$\text{Ent}(\rho(N\tau)|\rho) = \frac{(\beta_0 - \beta)(e^{\beta_0} - e^\beta)}{(e^{\beta_0} - 1)(e^\beta - 1)} (1 - |z|^{2N}) , \quad (4.7)$$

where $z := z(\tau)$ is defined by (1.11) and (1.20).

Proof : The trace cyclicity yields

$$\begin{aligned}
\text{Ent}(\rho(N\tau)|\rho) &= \text{Tr}_{\mathcal{H}}[\rho(N\tau)(\ln \rho(N\tau) - \ln \rho)] \\
&= \text{Tr}_{\mathcal{H}}[\rho(\ln \rho - e^{i\tau H_1} \dots e^{i\tau H_N} \ln \rho e^{-i\tau H_N} \dots e^{-i\tau H_1})] \\
&= \frac{\beta - \beta_0}{Z(\beta, \beta_0 - \beta)} \text{Tr}_{\mathcal{H}}[e^{-\beta_0 b_0^* b_0 - \beta \sum_{j=1}^N b_j^* b_j} (b_0^* b_0 - e^{i\tau H_1} \dots e^{i\tau H_N} b_0^* b_0 e^{-i\tau H_N} \dots e^{-i\tau H_1})] .
\end{aligned} \tag{4.8}$$

Note that one gets $b_0^* b_0 = \langle b, e \rangle \langle e, b \rangle$ by (2.4). Hence, (2.20) implies

$$e^{i\tau H_1} \dots e^{i\tau H_N} b_0^* b_0 e^{-i\tau H_N} \dots e^{-i\tau H_1} = \sum_{k=0}^N (U_1 \dots U_N e)_k b_k^* \sum_{k'=0}^N \overline{(U_1 \dots U_N e)_{k'}} b_{k'} . \tag{4.9}$$

Note also that the gauge invariance of the state ρ implies the selection rule:

$$\frac{1}{Z(\beta, \beta_0 - \beta)} \text{Tr}_{\mathcal{H}}[e^{-\beta_0 b_0^* b_0 - \beta \sum_{j=1}^N b_j^* b_j} b_k^* b_{k'}] = 0 \quad \text{for } k \neq k' . \tag{4.10}$$

By this rule after injection of (4.9) into (4.8) only diagonal terms with $k = k'$ survive in the expectation:

$$\begin{aligned}
\text{Ent}(\rho(N\tau)|\rho) &= \\
&= \frac{\beta - \beta_0}{Z(\beta, \beta_0 - \beta)} \text{Tr}_{\mathcal{H}}[e^{-\beta_0 b_0^* b_0 - \beta \sum_{j=1}^N b_j^* b_j} (b_0^* b_0 - \sum_{k=0}^N |(U_1 \dots U_N e)_k|^2 b_k^* b_k)] .
\end{aligned}$$

Finally, by Lemma 2.2, (2.16), (2.17), and by (3.3), we obtain

$$\begin{aligned}
\text{Ent}(\rho(N\tau)|\rho) &= \\
&= \frac{\beta - \beta_0}{Z(\beta, \beta_0 - \beta)} \text{Tr}_{\mathcal{H}}[e^{-\beta_0 b_0^* b_0 - \beta \sum_{j=1}^N b_j^* b_j} ((1 - |z|^{2N}) b_0^* b_0 - \sum_{k=1}^N |w|^2 |z|^{2N-2k} b_k^* b_k)] \\
&= \frac{(\beta_0 - \beta)(e^{\beta_0} - e^{\beta})}{(e^{\beta_0} - 1)(e^{\beta} - 1)} (1 - |z|^{2N}) ,
\end{aligned}$$

that proves (4.7). $\square$

The relative entropy defined by (4.6) is non-negative. In contrast to *invariant* total entropy (Lemma 4.2), the relative entropy (4.7) is a monotonously increasing function of time $t = N\tau$, for $|z| < 1$ (see Lemma 1.1, Remark 1.2). It converges to the limit:

$$\lim_{N \rightarrow \infty} \text{Ent}(\rho(N\tau)|\rho) = (\beta - \beta_0) \left[\frac{1}{e^{\beta_0} - 1} - \frac{1}{e^{\beta} - 1} \right] \geq 0 , \tag{4.11}$$

which is positive for $\beta_0 \neq \beta$. The limit (4.11) gives asymptotic amount of the entropy production, when one starts with the initial product state corresponding to (4.1) and then consider $N\tau \rightarrow \infty$, see [BJM].

5 Evolution of Subsystems

Subsystem $\mathcal{S}$. We start with the simplest subsystem $\mathcal{S}$. Let the initial state of the total system $\mathcal{S} + \mathcal{C}$ in (1.1) be a tensor-product of the corresponding density matrices $\rho = \rho_S \otimes \rho_C$, see (H1). Then for $t \geq 0$ the state $\omega_S^t(\cdot)$ of the subsystem $\mathcal{S}$ is given on the Weyl C^* -algebra $\mathcal{A}(\mathcal{H}_0)$ by

$$\omega_S^t(\cdot) := \omega_{\rho(t)}(\cdot \otimes \mathbb{1}) . \quad (5.1)$$

For $\zeta^{\text{tr}} = (\alpha, 0, \dots, 0) \in \mathbb{C}^{N+1}$, we consider the Weyl operator $W(\zeta) = \widehat{w}(\alpha) \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}$ (2.3). By virtue of (2.8), (2.21) and (5.1), we obtain for $t = m\tau$ ($1 \leq m \leq N$):

$$\omega_S^{m\tau}(\widehat{w}(\alpha)) = \omega_{\rho(m\tau)}(W(\zeta)) = \omega_{\rho}(W(U_1 \dots U_m \zeta)) . \quad (5.2)$$

Then for components $\{(U_1 \dots U_m \zeta)_k\}_{k=0}^N$ of the vector $U_1 \dots U_m \zeta$ in (5.2), one obtains the expression:

$$(U_1 \dots U_m \zeta)_k = \begin{cases} e^{im\tau\epsilon}(gz)^m \alpha & (k=0) \\ e^{im\tau\epsilon} g w(gz)^{m-k} \alpha & (1 \leq k < m) \\ e^{im\tau\epsilon} g w \alpha & (k=m) \\ 0 & (m < k \leq N) , \end{cases} \quad (5.3)$$

which follows from Remark 2.3.

