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SPHERICAL SQUARE FUNCTIONS OF MARCINKIEWICZ TYPE WITH RIESZ POTENTIALS

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ABSTRACT. We prove a pointwise equivalence between a spherical square function composed with the Riesz potential and a Littlewood-Paley function arising from the Bochner-Riesz operators. Also, its application to the theory of Sobolev spaces will be given.

1. INTRODUCTION

Let

$$\nu(f)(x) = \left(\int_0^\infty |f(x+t) + f(x-t) - 2f(x)|^2 \frac{dt}{t^3} \right)^{1/2}.$$

Then $\mu(f) = \nu(\mathcal{I}(f))$ is the function of Marcinkiewicz, where $\mathcal{I}(f)(x) = \int_0^x f(y) dy$. If $f \in L^p(\mathbb{R})$, $1 < p < \infty$, we have

$$(1.1) \quad \|\mu(f)\|_p \simeq \|f\|_p,$$

which means that there exist positive constants c_1, c_2 independent of f such that

$$c_1 \|f\|_p \leq \|\mu(f)\|_p \leq c_2 \|f\|_p.$$

In other words, we have $\|\nu(f)\|_p \simeq \|f'\|_p$ if f is in the Sobolev space $W^{1,p}(\mathbb{R})$, $1 < p < \infty$.

The Marcinkiewicz function was introduced by J. Marcinkiewicz [5] in 1938 in the setting of periodic functions on the torus $\mathbb{T}$. It can be used to investigate properties of functions such as differentiability and finiteness of norms in function spaces. Zygmund [10] proved a periodic analogue of (1.1), which was conjectured in [5]. The non-periodic version (1.1) was proved by Waterman [8].

The Marcinkiewicz function is a kind of Littlewood-Paley functions; $\mu(f)$ can be realized as

$$\mu(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where $\psi_t(x) = t^{-1}\psi(t^{-1}x)$ with $\psi(x) = \chi_{[0,1]}(x) - \chi_{[-1,0]}(x)$; here χ_E denotes the characteristic function of a set E . We observe that

$$f(x+t) + f(x-t) - 2f(x) = \int_{S^0} (f(x-t\theta) - f(x)) d\sigma(\theta),$$

where $d\sigma$ is a uniform measure on $S^0 = \{-1, 1\}$ such that $\sigma(\{-1\}) = 1$, $\sigma(\{1\}) = 1$.

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In this note we assume that $n \geq 2$ and consider the square function

$$(1.2) \quad D^\alpha(f)(x) = \left(\int_0^\infty \left| t^{-\alpha} \int_{S^{n-1}} (f(x - t\theta) - f(x)) d\sigma(\theta) \right|^2 \frac{dt}{t} \right)^{1/2},$$

for appropriate functions f , where $d\sigma$ is the Lebesgue surface measure on S^{n-1} . Then $D^\alpha(f)$ with $\alpha = 1$ can be regarded as a generalization to higher dimensions of $\nu(f)$. We shall see that $D^\alpha(f)$ also can be used to characterize Sobolev norms. This will be accomplished through a relation between $D^\alpha(f)$ and another square function arising from the Bochner-Riesz operators. Let

$$S_R^\beta(f)(x) = \int_{|\xi| < R} \hat{f}(\xi) (1 - R^{-2}|\xi|^2)^\beta e^{2\pi i \langle x, \xi \rangle} d\xi = H_{R^{-1}}^\beta * f(x)$$

be the Bochner-Riesz mean of order β on $\mathbb{R}^n$, where

$$H_{R^{-1}}^\beta(x) = R^n H^\beta(Rx), \quad H^\beta(x) = \pi^{-\beta} \Gamma(\beta + 1) |x|^{-(n/2+\beta)} J_{n/2+\beta}(2\pi|x|)$$

with J_ν denoting the Bessel function of the first kind of order ν and $\hat{f}$ is the Fourier transform defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx, \quad \langle x, \xi \rangle = \sum_{k=1}^n x_k \xi_k.$$

