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Pointwise convergence of Cesàro and Riesz means on certain function spaces

SHUICHI SATO

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Abstract. We consider a function space $\mathcal{QA}$ on the unit sphere of $\mathbb{R}^3$, which contains $L \log L \log \log L$, and prove the spherical harmonics expansions of functions in $\mathcal{QA}$ are summable a.e. with respect to the Cesàro means of the critical order $1/2$. We also prove that a similar result holds for the Bochner–Riesz means of multiple Fourier series of periodic functions on $\mathbb{R}^d$, $d \geq 2$.

1. Introduction

Let

$$Q_d = \{x \in \mathbb{R}^d : -1/2 < x_i \leq 1/2, i = 1, 2, \dots, d\}, \quad x = (x_1, \dots, x_d),$$

be the fundamental cube in the d -dimensional Euclidean space $\mathbb{R}^d$. For $f \in L^1(Q_d)$ we consider the Fourier series

$$f(x) \sim \sum a_n e^{2\pi i \langle n, x \rangle}, \quad n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d,$$

where $\langle n, x \rangle = n_1 x_1 + \dots + n_d x_d$ and

$$a_n = \int_{Q_d} f(x) e^{-2\pi i \langle n, x \rangle} dx, \quad dx = dx_1 \dots dx_d,$$

is the Fourier coefficient. The Bochner–Riesz means of order δ of the series are defined by

$$T_R^\delta(f)(x) = \sum_{|n| < R} \left(1 - \frac{|n|^2}{R^2}\right)^\delta a_n e^{2\pi i \langle n, x \rangle},$$

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where $|n| = (n_1^2 + \cdots + n_d^2)^{1/2}$.

According to [2], we define a space $\mathcal{QA}(Q_d)$ to be the collection of measurable functions f for which we can find a sequence $\{f_j\}$ of non-negative measurable functions such that

$$|f| \leq \sum_{j=1}^{\infty} f_j, \quad N(\{f_j\}) := \sum_{j=1}^{\infty} (1 + \log j) \|f_j\|_1 \log \left(\frac{e \|f_j\|_{\infty}}{\|f_j\|_1} \right) < \infty; \quad (1.1)$$

let $\|f\|_{\mathcal{QA}} = \inf N(\{f_j\})$, where the infimum is taken over all such $\{f_j\}$. Then, the space $\mathcal{QA}$ is a logconvex quasi-Banach space and a subspace of $L \log L$ (see [2, 9]).

Define $T_*^{\delta}(f)(x) = \sup_{R>0} |T_R^{\delta}(f)(x)|$. Let $\alpha = (d-1)/2$ (the critical index). Then we shall prove the following.

Theorem 1. *There exists a positive constant C such that*

$$\|T_*^{\alpha}(f)\|_{1,\infty} = \sup_{\lambda>0} \lambda |\{x \in Q_d : T_*^{\alpha}(f)(x) > \lambda\}| \leq C \|f\|_{\mathcal{QA}};$$

consequently,

$$\lim_{R \rightarrow \infty} T_R^{\alpha}(f)(x) = f(x) \quad \text{a.e.} \quad \text{for } f \in \mathcal{QA}(Q_d).$$

It is known that $L \log L \log \log L$ is a proper subspace of $\mathcal{QA}$ (see [2]). Thus, Theorem 1 implies the following.

Theorem 2. *If $f \in L \log L \log \log L(Q_d)$, then*

$$\lim_{R \rightarrow \infty} T_R^{\alpha}(f)(x) = f(x) \quad \text{a.e.}$$

The convergence a.e. for $f \in L \log L \log \log L(Q_d)$ was proved in [17].