If the initial density matrices: $\rho = \rho_S \otimes \rho_C$ corresponds to the product of Gibbs quasi-free states for different temperatures as in (4.1), then (5.2) and Lemma 4.1 yield

$$\omega_S^{m\tau}(\widehat{w}(\alpha)) = \exp \left[-\frac{|\alpha|^2}{4} \frac{1+e^{-\beta}}{1-e^{-\beta}} - \frac{|z^m \alpha|^2}{4} \left(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}} \right) \right] \quad (5.4)$$

Note that for any moment $t = m\tau$ the state $\omega_S^{m\tau}(\cdot)$ is a quasi-free Gibbs equilibrium state with parameter $\beta^*(m\tau)$ which satisfies the equation

$$\frac{1+e^{-\beta^*(m\tau)}}{1-e^{-\beta^*(m\tau)}} = |z|^{2m} \frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} + (1-|z|^{2m}) \frac{1+e^{-\beta}}{1-e^{-\beta}} . \quad (5.5)$$

This equation yields that either $\beta \leq \beta^*(m\tau) \leq \beta_0$, or $\beta_0 \leq \beta^*(m\tau) \leq \beta$.

For $m \rightarrow \infty$ ($N \rightarrow \infty$) the Weyl characteristic function (5.4) has the limit

$$\lim_{m \rightarrow \infty} \omega_S^{m\tau}(\widehat{w}(\alpha)) = \exp \left[-\frac{|\alpha|^2}{4} \frac{1+e^{-\beta}}{1-e^{-\beta}} \right] . \quad (5.6)$$

Hence, in the limit $t \rightarrow \infty$ the subsystem $\mathcal{S}$ evolves from the Gibbs equilibrium state with parameter β_0 to another equilibrium state with parameter β imposed by the chain $\mathcal{C}$.

Subsystem $\mathcal{S}_1$. The initial state $\omega_{\mathcal{S}_1}^0(\cdot) = \omega_{\mathcal{S}_1}^t(\cdot)|_{t=0}$ of this subsystem corresponds to a one-point reduced density matrix or to the partial trace on the CCR Weyl algebra $\mathcal{A}(\mathcal{H}_1)$:

$$\omega_{\mathcal{S}_1}^0(\widehat{w}(\alpha)) = \omega_{\rho}(\mathbb{1} \otimes \widehat{w}(\alpha) \otimes \bigotimes_{k=2}^N \mathbb{1}) = \exp \left[-\frac{|\alpha|^2}{4} \frac{1+e^{-\beta}}{1-e^{-\beta}} \right] . \quad (5.7)$$

Now we choose vector $(\zeta^1)^{\text{tr}} := (0, \alpha, 0, \dots, 0) \in \mathbb{C}^{N+1}$. Then

$$\omega_{\mathcal{S}_1}^{m\tau}(\widehat{w}(\alpha)) = \omega_{\rho(m\tau)}(W(\zeta^{(1)})) = \omega_{\rho_S \otimes \rho_C}(W(U_1 \dots U_m \zeta^{(1)})) \quad (5.8)$$

for $1 < m \leq N$. By Remark 2.3, the components $\{(U_1 \dots U_m \zeta^{(1)})_k\}_{k=0}^N$ are:

$$(U_1 \dots U_m \zeta)_k = \begin{cases} e^{im\tau\epsilon} g w \alpha & (k = 0) \\ e^{im\tau\epsilon} \delta_{k,1} g \bar{z} \alpha & (1 \leq k < m) \\ 0 & (m \leq k \leq N). \end{cases} \quad (5.9)$$

Then, we obtain

$$\omega_{\mathcal{S}_1}^{m\tau}(\widehat{w}(\alpha)) = \exp \left[-\frac{|\alpha|^2}{4} \frac{1+e^{-\beta}}{1-e^{-\beta}} - \frac{|w\alpha|^2}{4} \left(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}} \right) \right] \quad (5.10)$$

for any $1 < m \leq N$. Therefore, the initial state (5.7) changes to (5.10) after the first act of interaction on the interval $[0, \tau)$ and there is no further evolution of this state for $t > \tau$.

Note that (5.10) is characteristic function of a quasi-free Gibbs equilibrium state with parameter β^* , which satisfies the equation

$$\frac{1+e^{-\beta^*}}{1-e^{-\beta^*}} = |w|^2 \frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} + (1-|w|^2) \frac{1+e^{-\beta}}{1-e^{-\beta}}.$$

Again, this equation implies that either $\beta \leq \beta^* \leq \beta_0$, or $\beta_0 \leq \beta^* \leq \beta$.

Evolution of $\mathcal{S}_1$ has a transparent physical interpretation: after the one act of interaction during the time $t \in [0, \tau)$, subsystem $\mathcal{S}_1$ relaxes to an *intermediate* equilibrium with the subsystem $\mathcal{S}$. This manifests in a shift of initial parameter β to β^* , which rests unchangeable since there is no perturbations of subsystem $\mathcal{S}_1$ for $t > \tau$.

Subsystem $\mathcal{S}_m$. For $1 < m \leq N$ the initial state $\omega_{\mathcal{S}_m}^0(\cdot) = \omega_{\mathcal{S}_m}^t(\cdot)|_{t=0}$ of this subsystem is defined on the CCR Weyl algebra $\mathcal{A}(\mathcal{H}_m)$ by the partial trace :

$$\omega_{\mathcal{S}_m}^0(\widehat{w}(\alpha)) = \omega_{\rho} \left(\bigotimes_{k=0}^{m-1} \mathbb{1} \otimes \widehat{w}(\alpha) \otimes \bigotimes_{k=m+1}^N \mathbb{1} \right) = \exp \left[-\frac{|\alpha|^2}{4} \frac{1+e^{-\beta}}{1-e^{-\beta}} \right]. \quad (5.11)$$