We recall a Littlewood-Paley operator σ_β , $\text{Re}(\beta) > 0$, defined from the Bochner-Riesz means as

$$(1.3) \quad \begin{aligned} \sigma_\beta(f)(x) &= \left(\int_0^\infty \left| R \partial_R S_R^\beta(f)(x) \right|^2 \frac{dR}{R} \right)^{1/2} \\ &= \left(\int_0^\infty \left| -2\beta \left(S_R^\beta(f)(x) - S_R^{\beta-1}(f)(x) \right) \right|^2 \frac{dR}{R} \right)^{1/2}, \end{aligned}$$

where $\text{Re}(\beta)$ denotes the real part of the complex number β and $\partial_R = \partial/\partial R$. Also, let I_α be the Riesz potential operator defined as

$$(1.4) \quad \widehat{I_\alpha(f)}(\xi) = |\xi|^{-\alpha} \hat{f}(\xi).$$

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing smooth functions on $\mathbb{R}^n$. Let $\mathcal{S}_0(\mathbb{R}^n)$ be the subspace of $\mathcal{S}(\mathbb{R}^n)$ consisting of functions f with $\hat{f}$ vanishing in a neighborhood of the origin. We shall prove the following.

Theorem 1.1. *Let $0 < \alpha < 2$. Then if $\beta = \alpha + \frac{n}{2}$, we have*

$$\sigma_\beta(f)(x) \approx D^\alpha(I_\alpha f)(x)$$

for $f \in \mathcal{S}_0(\mathbb{R}^n)$, where D^α , I_α and σ_β are as in (1.2), (1.4) and (1.3), respectively.

Here $\sigma_\beta(f)(x) \approx D^\alpha(I_\alpha f)(x)$ means that there exist positive constants A, B independent of f and x such that

$$A\sigma_\beta(f)(x) \leq D^\alpha(I_\alpha f)(x) \leq B\sigma_\beta(f)(x).$$

A version of Theorem 1.1 was shown in [4] for the range $0 < \alpha < 1$. In this note we shall extend this range of α to $0 < \alpha < 2$. The difference from [4] that enables us to improve the range of α mainly comes from the estimate in part (1) of Lemma 2.5.

In Section 2, we shall give an almost self-contained proof of Theorem 1.1 except that Lemmas 2.2, 2.3 from [9] and the formula (2.9) are taken for granted. In

Section 3, applications of Theorem 1.1 to the theory of Sobolev spaces will be given.

2. PROOF OF THEOREM 1.1

For a fixed function $f \in \mathcal{S}_0(\mathbb{R}^n)$ and a fixed point $x \in \mathbb{R}^n$, let

$$(2.1) \quad \varphi(t) = \varphi(t; x, f) = \int_{S^{n-1}} f(x - ty') d\sigma(y'),$$

$$(2.2) \quad \theta(t) = \theta(t; x, f) = t \frac{\partial}{\partial t} \varphi(t; x, f) = - \int_{S^{n-1}} \langle ty', \nabla f(x - ty') \rangle d\sigma(y'),$$

where $\nabla f(x) = (\partial_1 f(x), \dots, \partial_n f(x))$, $\partial_j = \partial / \partial x_j$.

Let $\operatorname{Re} \alpha > -n$. Define

$$(2.3) \quad \widehat{I^\alpha(f)}(\xi) = |\xi|^\alpha \hat{f}(\xi),$$

and for $\operatorname{Re} \beta > -1$ consider

$$(2.4) \quad S_R^\beta(I^\alpha f)(x) = \int_{|\xi| < R} \hat{f}(\xi) |\xi|^\alpha (1 - R^{-2}|\xi|^2)^\beta e^{2\pi i \langle x, \xi \rangle} d\xi = R^\alpha L_{R^{-1}}^{\alpha, \beta} * f(x),$$

where

$$\begin{aligned} L^{\alpha, \beta}(x) &= \int_{|\xi| < 1} |\xi|^\alpha (1 - |\xi|^2)^\beta e^{2\pi i \langle x, \xi \rangle} d\xi \\ &= \int_0^1 r^{n+\alpha} (1 - r^2)^\beta 2\pi(r|x|)^{-(n-2)/2} J_{(n-2)/2}(2\pi r|x|) \frac{dr}{r} \\ &= (2\pi)^{n/2} \int_0^1 r^{n+\alpha} (1 - r^2)^\beta V_{(n-2)/2}(2\pi r|x|) \frac{dr}{r}, \end{aligned}$$

with

$$V_\nu(r) = r^{-\nu} J_\nu(r).$$

We write the formula in (2.4) by using φ in (2.1).