If we write $T_N(f) = T_N^{\alpha}(f)$ when $d = 1$, then $T_{N+1}(f)$ is the N th partial sum of the Fourier series of f . For $f \in L^2(Q_1)$, there is a result of L. Carleson [5] which shows that $\{T_N f\}$ converges a.e. (see also [7]). Let $T_* f = \sup_{N \geq 1} |T_N f|$. R. Hunt [8] proved the restricted weak type estimates:

$$\sup_{\lambda>0} \lambda |\{x \in Q_1 : T_*(\chi_A)(x) > \lambda\}|^{1/p} \leq C p^2 (p-1)^{-1} |A|^{1/p}, \quad 1 < p < \infty, \quad (1.2)$$

where χ_A denotes the characteristic function of a set $A \subset Q_1$. By (1.2) R. Hunt [8] proved the convergence a.e. of $\{T_N f\}$ for $f \in L(\log L)^2(Q_1)$. P. Sjölén [12] showed that (1.2) can be used to prove the convergence a.e. for the class $L \log L \log \log L(Q_1)$. Applying (1.2) more efficiently, N. Yu. Antonov [1] proved that $\{T_N f\}$ converges a.e. if $f \in L \log L \log \log L(Q_1)$. Theorem 2 can be regarded as a generalization of this result to higher dimensions.

To prove Theorem 1 for $d \geq 2$ we use the following estimates:

Lemma 1. *Let $1 < p \leq 2$, $d \geq 2$. Then there exists a constant C independent of p such that*

$$\sup_{\lambda>0} \lambda |\{x \in Q_d : T_*^\alpha(f)(x) > \lambda\}|^{1/p} \leq C(p-1)^{-1} \|f\|_p.$$

We write $\delta = \sigma + i\tau$, $\sigma, \tau \in \mathbb{R}$. Lemma 1 was proved in [17] by using the following two results and analytic interpolation.

Lemma 2. *Suppose $f \in L^1(Q_d)$, $d \geq 2$ and $\sigma > \alpha$. Then*

$$\|T_*^\delta(f)\|_{1,\infty} \leq A_\sigma e^{\pi|\tau|} (\sigma - \alpha)^{-1} \|f\|_1,$$

where A_σ remains bounded as $\sigma \rightarrow \alpha$.

Lemma 3. *Suppose that $f \in L^2(Q_d)$, $d \geq 2$. Then*

$$\|T_*^\delta(f)\|_2 \leq A_\sigma e^{\pi|\tau|} \|f\|_2, \quad \sigma > 0.$$

See Lemma 12 and Theorem 7 of [15] for Lemmas 2 and 3, respectively.

Sjölin–Soria [13] extended results of [1] to more general settings. We can apply results of [13] to prove Theorem 2 for $d \geq 2$. Indeed, we easily see that Theorem 2 for $d \geq 2$ follows from Lemma 1 and methods of [13, Section 3] (see Remark at the end of Section 3 of [13]). When $d = 1$, Theorem 1 is due to [2]. The result also can be proved by using the estimate (1.2) and Antonov's idea. More precisely, when $d = 1$, Lemma 7 (a key estimate) below is first proved for characteristic functions by applying (1.2) and the transition from characteristic functions to general functions f can be carried out by Antonov's idea. We can prove Theorem 1 by Lemma 1 in the same way in higher dimensions. In fact, our proof of Theorem 1 for $d \geq 2$ is more straightforward; to prove Lemma 7 the application of the idea of Antonov is not needed, since the estimate of Lemma 1 is not restricted to characteristic functions (see Section 2).

We have analogous results for the Cesàro means of spherical harmonics expansions. Let $\mathcal{H}_k$ be the space of the spherical harmonics of degree k on Σ_d , where $\Sigma_d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$ is the unit sphere in $\mathbb{R}^{d+1}$. We recall that the space $\mathcal{H}_k$ consists of the restrictions to Σ_d of harmonic homogeneous polynomials of degree k . Let

$$H_k f(x) = \int_{\Sigma_d} Z_x^{(k)}(y) f(y) d\mu(y),$$

where $d\mu$ is the Lebesgue surface measure on Σ_d normalized as $\mu(\Sigma_d) = 1$ (we also write $|E| = \mu(E)$ for a set $E \subset \Sigma_d$), and $Z_x^{(k)} \in \mathcal{H}_k$ is the zonal harmonic of degree

k with pole $x \in \Sigma_d$:

$$\begin{aligned} Z_x^{(k)}(y) &= \left(\frac{2k}{d-1} + 1 \right) \frac{\Gamma(d/2)\Gamma(d+k-1)}{\Gamma(d-1)\Gamma(k+d/2)} P_k^{((d-2)/2, (d-2)/2)}(\langle x, y \rangle) \\ &= \left(\frac{2k}{d-1} + 1 \right) P_k^{((d-1)/2)}(\langle x, y \rangle). \end{aligned}$$