Now we choose vector $(\zeta^{(m)})^{\text{tr}} := (0, \dots, 0, \alpha, 0, \dots, 0) \in \mathbb{C}^{N+1}$, where α occupies the $m+1$ position. Consequently

$$\omega_{\mathcal{S}_m}^{m\tau}(\widehat{w}(\alpha)) = \omega_{\rho(m\tau)}(W(\zeta^{(m)})) = \omega_{\rho_S \otimes \rho_C}(W(U_1 \dots U_m \zeta^{(m)})) . \quad (5.12)$$

The components $\{(U_1 \dots U_m \zeta^{(m)})_k\}_{k=0}^N$ are:

$$(U_1 \dots U_m \zeta^{(m)})_k = \begin{cases} e^{im\tau\epsilon} g w (gz)^{m-1} \alpha & (k = 0) \\ e^{im\tau\epsilon} g^2 w^2 (gz)^{m-k-1} \alpha & (1 \leq k < m) \\ e^{im\tau\epsilon} g \bar{z} \alpha, & (k = m) \\ 0 & (m < k \leq N). \end{cases} \quad (5.13)$$

which again follows from Remark 2.3. Then evolution of the state of subsystem $\mathcal{S}_m$ is:

$$\omega_{\mathcal{S}_m}^{m\tau}(\widehat{w}(\alpha)) = \exp \left[-\frac{|\alpha|^2}{4} \frac{1+e^{-\beta}}{1-e^{-\beta}} - \frac{|w\alpha|^2}{4} |z|^{2(m-1)} \left(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}} \right) \right]. \quad (5.14)$$

Note that interaction for $t \in [(m-1)\tau, m\tau)$ push out the subsystem $\mathcal{S}_m$ from the Gibbs equilibrium state (5.11), but its effect attenuates for large m :

$$\lim_{m \rightarrow \infty} \omega_{\mathcal{S}_m}^{m\tau}(\widehat{w}(\alpha)) = \exp \left[-\frac{|\alpha|^2}{4} \frac{1+e^{-\beta}}{1-e^{-\beta}} \right]. \quad (5.15)$$

Again, this is evolution of a quasi-free Gibbs equilibrium state with time-dependent inverse temperature parameter $\beta^{**}(m\tau)$, which satisfies the equation

$$\frac{1+e^{-\beta^{**}(m\tau)}}{1-e^{-\beta^{**}(m\tau)}} = |w|^2 |z|^{2(m-1)} \frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} + (1-|w|^2 |z|^{2(m-1)}) \frac{1+e^{-\beta}}{1-e^{-\beta}}. \quad (5.16)$$

As above, the value of the parameter $\beta^{**}(m\tau)$ is always between β_0 and β .

To interpret the evolution of $\mathcal{S}_m$ and the coincidence between (5.15) and (5.6) note that the state of the subsystem $\mathcal{S}$ relaxes to that of initial state of the chain $\mathcal{C}$, see (5.6). Therefore, after interaction of the subsystem $\mathcal{S}_m$, i.e. at the moment $t = m\tau$, its parameter $\beta^{**}(m\tau)$ has a value between β and $\beta^*((m-1)\tau)$ since (5.5) and (5.16) yield

$$\frac{1+e^{-\beta^{**}(m\tau)}}{1-e^{-\beta^{**}(m\tau)}} = |w|^2 \frac{1+e^{-\beta^*((m-1)\tau)}}{1-e^{-\beta^*((m-1)\tau)}} + (1-|w|^2) \frac{1+e^{-\beta}}{1-e^{-\beta}}.$$

As in the case $m = 1$, there is no further evolution: $\omega_{\mathcal{S}_m}^{n\tau} = \omega_{\mathcal{S}_m}^{m\tau}$ for $n \geq m$.

Next, we consider the composed subsystems $\mathcal{S} + \mathcal{S}_m$ and $\mathcal{S}_{m-n} + \mathcal{S}_m$. Our aim is to study the indirect *correlations* imposed by repeated interaction via $\mathcal{S}$.

Subsystem $\mathcal{S} + \mathcal{S}_m$. For $1 < m \leq N$ the initial state $\omega_{\mathcal{S}+\mathcal{S}_m}^0(\cdot) = \omega_{\mathcal{S}+\mathcal{S}_m}^t(\cdot)|_{t=0}$ of this *composed* subsystem is defined by the partial trace on the Weyl C^* -algebra $\mathcal{A}(\mathcal{H}_0 \otimes \mathcal{H}_m) \approx \mathcal{A}(\mathcal{H}_0) \otimes \mathcal{A}(\mathcal{H}_m)$ by:

$$\begin{aligned} \omega_{\mathcal{S}+\mathcal{S}_m}^0(\widehat{w}(\alpha_0) \otimes \widehat{w}(\alpha_1)) &:= \omega_\rho(\widehat{w}(\alpha_0) \otimes \bigotimes_{k=1}^{m-1} \mathbb{1} \otimes \widehat{w}(\alpha_1) \otimes \bigotimes_{k=m+1}^N \mathbb{1}) \\ &= \exp \left[-\frac{|\alpha_0|^2}{4} \frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} \right] \exp \left[-\frac{|\alpha_1|^2}{4} \frac{1+e^{-\beta}}{1-e^{-\beta}} \right]. \end{aligned} \quad (5.17)$$

This is the characteristic function of the product state corresponding to two isolated systems with different temperatures. If one defines vector $(\zeta^{(0,m)})^{\text{tr}} := (\alpha_0, 0, \dots, 0, \alpha_1, 0, \dots, 0) \in \mathbb{C}^{N+1}$, where α_1 occupies the $m+1$ position, then

$$\omega_{\mathcal{S}+\mathcal{S}_m}^{m\tau}(\widehat{w}(\alpha_0) \otimes \widehat{w}(\alpha_1)) = \omega_{\rho(m\tau)}(W(\zeta^{(0,m)})) = \omega_{\rho_S \otimes \rho_C}(W(U_1 \dots U_m \zeta^{(0,m)})). \quad (5.18)$$