Lemma 2.1. *Suppose that $\operatorname{Re} \alpha > -n$, $\operatorname{Re} \beta > -1$. Let*

$$l_{\alpha, \beta}(s) = (2\pi)^{n/2} \int_0^1 r^{n+\alpha} (1 - r^2)^\beta V_{(n-2)/2}(2\pi rs) \frac{dr}{r}.$$

Then we have

$$S_R^\beta(I^\alpha f)(x) = R^{n+\alpha} \int_0^\infty l_{\alpha, \beta}(Rs) \varphi(s; x, f) s^n \frac{ds}{s}.$$

Proof. By (2.4) we have

$$S_R^\beta(I^\alpha f)(x) = R^\alpha L_{R^{-1}}^{\alpha, \beta} * f(x) = R^\alpha \int_{\mathbb{R}^n} f(x - y) L_{R^{-1}}^{\alpha, \beta}(y) dy.$$

Using polar coordinates and recalling the definition of $L^{\alpha, \beta}$ in the last integral, by (2.1) we reach the conclusion. □

We use the following formulas.

Lemma 2.2. *If $\operatorname{Re}(\mu) > -1$, $\operatorname{Re}(\nu) > -1$, $t > 0$,*

$$J_{\mu+\nu+1}(t) = \frac{t^{\nu+1}}{2^\nu \Gamma(\nu+1)} \int_0^1 J_\mu(ts) s^{\mu+1} (1-s^2)^\nu ds.$$

This can be rewritten as

$$V_{\mu+\nu+1}(t) = \frac{1}{2^\nu \Gamma(\nu+1)} \int_0^1 V_\mu(ts) s^{2\mu+1} (1-s^2)^\nu ds.$$

Lemma 2.3. *If $0 < \operatorname{Re}(\mu) < \operatorname{Re}(\nu) + 1/2$,*

$$\int_0^\infty t^{\mu-1} V_\nu(t) dt = \frac{\Gamma(\mu/2)}{2^{\nu-\mu+1} \Gamma(\nu - \mu/2 + 1)}.$$

See [9, p.373] and [9, p.391] for Lemma 2.2 and Lemma 2.3, respectively.

We need the expression of $\sigma_\beta(I^\alpha f)$ in (2.7) below. To obtain it we show the following.

Lemma 2.4. *Let $\operatorname{Re} \alpha > -n$, $\operatorname{Re} \beta > 0$. Then*

$$\begin{aligned} R\partial_R S_R^\beta(I^\alpha f)(x) \\ = -2\beta(2\pi)^{n/2} R^\alpha \int_0^\infty \theta(s/R) s^{n-1} ds \int_0^1 r^{n+\alpha+2} (1-r^2)^{\beta-1} V_{n/2}(2\pi rs) \frac{dr}{r}. \end{aligned}$$

Proof. We first compute $R\partial_R(R^{n+\alpha} l_\beta(Rs))$ as follows.

$$\begin{aligned} R\partial_R(R^{n+\alpha} l_{\alpha,\beta}(Rs)) &= R\partial_R \left[(2\pi)^{n/2} R^{n+\alpha} \int_0^1 r^{n+\alpha} (1-r^2)^\beta V_{(n-2)/2}(2\pi Rrs) \frac{dr}{r} \right] \\ &= R\partial_R \left[(2\pi)^{n/2} \int_0^R r^{n+\alpha} \left(1 - \frac{r^2}{R^2}\right)^\beta V_{(n-2)/2}(2\pi rs) \frac{dr}{r} \right] \\ &= R(2\pi)^{n/2} \int_0^R r^{n+\alpha} \beta \left(1 - \frac{r^2}{R^2}\right)^{\beta-1} (2r^2 R^{-3}) V_{(n-2)/2}(2\pi rs) \frac{dr}{r} \\ &= 2\beta(2\pi)^{n/2} R^{n+\alpha} \int_0^1 r^{n+\alpha} (1-r^2)^{\beta-1} r^2 V_{(n-2)/2}(2\pi Rrs) \frac{dr}{r}. \end{aligned}$$

Therefore, from Lemma 2.1 it follows that

$$\begin{aligned} (2.5) \quad R\partial_R S_R^\beta(I^\alpha f)(x) \\ = 2\beta(2\pi)^{n/2} R^{n+\alpha} \int_0^\infty \varphi(s; x, f) s^{n-1} ds \int_0^1 r^{n+\alpha} (1-r^2)^{\beta-1} r^2 V_{(n-2)/2}(2\pi Rrs) \frac{dr}{r}. \end{aligned}$$