Here $P_k^{(\alpha, \beta)}$ is the Jacobi polynomial and $P_k^{(\lambda)}$ is the Gegenbauer polynomial defined by $(1 - 2tr + r^2)^{-\lambda} = \sum_{k=0}^{\infty} P_k^{(\lambda)}(t)r^k$. We consider the spherical harmonics expansion $f \sim \sum_{k=0}^{\infty} H_k f$ and the Cesàro means of order δ defined by

$$S_n^\delta f = \frac{1}{A_n^{(\delta)}} \sum_{k=0}^n A_{n-k}^{(\delta)} H_k f, \quad n = 0, 1, 2, \dots, \quad \delta = \sigma + i\tau,$$

where

$$A_k^{(\delta)} = \frac{\Gamma(k + \delta + 1)}{\Gamma(k + 1)\Gamma(\delta + 1)} = \binom{k + \delta}{k}, \quad \sigma > -1 \quad (1.3)$$

(see [19, Chap. III]). We refer to [4, 6, 14, 18] and [16, Chap. IV] for relevant results.

Let $S_*^\delta(f)(x) = \sup_{n>0} |S_n^\delta(f)(x)|$. If we define the space $\mathcal{QA}(\Sigma_d)$ analogously to $\mathcal{QA}(Q_d)$, we have the following result (we focus on the case $d = 2$).

Theorem 3. *There exists a positive constant C such that*

$$\sup_{\lambda>0} \lambda |\{x \in \Sigma_2 : S_*^{1/2}(f)(x) > \lambda\}| \leq C \|f\|_{\mathcal{QA}}$$

for $f \in \mathcal{QA}(\Sigma_2)$, which implies

$$\lim_{n \rightarrow \infty} S_n^{1/2}(f)(x) = f(x) \quad \text{a.e.} \quad \text{for } f \in \mathcal{QA}(\Sigma_2).$$

Theorem 3 implies the following result as Theorem 1 implies Theorem 2.

Theorem 4. *If $f \in L \log L \log \log L(\Sigma_2)$, then*

$$\lim_{n \rightarrow \infty} S_n^{1/2} f(x) = f(x) \quad \text{a.e.}$$

See [4] for the convergence a.e. of $\{S_n^{1/2} f\}$ for $f \in L^p(\Sigma_2)$, $p > 1$. The proof of Theorem 3 is similar to that of Theorem 1, with the following estimates:

Lemma 4. *Let $1 < p \leq 2$. Then we have*

$$\sup_{\lambda>0} \lambda |\{x \in \Sigma_2 : S_*^{1/2}(f)(x) > \lambda\}|^{1/p} \leq C(p-1)^{-1} \|f\|_p$$

for a positive constant C independent of p .

Let

$$Mf(x) = \sup_{r>0} |B(x, r)|^{-1} \int_{B(x, r)} |f(y)| d\mu(y),$$

where $B(x, r) = \{y \in \Sigma_2 : |y - x| < r\}$, $x \in \Sigma_2$. To prove Lemma 4 we need the following two results.

Lemma 5. *Suppose that $f \in L^1(\Sigma_2)$ and $\alpha < \sigma < 1$, where $\alpha = 1/2$. Then*

$$S_*^\delta(f)(x) \leq A_\sigma e^{B\tau^2} (\sigma - \alpha)^{-1} (Mf(x) + Mf(-x)).$$

The constant A_σ remains bounded as $\sigma \rightarrow \alpha$.