The components $\{(U_1 \dots U_m \zeta^{(0,m)})_k\}_{k=0}^N$ are deduced from Remark 2.3:

$$(U_1 \dots U_m \zeta^{(0,m)})_k = \begin{cases} e^{im\tau\epsilon} (gz)^{m-1} [gz \alpha_0 + gw \alpha_1], & (k=0) \\ e^{im\tau\epsilon} (gz)^{m-k-1} g^2 [wz \alpha_0 + w^2 \alpha_1], & (1 \leq k < m) \\ e^{im\tau\epsilon} [gw \alpha_0 + g\bar{z} \alpha_1], & (k=m) \\ 0 & (m < k \leq N). \end{cases} \quad (5.19)$$

Together with (2.8), one gets for $m \rightarrow \infty$:

$$\begin{aligned} \omega_{\mathcal{S}+\mathcal{S}_m}^{m\tau}(\widehat{w}(\alpha_0) \otimes \widehat{w}(\alpha_1)) &= \exp \left[-\frac{1}{4}|z\alpha_0 + w\alpha_1|^2 |z|^{2(m-1)} \frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} \right] \\ &\times \exp \left[-\frac{1}{4}|z\alpha_0 + w\alpha_1|^2 (1-|z|^{2(m-1)}) \frac{1+e^{-\beta}}{1-e^{-\beta}} \right] \exp \left[-\frac{1}{4}|w\alpha_0 + \bar{z}\alpha_1|^2 \frac{1+e^{-\beta}}{1-e^{-\beta}} \right] \\ &\longrightarrow \exp \left[-\frac{1}{4}(|\alpha_0|^2 + |\alpha_1|^2) \frac{1+e^{-\beta}}{1-e^{-\beta}} \right]. \end{aligned} \quad (5.20)$$

Hence, in this limit the composed subsystem $\mathcal{S} + \mathcal{S}_m$ evolves from the product of two quasi-free equilibrium states (5.17) with different parameters β_0 and β to the product of quasi-free equilibrium states for the same parameter β imposed by repeated interaction with the chain $\mathcal{C}$, when $m \rightarrow \infty$. Interpretation is similar to the case *Subsystem* $\mathcal{S}_m$.

Subsystem $\mathcal{S}_{m-n} + \mathcal{S}_m$. We suppose that $1 < (m-n) < m \leq N$. Then the initial state $\omega_{\mathcal{S}_{m-n}+\mathcal{S}_m}^t(\cdot)|_{t=0}$ of this *composed* subsystem is the partial trace over the Weyl C^* -algebra $\mathcal{A}(\mathcal{H}_{m-n} \otimes \mathcal{H}_m) \approx \mathcal{A}(\mathcal{H}_{m-n}) \otimes \mathcal{A}(\mathcal{H}_m)$:

$$\begin{aligned} \omega_{\mathcal{S}_{m-n}+\mathcal{S}_m}^0(\widehat{w}(\alpha_1) \otimes \widehat{w}(\alpha_2)) &:= \\ \omega_\rho \left(\bigotimes_{k=0}^{m-n-1} \mathbb{1} \otimes \widehat{w}(\alpha_1) \otimes \bigotimes_{k=m-n+1}^{m-1} \mathbb{1} \otimes \widehat{w}(\alpha_2) \otimes \bigotimes_{k=m+1}^N \mathbb{1} \right) &= \\ = \exp \left[-\frac{|\alpha_1|^2}{4} \frac{1+e^{-\beta}}{1-e^{-\beta}} \right] \exp \left[-\frac{|\alpha_2|^2}{4} \frac{1+e^{-\beta}}{1-e^{-\beta}} \right]. \end{aligned} \quad (5.21)$$

This is the characteristic function of the product state corresponding to two isolated systems with the same temperatures.

We define the vector $(\zeta^{(m-n,m)})^{\text{tr}} := (0, 0, \dots, 0, \alpha_1, 0, \dots, 0, \alpha_2, 0, \dots, 0) \in \mathbb{C}^{N+1}$. Here α_1 and α_2 occupy respectively the $(m-n+1)$ th and the $(m+1)$ th positions, then

$$\begin{aligned} \omega_{\mathcal{S}_{m-n}+\mathcal{S}_m}^{m\tau}(\widehat{w}(\alpha_1) \otimes \widehat{w}(\alpha_2)) &= \\ \omega_{\rho(m\tau)}(W(\zeta^{(m-n,m)})) &= \omega_{\rho_S \otimes \rho_C}(W(U_1 \dots U_m \zeta^{(m-n,m)})) . \end{aligned} \quad (5.22)$$

By Remark 2.3 we obtain for the values of components $\{(U_1 \dots U_m \zeta^{(m-n,m)})_k\}_{k=0}^N$:

$$(U_1 \dots U_m \zeta^{(m-n,m)})_k = \quad (5.23)$$

$$= \begin{cases} e^{im\tau\epsilon} (gz)^{m-n-1} gw[\alpha_1 + (gz)^n \alpha_2] & (k=0) \\ e^{im\tau\epsilon} [g^2 w^2 (gz)^{m-n-k-1} \alpha_1 + g^2 w^2 (gz)^{m-k-1} \alpha_2] & (1 \leq k < m-n) \\ e^{im\tau\epsilon} [g\bar{z} \alpha_1 + g^2 w^2 (gz)^{m-k-1} \alpha_2] & (k=m-n) \\ e^{im\tau\epsilon} g^2 w^2 (gz)^{m-k-1} \alpha_2 & (m-n < k < m) \\ e^{im\tau\epsilon} g\bar{z} \alpha_2 & (k=m) \\ 0 & (m < k \leq N) \end{cases} .$$

When $m \rightarrow \infty$, then for any fixed n we obtain for (5.22):

$$\begin{aligned} \omega_{\mathcal{S}_{m-n} + \mathcal{S}_m}^{m\tau}(\widehat{w}(\alpha_1) \otimes \widehat{w}(\alpha_2)) &= \exp \left[-\frac{1}{4}|w|^2|\alpha_1 + (gz)^{n+1}\alpha_2|^2|z|^{2(m-n-1)}\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} \right] \\ &\times \exp \left[-\frac{1}{4}(\{|w|^2(1-|z|^{2(m-n-1)}) + |z|^2\}|\alpha_1|^2 + (1-|w|^2|z|^{2(m-1)})|\alpha_2|^2)\frac{1+e^{-\beta}}{1-e^{-\beta}} \right] \\ &\longrightarrow \exp \left[-\frac{1}{4}(|\alpha_1|^2 + |\alpha_2|^2)\frac{1+e^{-\beta}}{1-e^{-\beta}} \right]. \end{aligned} \quad (5.24)$$

Therefore, in this limit, the composed subsystem $\mathcal{S}_{m-n} + \mathcal{S}_m$ evolves from the initial product of two quasi-free equilibrium states (5.21) to the *same* final state, although for a finite m the evolution (5.24) is *nontrivial*. This again easily understandable taking into account our analysis of *Subsystem* $\mathcal{S}_m$ and *Subsystem* $\mathcal{S} + \mathcal{S}_m$.