By Lemma 2.2 we have

$$\begin{aligned} \int_0^u V_{(n-2)/2}(2\pi Rrs) s^{n-1} ds &= u^n \int_0^1 V_{(n-2)/2}(2\pi Rru) s^{n-1} ds \\ &= u^n \Gamma(1) V_{n/2}(2\pi Ru) = u^n V_{n/2}(2\pi Ru). \end{aligned}$$

Thus applying integration by parts in (2.5), we see that

$$\begin{aligned}
& R\partial_R S_R^\beta(I^\alpha f)(x) \\
&= -2\beta(2\pi)^{n/2} R^{n+\alpha} \int_0^\infty \varphi'(s) ds \int_0^1 r^{n+\alpha} (1-r^2)^{\beta-1} r^2 s^n V_{n/2}(2\pi Rrs) \frac{dr}{r} \\
&= -2\beta(2\pi)^{n/2} R^{n+\alpha} \int_0^\infty \theta(s) s^{n-1} ds \int_0^1 r^{n+\alpha} (1-r^2)^{\beta-1} r^2 V_{n/2}(2\pi Rrs) \frac{dr}{r} \\
&= -2\beta(2\pi)^{n/2} R^\alpha \int_0^\infty \theta(s/R) s^{n-1} ds \int_0^1 r^{n+\alpha+2} (1-r^2)^{\beta-1} V_{n/2}(2\pi rs) \frac{dr}{r},
\end{aligned}$$

where $\varphi(s) = \varphi(s; x, f)$ and $\theta(s) = \theta(s; x, f)$ are as in (2.1) and (2.2), respectively. This completes the proof. $\square$

For appropriate complex numbers α, β , let

$$(2.6) \quad \Phi_{\alpha,\beta}(s) = s^{\alpha+n} \int_0^1 r^{n+\alpha+1} (1-r^2)^{\beta-1} V_{n/2}(2\pi rs) dr$$

and

$$\theta_\alpha(t) = t^{-\alpha} \theta(t) = t^{-\alpha} \theta(t; x, f).$$

Then Lemma 2.4 implies that

$$R\partial_R S_R^\beta(I^\alpha f)(x) = -2\beta(2\pi)^{n/2} \int_0^\infty \Phi_{\alpha,\beta}(s) \theta_\alpha(sR^{-1}) \frac{ds}{s}$$

for $\operatorname{Re} \alpha > -n$, $\operatorname{Re} \beta > 0$. Define

$$\begin{aligned}
K_{\alpha,\beta}(u) &= -2\beta(2\pi)^{n/2} \Phi_{\alpha,\beta}(e^u), \\
\Theta_\alpha(u) &= \Theta_\alpha(u, x, f) = \theta_\alpha(e^{-u}), \quad \Theta(u) = \theta(e^{-u}).
\end{aligned}$$

Then by change of variables $s = e^v$, $R = e^u$ we have

$$R\partial_R S_R^\beta(I^\alpha f)(x) = \int_{-\infty}^\infty K_{\alpha,\beta}(v) \Theta_\alpha(u-v) dv,$$

and hence by (1.3),

$$(2.7) \quad \sigma_\beta(I^\alpha f)(x)^2 = \int_{-\infty}^\infty |K_{\alpha,\beta} * \Theta_\alpha(u)|^2 du.$$

In proving Theorem 1.1 we need Proposition 2.10 below. To show it we first state some properties of Θ_α and $K_{\alpha,\beta}$ (Lemmas 2.5 and 2.6).

Lemma 2.5. *We have the following estimates for Θ_α , $\alpha \in \mathbb{C}$.*

- (1) $|\Theta_\alpha(u)| \leq C e^{u \operatorname{Re}(\alpha)} e^{-2u}$ for $u \geq 0$;
- (2) if $u < 0$, $|\Theta_\alpha(u)| \leq C_m e^{u \operatorname{Re}(\alpha)} e^{mu}$ for any $m > 0$.

Further, we have similar estimates for the derivatives $(d/du)^k \Theta_\alpha$, $k = 1, 2, \dots$. In particular, $\Theta_\alpha \in \mathcal{S}(\mathbb{R})$ if $\operatorname{Re}(\alpha) < 2$.

Proof. Recall that $\theta(t) = -\int_{S^{n-1}} \langle ty', \nabla f(x - ty') \rangle d\sigma(y')$ and $\Theta_\alpha(u) = e^{u\alpha} \theta(e^{-u})$. Thus, part (2) follows easily since $|\nabla f(x)| \leq C_m (1 + |x|)^{-m-1}$ and $|x - e^{-u}y'| \geq e^{-u} - |x| \geq e^{-u}/2$ if $e^{-u} \geq 2|x|$.