Lemma 6. *Suppose that $f \in L^2(\Sigma_2)$. Then*

$$\|S_*^\delta(f)\|_2 \leq A_\sigma e^{B\sigma\tau^2} \|f\|_2, \quad \sigma > 0.$$

The constants A_σ and B_σ are bounded on any compact subinterval of $(0, \infty)$.

We can find Lemma 6 in [4]. Using Lemmas 5 and 6, we can prove Lemma 4 by analytic interpolation (see Section 4). We shall prove Lemma 5 in Section 3 by applying methods of [10].

2. Proof of Theorem 1

We assume that $d \geq 2$. In proving Theorem 1 we use the following result.

Lemma 7. *Suppose that $f \in L^\infty(Q_d)$, $f \neq 0$. Then*

$$\|T_*^\alpha(f)\|_{1,\infty} \leq C \|f\|_1 \log \left(\frac{e \|f\|_\infty}{\|f\|_1} \right).$$

Proof. By homogeneity we may assume that $\|f\|_\infty = 1$. For $\lambda > 0$, let $m(\lambda) = \inf_{1 < p \leq 2} \lambda^{-p} (p-1)^{-p}$. Then, observing that $\|f\|_p^p \leq \|f\|_1$, by Lemma 1 we have

$$|\{x \in Q_d : T_*^\alpha(f)(x) > \lambda\}| \leq C \min(1, m(\lambda) \|f\|_1).$$

This will imply the conclusion, if we note that $m(\lambda) = \lambda^{-2}$ when $\lambda \geq e^{-2}$ and $m(\lambda) \sim \lambda^{-1} \log(1/\lambda)$ when $\lambda < e^{-2}$. ■

Let $f \in \mathcal{QA}(Q_d)$. To prove Theorem 1, we may assume that $f \geq 0$. For any $\epsilon > 0$ there exists a sequence $\{f_j\}$ of non-negative bounded functions such that $f = \sum f_j$ and $N(\{f_j\}) \leq \|f\|_{\mathcal{QA}} + \epsilon$ (see [2, p. 149]). Since $L^{1,\infty}$ is a logconvex

quasi-Banach space (see [9]) and T_*^α is a sublinear operator, using Lemma 7 we have

$$\begin{aligned} \|T_*^\alpha(f)\|_{1,\infty} &\leq C \sum_j (1 + \log j) \|T_*^\alpha(f_j)\|_{1,\infty} \\ &\leq C \sum_j (1 + \log j) \|f_j\|_1 \log \left(\frac{e\|f_j\|_\infty}{\|f_j\|_1} \right) = CN(\{f_j\}) \leq C(\|f\|_{\mathcal{Q}\mathcal{A}} + \epsilon). \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we get the conclusion.

3. Proof of Lemma 5

Let

$$S_n^{(\delta,\lambda)}(\cos v) = (A_n^{(\delta)})^{-1} \sum_{k=0}^n A_{n-k}^{(\delta)} 2(k+\lambda) P_k^{(\lambda)}(\cos v),$$

where $0 < \lambda < 1$, $0 \leq v \leq \pi$, $0 < \sigma < 1$, $\delta = \sigma + i\tau$. Then, $S_n^{(\delta,1/2)}(\langle x, y \rangle)$ is the kernel of the operator S_n^δ . In [10, p. 121], $S_n^{(\delta,\lambda)}(\cos v)$ was represented by the contour integrals as follows:

$$\frac{1}{2} A_n^{(\delta)} S_n^{(\delta,\lambda)}(\cos v) = \frac{1}{2\pi i} \int_{L_1} \varphi(z) dz + \frac{1}{2\pi i} \int_{L_2} \varphi(z) dz + \frac{1}{2\pi i} \int_{L_3} \varphi(z) dz, \quad (3.1)$$

where

$$\varphi(z) = \frac{\lambda(1+z)z^{n+\delta+2\lambda}}{(z-1)^\delta(1-2z\cos v+z^2)^{\lambda+1}}.$$