Consider now the case of a *fixed* $s := m - n \geq 1$. Then the limit in (5.24) is

$$\begin{aligned} \lim_{m \rightarrow \infty} \omega_{\mathcal{S}_s + \mathcal{S}_m}^{m\tau}(\widehat{w}(\alpha_1) \otimes \widehat{w}(\alpha_2)) &= \quad (5.25) \\ \exp \left[-\frac{1}{4}|w|^2|z|^{2(s-1)}|\alpha_1|^2 \left\{ \frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}} \right\} \right] \exp \left[-\frac{1}{4}(|\alpha_1|^2 + |\alpha_2|^2)\frac{1+e^{-\beta}}{1-e^{-\beta}} \right] \\ &= \exp \left[-\frac{1}{4}|\alpha_1|^2 \frac{1+e^{-\beta^{**}(s\tau)}}{1-e^{-\beta^{**}(s\tau)}} \right] \exp \left[-\frac{1}{4}|\alpha_2|^2 \frac{1+e^{-\beta}}{1-e^{-\beta}} \right], \end{aligned}$$

where $\beta^{**}(s\tau)$ verifies equation (5.16). Hence, in this case the limit state (5.25) is the product of quasi-free Gibbs states with *different* parameters $\beta^{**}(s\tau)$ and β . This means that subsystem $\mathcal{S}_s$ keeps a *memory* about perturbation at the moment $t = s\tau$, when the parameter $\beta^*(s\tau)$ (5.5) of subsystem $\mathcal{S}$ was still different from β .

Note that (5.25) coincides with the product state (5.21) when $s \rightarrow \infty$.

Subsystem $\mathcal{S}_{\sim n}$. To define $\mathcal{S}_{\sim n}$ for $0 \leq n \leq k \leq N$, we divide the total system at the moment $t = k\tau$ into two subsystems: $\mathcal{S}_{n,k} + \mathcal{C}_{n,k}$. Here

$$\mathcal{S}_{n,k} := \mathcal{S} + \mathcal{S}_k + \mathcal{S}_{k-1} + \cdots + \mathcal{S}_{k-n+1}, \quad (\mathcal{S}_{0,k} := \mathcal{S}), \quad (5.26)$$

whereas

$$\mathcal{C}_{n,k} := \mathcal{S}_N + \cdots + \mathcal{S}_{k+1} + \mathcal{S}_{k-n} + \cdots + \mathcal{S}_1. \quad (5.27)$$

We interpret $\mathcal{S}_{\sim n}$ is an entire “object” whose entity is $\mathcal{S}_{n,k}$ at the moment $t = k\tau$ ($k = n, n+1, \dots, N$). As time is running, the elementary subsystems $\mathcal{S}_k$ in $\mathcal{S}_{\sim n}$ are replacing. We study the behaviour of $\mathcal{S}_{\sim n}$ for large $t = k\tau$, i.e., we analyse the k -dependence of the “state” of $\mathcal{S}_{n,k}$ at $t = k\tau$.

For any fixed $t = k\tau$ we can decompose the Hilbert space $\mathcal{H}$ into tensor product $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_c$. Here $\mathcal{H}_s$ is the Hilbert space of subsystem (5.26) and $\mathcal{H}_c$ corresponds to subsystem (5.27):

$$\mathcal{H}_s := \mathcal{H}_0 \otimes \bigotimes_{j=1}^n \mathcal{H}_{k-j+1}, \quad \mathcal{H}_c := \bigotimes_{j=1}^{k-n} \mathcal{H}_j \otimes \bigotimes_{j=k+1}^N \mathcal{H}_j. \quad (5.28)$$

For a density matrix ϱ on $\mathcal{H}$, we introduce the *reduced* density matrix ϱ_s on $\mathcal{H}_s$ as the partial trace over $\mathcal{H}_c$:

$$\varrho_s := \text{Tr}_{\mathcal{H}_c} \varrho. \quad (5.29)$$

To avoid a possible confusion causing by the fact that all $\mathcal{H}_j, j = 0, 1, \dots$ are identical to $\mathcal{F}$ and by the change of components with time, we treat the Weyl algebra on the subsystem and the corresponding reduced density matrix of $\rho \in \mathfrak{C}_1(\mathcal{H})$ in the following way. For $n \leq N$ on the Fock space $\mathcal{F}^{\otimes(n+1)}$ we consider the Weyl operator

$$W_n(\zeta) = \exp \left[i \frac{\langle \zeta, \tilde{b} \rangle + \langle \tilde{b}, \zeta \rangle}{\sqrt{2}} \right], \quad (5.30)$$

where $\zeta \in \mathbb{C}^{n+1}$, $\tilde{b}_0, \dots, \tilde{b}_n$ and $\tilde{b}_0^*, \dots, \tilde{b}_n^*$ are the annihilation and the creation operators in $\mathcal{F}^{\otimes(n+1)}$ satisfying the corresponding CCR, and

$$\langle \zeta, \tilde{b} \rangle = \sum_{j=0}^n \bar{\zeta}_j \tilde{b}_j, \quad \langle \tilde{b}, \zeta \rangle = \sum_{j=0}^n \zeta_j \tilde{b}_j^*.$$

By $\mathcal{A}(\mathcal{F}^{\otimes(n+1)})$, we denote the C^* -algebra generated by the Weyl operators (5.30). For any subset $J \subset \{1, 2, \dots, N\}$, we define the operation of taking the partial trace

$$R^J : \mathfrak{C}_1(\mathcal{F}^{\otimes(N+1)}) \ni \rho \longmapsto R^J \rho \in \mathfrak{C}_1(\mathcal{F}^{\otimes(N+1-|J|)})$$

by

$$\omega_{R^J \rho}(W_{N-|J|}(\zeta)) = \omega_\rho(W_N(r_J \zeta)).$$

Here the mapping

$$r_J : \mathbb{C}^{N+1-|J|} \ni \zeta \longmapsto r_J \zeta \in \mathbb{C}^{N+1}$$

is defined by

$$(r_J \zeta)_j := \begin{cases} \zeta_0 & (j = 0) \\ 0 & (j \in J) \\ \zeta_{j - |\{i \in J \mid i < j\}|} & (\text{otherwise}) \end{cases},$$

where $|A|$ denotes the cardinality of the set A .

Since all $\mathcal{H}_1, \mathcal{H}_2, \dots$ are identical to $\mathcal{F}$, we do not care to distinguish the spaces

$$\bigotimes_{j \in \{0, 1, \dots, N\} \setminus J} \mathcal{H}_j \quad \text{and} \quad \bigotimes_{j \in \{0, 1, \dots, N\} \setminus J'} \mathcal{H}_j$$

when $J \neq J'$, but $|J| = |J'|$, and consider them as the same space $\mathcal{F}^{\otimes(N+1-|J|)}$. Instead, we pay attention to distinguishing projections

$$\bigotimes_{j=0}^N \mathcal{H}_j \longrightarrow \bigotimes_{j \in \{0, 1, \dots, N\} \setminus J} \mathcal{H}_j$$

for different subsets $J \subset \{1, 2, \dots, N\}$ with same $|J|$.

Since we treat $\mathcal{S}_{n,k}$ at time $t = k\tau$ for $k = n, n+1, \dots$ as the result of the time evolution of a *single* subsystem $\mathcal{S}_{\sim n}$, we define its state at the moment $t = k\tau$ by the reduced density matrix $\{\rho_s(k\tau)\}_{k \geq n}$ of this subsystem as follows:

$$\rho_s(k\tau) := R^{\{1, \dots, k-n, k+1, \dots, N\}}(\rho(k\tau)) = R^{\{1, \dots, k-n, k+1, \dots, N\}} T_{k\tau}(\rho), \quad (5.31)$$

see (2.12). Taking into account Lemma 4.2 and identity $\langle r_J \zeta, r_J \zeta \rangle_{\mathbb{C}^{N+1}} = \langle \zeta, \zeta \rangle_{\mathbb{C}^{N+1-|J|}}$, one readily obtains the following result.

Lemma 5.1 For the initial density matrix (4.1),

$$\begin{aligned} \omega_{\rho_s(k\tau)}(W_n(\zeta)) &= \omega_{R^{J_{n,k}}\rho(k\tau)}(W_n(\zeta)) \\ &= \exp \left[-\frac{|(U_1 \dots U_k r_{J_{n,k}} \zeta)_0|^2}{4} \left(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}} \right) - \frac{\langle \zeta, \zeta \rangle}{4} \frac{1+e^{-\beta}}{1-e^{-\beta}} \right] \end{aligned}$$

holds, where $J_{n,k} = \{1, 2, \dots, k-n, k+1, \dots, N\}$.

To study the limit $k \rightarrow \infty$ and $N \rightarrow \infty$ ($k \leq N$) for a fixed n , we note that $(U_1 \dots U_k r_{J_{n,k}} \zeta)_0 \rightarrow 0$ follows from (2.16) and $|z| < 1$. Lemma 5.1 implies that

$$\lim_{k \rightarrow \infty} \omega_{\rho_s(k\tau)}(W_n(\zeta)) = \exp \left[-\frac{\langle \zeta, \zeta \rangle}{4} \frac{1+e^{-\beta}}{1-e^{-\beta}} \right] = \omega_{\rho_n^{(\beta)}}(W_n(\zeta)) , \quad (5.32)$$

where by the Araki-Segal theorem and irreducibility of the CCR algebra $\mathcal{A}(\mathcal{F}^{\otimes(n+1)})$

$$\rho_n^{(\beta)} = \exp \left[-\beta \sum_{j=0}^n \tilde{b}_j^* \tilde{b}_j \right] / Z(\beta)^{n+1} , \quad Z(\beta) = (1 - e^{-\beta})^{-1} . \quad (5.33)$$

Therefore, we proved the following statement:

Theorem 5.2 Let the initial state of the total system $\mathcal{S} + \mathcal{C}$ is defined by the density matrix (4.2): $\rho = \rho(\beta, \beta_0 - \beta; e)$. Then for any fixed n , the state $\omega_{\rho_s(k\tau)}(\cdot)$ of subsystem $\mathcal{S}_{n,k}$ converges to the equilibrium Gibbs state $\omega_{\rho_n^{(\beta)}}(\cdot)$ as $k \rightarrow \infty$ in the weak*-topology for the states on $\mathcal{A}(\mathcal{F}^{\otimes(n+1)})$.

Theorem 5.3 Under the same conditions as in Theorem 5.2, we obtain

$$\lim_{k \rightarrow \infty} S(\rho_s(k\tau)) = S(\rho_n^{(\beta)}) .$$

Proof : Let the vector $\xi_{n,k} \in \mathbb{C}^{n+1}$ be defined by $(U_1 \dots U_k r_{J_{n,k}} \zeta)_0 =: \langle \xi_{n,k}, \zeta \rangle$. Then $k \rightarrow \infty$, for a fixed n , implies $\langle \xi_{n,k}, \xi_{n,k} \rangle \rightarrow 0$. By Proposition 3.3 and Lemma 5.1 we obtain that in this limit

$$\begin{aligned} S(\rho_s(k\tau)) &= n\sigma\left(\frac{1+e^{-\beta}}{1-e^{-\beta}}\right) + \sigma\left(\frac{1+e^{-\beta}}{1-e^{-\beta}} + \langle \xi_{n,k}, \xi_{n,k} \rangle \left(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\right)\right) \\ &\longrightarrow (n+1)\sigma\left(\frac{1+e^{-\beta}}{1-e^{-\beta}}\right) = S(\rho_n^{(\beta)}) . \end{aligned}$$

□

Remark 5.4 The local entropy decreases or increases with $k\tau$ according to $\beta > \beta_0$ or $\beta < \beta_0$, respectively.