Let

$$\begin{aligned} i_n^{(\delta,\lambda)}(v) &= \frac{\lambda \sin(\delta\pi)}{\pi} \int_0^1 \frac{u^{n+\delta+2\lambda}}{(1-u)^\delta(1-2u\cos v+u^2)^{\lambda+1}} du, \\ \mathcal{J}_n^{(\delta,\lambda)}(v) &= \frac{\exp(-i[(n+\lambda+(\delta+1)/2)v-(\lambda+\delta+1)\pi/2])}{(2\sin v)^\lambda(2\sin(v/2))^{\delta+1}} \frac{\sin(\lambda\pi)}{\pi} \times \\ &\quad \times \int_0^1 \frac{u^{-\lambda}(1-u)^{n+\delta+2\lambda}}{(1-u\tau(v/2))^{\delta+1}(1-u\tau(v))^\lambda} du, \\ \mathcal{J}_n^{(\delta,\lambda)}(v) &= \frac{\exp(i[(n+\lambda+(\delta+1)/2)v-(\lambda+\delta+1)\pi/2])}{(2\sin v)^\lambda(2\sin(v/2))^{\delta+1}} \frac{\sin(\lambda\pi)}{\pi} \times \\ &\quad \times \int_0^1 \frac{u^{-\lambda}(1-u)^{n+\delta+2\lambda}}{(1-u\tau(-v/2))^{\delta+1}(1-u\tau(-v))^\lambda} du, \end{aligned}$$

where $\tau(v) = (1 + i \cot v)/2$. Then, according to (3.1), it follows that

$$\begin{aligned} \frac{1}{2} A_n^{(\delta)} S_n^{(\delta, \lambda)}(\cos v) &= (n + \lambda) \mathcal{J}_n^{(\delta, \lambda)}(v) - (\delta + 1) \mathcal{J}_{n-1}^{(\delta+1, \lambda)}(v) + i_{n+1}^{(\delta, \lambda)}(v) + \\ &\quad + i_n^{(\delta, \lambda)}(v) + (n + \lambda) \mathcal{J}_n^{(\delta, \lambda)}(v) - (\delta + 1) \mathcal{J}_{n-1}^{(\delta+1, \lambda)}(v) \end{aligned} \quad (3.2)$$

(see [10]). Put

$$\begin{aligned} K(n, \delta, \lambda, v) &= \frac{4(n + \lambda)}{\Gamma(\lambda)} C(n, \delta, \lambda) \frac{\cos[(n + \lambda + (\delta + 1)/2)v - (\lambda + \delta + 1)\pi/2]}{(2 \sin v)^\lambda (2 \sin(v/2))^{\delta+1}}, \\ L(n, \delta, \lambda, v) &= \frac{-4(\delta + 1)}{\Gamma(\lambda)} C(n, \delta, \lambda) \frac{\cos[(n + \lambda + \delta/2)v - (\lambda + \delta + 2)\pi/2]}{(2 \sin v)^\lambda (2 \sin(v/2))^{\delta+2}}, \end{aligned}$$

where

$$C(n, \delta, \lambda) = \frac{\Gamma(n + \delta + 2\lambda + 1)}{\Gamma(n + \delta + \lambda + 2)};$$

and also

$$\begin{aligned} R_1(n, \delta, \lambda, v) &= 2(n + \lambda) \mathcal{J}_n^{(\delta, \lambda)}(v) + 2(n + \lambda) \mathcal{J}_n^{(\delta, \lambda)}(v) - K(n, \delta, \lambda, v), \\ R_2(n, \delta, \lambda, v) &= -2(\delta + 1) \mathcal{J}_{n-1}^{(\delta+1, \lambda)}(v) - 2(\delta + 1) \mathcal{J}_{n-1}^{(\delta+1, \lambda)}(v) - L(n, \delta, \lambda, v), \\ R_3(n, \delta, \lambda, v) &= 2i_{n+1}^{(\delta, \lambda)}(v) + 2i_n^{(\delta, \lambda)}(v). \end{aligned}$$

Then (3.2) implies that

$$\begin{aligned} S_n^{(\delta, \lambda)}(\cos v) &= (A_n^{(\delta)})^{-1} (K(n, \delta, \lambda, v) + L(n, \delta, \lambda, v) + R_1(n, \delta, \lambda, v) + \\ &\quad + R_2(n, \delta, \lambda, v) + R_3(n, \delta, \lambda, v)). \end{aligned} \quad (3.3)$$

We need the following results.