6 A Short-Time Limit for Repeated Perturbation

The results in the Section 5 are essentially due explicit knowledge of the initial density matrix (4.1) of the total system $\mathcal{S} + \mathcal{C}$. In this section, we show that the lack of this information is not decisive for certain results concerning the convergence to equilibrium if one considers the repeated perturbation in a *short-time* limit.

We study this limit for the subsystem $\mathcal{S}$. We keep to consider the initial state of the system $\mathcal{S} + \mathcal{C}$ to be a product state with the density matrix

$$\rho = \rho_0 \otimes \bigotimes_{k=1}^N \rho_k \in \mathfrak{C}_1(\mathcal{H}) , \quad (6.1)$$

see (2.7), but we essentially relax the conditions on ρ_0 and on $\{\rho_k\}_{k=1}^N$ (cf.(4.1)):

$$\begin{aligned} (h1) \quad & \rho_1 = \rho_2 = \cdots = \rho_N \in \mathfrak{C}_1(\mathcal{F}) ; \\ (h2) \quad & \text{Tr}_{\mathcal{F}}(\rho_1 a) = \text{Tr}_{\mathcal{F}}(\rho_1 a^2) = \text{Tr}_{\mathcal{F}}(\rho_1 a^*) = \text{Tr}_{\mathcal{F}}(\rho_1 a^{*2}) = 0 ; \\ (h3) \quad & \text{Tr}_{\mathcal{F}}[\rho_1 (a^* a)^2] < \infty . \end{aligned}$$

Remark 6.1 *Note that hypothesis (h1)-(h3) are satisfied when the density matrices $\{\rho_k\}_{k=0}^N$ correspond to the gauge-invariant quasi-free states with parameter β_0 for $k = 0$ and β for $k = 1, 2, \dots, N$, see (4.1). Then (h2) is due to the gauge invariance and one gets for (h3):*

$$\text{Tr}_{\mathcal{F}}[\rho_k (a^* a)^2] = (2n_\beta^2 + n_\beta) , \quad (6.2)$$

where $n_\beta = \text{Tr}_{\mathcal{F}}(\rho_k a^* a) = (e^\beta - 1)^{-1}$, $k = 1, \dots, N$.

Below we denote by $|ya^* + \bar{y}a|$ the operator originated from the *polar decomposition* of the operator $ya^* + \bar{y}a = U|ya^* + \bar{y}a|$, where U is the partial isometry on $\mathcal{F}$.

Lemma 6.2 *Under hypothesis (h1)-(h3), the following bounds hold:*

$$\begin{aligned} (i) \quad & \text{Tr}_{\mathcal{F}}(\rho_k a^* a) < \infty, \\ (ii) \quad & \text{Tr}_{\mathcal{F}}(\rho_k |ya^* + \bar{y}a|^2) \leq C|y|^2, \\ (iii) \quad & \text{Tr}_{\mathcal{F}}(\rho_k |ya^* + \bar{y}a|^3) \leq C'|y|^3, \\ (iv) \quad & \text{Tr}_{\mathcal{F}}(\rho_k |ya^* + \bar{y}a|^4) \leq C''|y|^4, \end{aligned}$$

for all $k = 1, \dots, N$. Here C, C', C'' are positive constants, which depend only on $\text{Tr}[\rho_1 (a^* a)^2]$.

Proof : The first bound (i) is a consequence of the Cauchy-Schwarz inequality and (h3). Applying the inequalities

$$\begin{aligned} |A + A^*|^2 & \leq |A + A^*|^2 + |A - A^*|^2 = 2(AA^* + A^*A), \\ |A + A^*|^4 & \leq |A + A^*|^4 + |A - A^*|^4 + |A + iA^*|^4 + |A - iA^*|^4 \\ & = 4(AA^* + A^*A)^2 + 4(A^2 A^{*2} + A^{*2} A^2), \end{aligned}$$

to $A = \bar{y}a$, we obtain (ii) and (iv). Finally, a combination of (ii), (iv) with the Cauchy-Schwarz inequality yields (iii). $\square$

Theorem 6.3 *Let $\tau \rightarrow 0$, $N \rightarrow \infty$ be short-time perturbation limit subjected to demands: $\tau^2 N \rightarrow \infty$ and $\tau^3 N \rightarrow 0$. Then for any initial condition (6.1) verifying (h1)-(h3), the characteristic function $\omega_S^{N\tau}(\widehat{w}(\theta))$ of the state for subsystem $\mathcal{S}$ at $t = N\tau$, converges to*

$$\omega_S(\widehat{w}(\theta)) := \lim_{\tau \rightarrow 0, N \rightarrow \infty} \omega_{\rho(N\tau)}(W(\zeta_\theta)) = e^{-|\theta|^2 \text{Tr}_{\mathcal{F}}[\rho_1(a^*a + aa^*)]/4}. \quad (6.3)$$

Here $\theta \in \mathbb{C}$ and the $(N+1)$ -component vector is $(\zeta_\theta)^{\text{tr}} := (\theta, 0, 0, \dots, 0) \in \mathbb{C}^{N+1}$.

By (6.3) the state $\omega_S^{N\tau}$ converges to ω_S in the weak*-topology. From the right-hand side of (6.3) and Definition 3.1 we deduce that the limit state is gauge-invariant and quasi-free with $h(\theta) := |\theta|^2 \text{Tr}_{\mathcal{F}}(\rho_1 a^* a)$.