Lemma 8. *Let $x > -1$, $y \in \mathbb{R}$. Then $|A_n^{(x+iy)}| \geq |A_n^{(x)}|$ and $|A_n^{(x+iy)}| \leq e^{c(x)y^2} A_n^{(x)}$, where $c(x) = (1/2) \sum_{k=1}^{\infty} (x+k)^{-2}$ and $A_n^{(x+iy)}$ is as in (1.3).*

Lemma 9. *Suppose $0 < \lambda < 1$, $0 < \sigma < 1$. Let $C(n, \delta, \lambda)$ be as above. Then*

$$|C(n, \delta, \lambda)| \leq C(n+1)^{\lambda-1},$$

where the constant C is independent of δ and λ .

Lemma 8 is in [3]. Lemma 9 can be proved by using the formula

$$\lim_{\operatorname{Re}(z) \geq c > 0, |z| \rightarrow \infty} \frac{\Gamma(z)}{\sqrt{2\pi} e^{-z} z^{z-1/2}} = 1.$$

Let $|\pi/2 - v| \leq (\pi/2)(n/(n+1))$. By [10, pp. 130–133] and Lemma 9 we have

$$\begin{aligned} |R_1(n, \delta, \lambda, v)| &\leq Ce^{B|\tau|} \frac{C(n, \sigma, \lambda)}{\Gamma(\lambda)|n + \sigma + \lambda + 2|} \frac{n+1}{(\sin v)^{\lambda+1}(\sin(v/2))^{\sigma+1}} \\ &\leq Ce^{B|\tau|} \frac{(n+1)^{\lambda-1}}{(\sin v)^{\lambda+1}(\sin(v/2))^{\sigma+1}}, \\ |R_2(n, \delta, \lambda, v)| &\leq Ce^{B|\tau|} \frac{C(n, \sigma, \lambda)}{\Gamma(\lambda)|n + \sigma + \lambda + 2|} \frac{1}{(\sin v)^{\lambda+1}(\sin(v/2))^{\sigma+2}} \\ &\leq Ce^{B|\tau|} \frac{(n+1)^{\lambda-1}}{(\sin v)^{\lambda+1}(\sin(v/2))^{\sigma+1}}. \end{aligned}$$

Also, by [10, pp. 122–123] and estimates similar to the one in Lemma 9

$$\begin{aligned} |R_3(n, \delta, \lambda, v)| &\leq C \frac{|\sin(\delta\pi)|\Gamma(1-\sigma)}{(\sin(v/2))^{2(\lambda+1)}} \left(\frac{\Gamma(n+\sigma+2\lambda+1)}{\Gamma(n+2\lambda+2)} + \frac{\Gamma(n+\sigma+2\lambda+2)}{\Gamma(n+2\lambda+3)} \right) \\ &\leq C(n+1)^{\sigma-1} \frac{|\sin(\delta\pi)|\Gamma(1-\sigma)}{(\sin(v/2))^{2(\lambda+1)}}. \end{aligned}$$

Since $|A_n^{(\delta)}| \geq |A_n^{(\sigma)}|$ and $A_n^{(\sigma)} \sim (n+1)^\sigma$ (see Lemma 8 and [19, Chap. III]), if $|\pi/2 - v| \leq (\pi/2)(n/(n+1))$, we have

$$|R_j(n, \delta, \lambda, v)/A_n^{(\delta)}| \leq Ce^{B|\tau|} \frac{(n+1)^{\lambda-1-\sigma}}{(\sin v)^{\lambda+1}(\sin(v/2))^{\sigma+1}}, \quad j = 1, 2, \quad (3.4)$$

$$|R_3(n, \delta, \lambda, v)/A_n^{(\delta)}| \leq C \frac{|\sin(\delta\pi)|\Gamma(1-\sigma)}{(n+1)(\sin(v/2))^{2(\lambda+1)}}. \quad (3.5)$$

By Lemma 9 we have

$$|K(n, \delta, \lambda, v)/A_n^{(\delta)}| \leq Ce^{(\pi/2)|\tau|} \frac{(n+1)^{\lambda-\sigma}}{(\sin v)^\lambda(\sin(v/2))^{\sigma+1}}. \quad (3.6)$$

Similarly,

$$|L(n, \delta, \lambda, v)/A_n^{(\delta)}| \leq C(1+|\tau|)e^{(\pi/2)|\tau|} \frac{(n+1)^{\lambda-\sigma-1}}{(\sin v)^\lambda(\sin(v/2))^{\sigma+2}}. \quad (3.7)$$

We also need the following.

Lemma 10. *Let $0 < \lambda < 1$, $0 < \sigma < 1$, $\delta = \sigma + i\tau$, $0 \leq v \leq \pi$. Then*

$$|S_n^{(\delta, \lambda)}(\cos v)| \leq Ce^{c\tau^2} (n+1)^{2\lambda+1}.$$

Proof. By [18, p. 168], we have $|P_n^{(\lambda)}| \leq CA_n^{(2\lambda-1)}$. Using this and Lemma 8, we see that

$$\begin{aligned} |S_n^{(\delta,\lambda)}(\cos v)| &\leq C|A_n^{(\delta)}|^{-1} \sum_{m=0}^n |A_{n-m}^{(\delta)}|(m+\lambda)A_m^{(2\lambda-1)} \\ &\leq C\lambda|A_n^{(\delta)}|^{-1} \sum_{m=0}^n \frac{m+\lambda}{m+2\lambda}|A_{n-m}^{(\delta)}|A_m^{(2\lambda)} \\ &\leq Ce^{c\tau^2}|A_n^{(\sigma)}|^{-1} \sum_{m=0}^n |A_{n-m}^{(\sigma)}|A_m^{(2\lambda)} \\ &\leq Ce^{c\tau^2}|A_n^{(\sigma)}|^{-1}A_n^{(\sigma+2\lambda+1)} \leq Ce^{c\tau^2}(n+1)^{2\lambda+1}. \end{aligned}$$

By (3.3)–(3.7) and Lemma 10, we have

$$|S_n^{(\delta,\lambda)}(\cos v)| \leq Ce^{B\tau^2}(n+1)^{\lambda-\sigma}((n+1)^{-1} + \sin v)^{-\lambda-\sigma-1}, \quad (3.8)$$

where $0 \leq v \leq \pi$, $\lambda = 1/2$, $1/2 < \sigma < 1$. Suppose $\langle x, y \rangle = \cos v$, $x, y \in \Sigma_2$. Then $\sin v \sim |x - y|$ if $\langle x, y \rangle \geq 0$ and $\sin v \sim |x + y|$ if $\langle x, y \rangle \leq 0$. Thus (3.8) implies

$$\begin{aligned} &|S_n^{(\delta,\lambda)}(\langle x, y \rangle)| \\ &\leq \begin{cases} Ce^{B\tau^2}(n+1)^{\lambda-\sigma}((n+1)^{-1} + |x - y|)^{-\lambda-\sigma-1}, & \text{if } \langle x, y \rangle \geq 0, \\ Ce^{B\tau^2}(n+1)^{\lambda-\sigma}((n+1)^{-1} + |x + y|)^{-\lambda-\sigma-1}, & \text{if } \langle x, y \rangle \leq 0. \end{cases} \end{aligned} \quad (3.9)$$

Since $S_n^\delta f(x) = \int_{\Sigma_2} S_n^{(\delta,1/2)}(\langle x, y \rangle) f(y) d\mu(y)$, the conclusion of Lemma 5 easily follows from (3.9).

Remark. In fact, we can prove estimates of the type in [6, Theorem (3.21)], partly improving (3.9). We do not need those estimates here; for our purpose (3.9) suffices.

4. Proofs of Lemmas 4, 6 and Theorem 3

We first prove Lemma 6.