Remark 6.4 *Recall that the state ω over the Weyl algebra $\mathcal{A}(\mathcal{F}) = \overline{\mathcal{A}_w(\mathcal{F})}$ is regular, C^n -smooth or analytic, if the function (see (2.1))*

$$s \mapsto \omega(\widehat{w}(s\theta)) = \omega(e^{i s \Phi(\theta)/\sqrt{2}}) \quad (6.4)$$

is respectively continuous, C^n -smooth or analytic in the vicinity of $s = 0$. In the last case the characteristic function $\omega(\widehat{w}(s\theta))$ (and therefore the state) is completely determined by

$$\omega(\widehat{w}(s\theta)) = \exp \left\{ \sum_{m=1}^{\infty} \frac{i^m s^m}{m!} 2^{-m/2} \omega^T(\Phi^m(\theta)) \right\}. \quad (6.5)$$

Here $\{\omega^T(\Phi^m(\theta))\}_{m=0}^{\infty}$ are truncated correlation functions defined recursively by relations

$$\begin{aligned} \omega^T(\Phi(\theta)) &:= \omega(\Phi(\theta)), \\ \omega^T(\Phi^2(\theta)) &:= \omega(\Phi^2(\theta)) - \omega(\Phi(\theta))^2, \\ \omega^T(\Phi^3(\theta)) &:= \omega(\Phi^3(\theta)) - 3\omega(\Phi^2(\theta))\omega(\Phi(\theta)) + 2\omega(\Phi(\theta))^3, \text{ etc} \end{aligned}$$

Lemma 6.2 implies that the states for density matrices $\rho_1 = \rho_2 = \dots$ are C^4 -smooth.

Proof (of Theorem 6.3): By (h2) and by Lemma 6.2 (i)-(iii) together with Remark 6.4, we obtain for the states $\omega(\cdot) = \omega_{\rho_k}(\cdot)$ the representation of (6.5) in the form:

$$C_k(\theta) = \omega_{\rho_k}(\widehat{w}(\theta)) = \exp\left[-\frac{1}{4} \omega_{\rho_k}^T(\Phi^2(\theta)) + R(\theta)\right], \quad k = 1, 2, \dots, N, \quad (6.6)$$

where $R(\theta) = O(|\theta|^3)$ in the vicinity of $\theta = 0$. For the self-adjoint operator $\Phi(\theta) = \bar{\theta}a + \theta a^*$, the hypothesis (h2) and Lemma 6.2 (i) imply

$$\omega_{\rho_k}^T(\Phi^2(\theta)) = |\theta|^2 \text{Tr}_{\mathcal{F}}[\rho_k(a^*a + aa^*)]. \quad (6.7)$$

Now, taking into account Lemma 2.2 for the vector ζ_θ , as well as (6.6) and (6.7), we obtain the representation:

$$\omega_S^{N\tau}(\widehat{w}(\theta)) = \omega_{\rho(N\tau)}(W(\zeta_\theta)) = C_0(e^{i\epsilon\tau N}(gz)^N\theta) \prod_{k=1}^N C_k(e^{i\epsilon\tau N}gz (gz)^{N-k}\theta)$$

$$= C_0(e^{i\epsilon\tau N}(gz)^N\theta) \exp\left(-\sum_{k=1}^N \frac{|\theta_k|^2}{4} \text{Tr}_{\mathcal{F}}[(a^*a + aa^*)\rho_k] + \widehat{R}\right). \quad (6.8)$$

Here by (2.17) and by (6.6) one has

$$\theta_k := e^{i\epsilon N\tau} gw (gz)^{N-k}\theta, \quad \sum_{k=1}^N |\theta_k|^2 = |\theta|^2 |w|^2 \frac{1-|z|^{2N}}{1-|z|^2}, \quad \widehat{R} = \sum_{k=1}^N O(|\theta_k|^3).$$

By virtue of (1.10) and (1.11), we get $|g(\tau)| = 1$, $|w(\tau)|^2 + |z(\tau)|^2 = 1$ and also

$$w(\tau) = i\eta\tau + O(\tau^3), \quad |z(\tau)| = 1 - \frac{|\eta|^2\tau^2}{2} + O(\tau^4),$$

for small τ . This yields for small $\tau > 0$ and large N , the estimates $|(gz)^N| \leq O(e^{-|\eta|^2\tau^2 N/2})$, $|\theta_k| \leq O(\tau)$, and $\widehat{R} = O(\tau^3 N)$ by virtue of (h1). Then taking into account the conditions $\tau^2 N \rightarrow \infty$ and $\tau^3 N \rightarrow 0$, we get the limits:

$$\lim_{\tau \rightarrow 0, N \rightarrow \infty} C_0(e^{i\epsilon\tau N}(gz)^N\theta) = 1, \quad \lim_{\tau \rightarrow 0, N \rightarrow \infty} \sum_{k=1}^N |\theta_k|^2 = |\theta|^2, \quad \lim_{\tau \rightarrow 0, N \rightarrow \infty} \widehat{R} = 0.$$

C_0 is a continuous function since it is defined by a normal state with density matrix ρ_0 . Inserting all these limits into (6.8), we obtain what is claimed as the limit (6.3). $\square$

Corollary 6.5 *Suppose that density matrices $\{\rho_k\}_{k=1}^N$ correspond to the gauge-invariant quasi-free Gibbs state with parameter β (4.1). These states satisfy (h1)-(h3). The statement of Theorem 6.3 is valid with the limit*

$$\omega_{\mathcal{S}}(\widehat{w}(\theta)) = \lim_{\tau \rightarrow 0, N \rightarrow \infty} \omega_{\mathcal{S}}^{N\tau}(\widehat{w}(\theta)) = \exp\left\{-\frac{|\theta|^2}{4} \frac{1+e^{-\beta}}{1-e^{-\beta}}\right\}. \quad (6.9)$$

It coincides with the result for equilibrium state (5.6) of the subsystem $\mathcal{S}$.

Hence, the short-time perturbation limit $\tau \rightarrow 0$, $N \rightarrow \infty$ subjected to $\tau^2 N \rightarrow \infty$ and $\tau^3 N \rightarrow 0$ gives a *universal* gauge-invariant quasi-free limiting state under hypothesis (h1)-(h3). The hypotheses (h2),(h3) control only first *two* moments of the initial states of the subsystem $\mathcal{C}$. Then stationarity and independence of repeated perturbation due to (h1), correspond to conditions for the non-commutative Central Limit Theorem [Ve]. Note also that the state ω_{ρ_0} of the subsystem $\mathcal{S}$ may be replaced by any *regular* state.

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