Proof of Lemma 6. When $\delta > 0$, we have $\|S_*^\delta(f)\|_2 \leq A_\delta\|f\|_2$ (see [4, Lemma (3.5)]). If $\delta = \sigma + i\tau$, $\sigma > 0$, $\tau \in \mathbb{R}$, we write

$$S_n^\delta(f) = (A_n^\delta)^{-1} \sum_{k=0}^n A_k^{(\sigma-\epsilon)} A_{n-k}^{(\epsilon-1+i\tau)} S_k^{\sigma-\epsilon}(f),$$

where $0 < \epsilon < \sigma$. Using Lemma 8 as in [4], we have $S_*^\delta(f) \leq e^{c(\epsilon-1)\tau^2} S_*^{\sigma-\epsilon}(f)$. Combining these results, we reach the conclusion of Lemma 6. ■

Proof of Lemma 4. Let $1 < p < 2$, $1/p = (1 - \theta)/2 + \theta$, $\alpha = (1 - \theta)c + \theta b$, where $c = \alpha - (1/2)(1/p - 1/2)$, $b = \alpha + (1/2)(1 - 1/p)$, $\alpha = 1/2$. We note that $\theta = 2(1/p - 1/2)$, $1/4 \leq c \leq \alpha$, $\alpha \leq b \leq 3/4$.

Define $T_z f = S_0^{\delta(z)} f$, $\delta(z) = (1 - z)c + zb$, $0 \leq \sigma \leq 1$, $z = \sigma + i\tau$, $\tau \in \mathbb{R}$. Here S_0^δ is a linear operator approximating S_*^δ defined by $S_0^\delta f(x) = S_{n(x)}^\delta f(x)$, where $n(x)$ is a suitable non-negative mapping from Σ_2 to $\mathbb{Z}$, so that $\{T_z\}$ is an analytic family of linear operators which is admissible in the sense of [11] (see also [16, Chap. V, Section 4]).

We apply the analytic interpolation theorem on the Lorentz spaces $L^{p,q}$ due to [11]. Note that $\operatorname{Re}(\delta(i\tau)) = c \in [1/4, 1/2]$. Thus Lemma 6 implies

$$\|T_{i\tau} f\|_{2,2} \leq C_0 e^{B_0 \tau^2} \|f\|_{2,2} \quad (4.1)$$

for some $B_0, C_0 > 0$. By Lemma 5 and the $L^1 - L^{1,\infty}$ boundedness of the maximal operator M we have

$$\|T_{1+i\tau} f\|_{1,\infty} \leq C_1 (p-1)^{-1} e^{B_1 \tau^2} \|f\|_{1,1} \quad (4.2)$$

for some $B_1, C_1 > 0$, since $\operatorname{Re}(\delta(1+i\tau)) = b$. Interpolating between (4.1) and (4.2), we get

$$\|S_0^\alpha f\|_{p,p'} = \|T_\theta f\|_{p,p'} \leq A_\theta \|f\|_{p,p},$$

where

$$A_\theta \leq C(p-1)^{-\theta} \leq C(p-1)^{-1}.$$

Therefore

$$\|S_0^\alpha f\|_{p,\infty} \leq C \|S_0^\alpha f\|_{p,p'} \leq C(p-1)^{-1} \|f\|_{p,p},$$

from which Lemma 4 follows. ■

To prove Theorem 3, we note that by Lemma 4, similarly to the case of T_*^α , we can prove

$$\|S_*^{1/2} f\|_{1,\infty} \leq C \|f\|_1 \log \left(\frac{e \|f\|_\infty}{\|f\|_1} \right) \quad (4.3)$$

if $f \in L^\infty(\Sigma_2)$, $f \neq 0$. Also, as in the case of T_*^α , the estimate (4.3) readily implies $\|S_*^{1/2} f\|_{1,\infty} \leq C \|f\|_{\mathcal{Q}_A}$, from which the almost everywhere convergence follows.

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S. SATO, Department of Mathematics, Faculty of Education, Kanazawa University, Kanazawa 920-1192, Japan; *e-mail*: shuichi@kenroku.kanazawa-u.ac.jp