To prove part (1), we note that

$$\theta(t) = -\int_{S^{n-1}} \langle ty', \nabla f(x - ty') - \nabla f(x) \rangle d\sigma(y'),$$

since $\int_{S^{n-1}} \langle ty', w \rangle d\sigma(y') = 0$ for any $w \in \mathbb{R}^n$. So $\theta(t) = O(t^2)$ as $t \rightarrow 0$, which proves part (1).

By a direct computation, we can prove the result for the derivatives $(d/du)^k \Theta_\alpha$ similarly. $\square$

Lemma 2.6. *The following results hold for $K_{\alpha,\beta}$, $\alpha, \beta \in \mathbb{C}$.*

(1) *If $\operatorname{Re}(\alpha) > -n - 2$ and $\operatorname{Re}(\beta) > 0$,*

$$|K_{\alpha,\beta}(u)| \leq C_{\alpha,\beta} e^{(n+\operatorname{Re}(\alpha))u}, \quad u \in \mathbb{R}.$$

(2) *If $\operatorname{Re}(\alpha) > -n/2 - 2$ and $\operatorname{Re}(\beta) > 0$,*

$$|K_{\alpha,\beta}(u)| \leq C_{\alpha,\beta} e^{(n/2+\operatorname{Re}(\alpha))u}, \quad u \in \mathbb{R}.$$

(3) *If $-n/2 > \operatorname{Re}(\alpha) > -n/2 - 1$ and $\operatorname{Re}(\beta) > 0$, then*

$$|K_{\alpha,\beta}(u)| \leq C_{\alpha,\beta} e^{-\delta|u|}, \quad u \in \mathbb{R},$$

where $\delta = \min(n + \operatorname{Re}(\alpha), -\operatorname{Re}(\alpha) - n/2) > 0$.

Proof. Since $V_{n/2}$ is bounded, by (2.6) we have part (1), where we assume that $\operatorname{Re}(\alpha) > -n - 2$ and $\operatorname{Re}(\beta) > 0$ for the integrabilities on $[0, 1]$ of $r^{n+\alpha+1}$ and $(1 - r^2)^{\beta-1}$, respectively. By (2.6), we also have

$$\Phi_{\alpha,\beta}(s) = (2\pi)^{-n/2} s^{\alpha+n/2} \int_0^1 r^{n/2+\alpha+1} (1 - r^2)^{\beta-1} J_{n/2}(2\pi rs) dr.$$

This implies part (2). Part (3) follows from the estimates of (1) and (2). $\square$

Let

$$(2.8) \quad G(\alpha, \beta) = \int_{-\infty}^{\infty} K_{\alpha,\beta} * \Theta_\alpha(u) g(u) du,$$

where $g \in C_0^\infty(\mathbb{R})$. By Lemmas 2.5 and 2.6, the convolution $K_{\alpha,\beta} * \Theta_\alpha$ can be defined and $G(\alpha, \beta)$ is analytic in α and β if $\operatorname{Re}(\alpha) > -n - 2$, $\operatorname{Re}(\beta) > 0$. Also, if $-n/2 > \operatorname{Re}(\alpha) > -n/2 - 1$ and $\operatorname{Re}(\beta) > 0$, then $K_{\alpha,\beta}, \Theta_\alpha \in L^1(\mathbb{R})$; in this case we have

$$G(\alpha, \beta) = \int_{-\infty}^{\infty} \widehat{K}_{\alpha,\beta}(\xi) \widehat{\Theta}_\alpha(\xi) \widehat{g}(-\xi) d\xi.$$

An explicit form of the Fourier transform of $K_{\alpha,\beta}$ needed is stated in the next result.

Lemma 2.7. *If $-n/2 - 1 < \operatorname{Re}(\alpha) < -n/2$ and $\operatorname{Re}(\beta) > 0$, then $\widehat{K}_{\alpha,\beta}(\xi) = \Psi_{\alpha,\beta}(\xi)$, where*

$$\Psi_{\alpha,\beta}(\xi) = -2^{-1} \beta \pi^{-n/2-\alpha+2\pi i \xi} \frac{\Gamma((n+\alpha-2\pi i \xi)/2) \Gamma(\pi i \xi + 1) \Gamma(\beta)}{\Gamma(-\alpha/2 + \pi i \xi + 1) \Gamma(\pi i \xi + 1 + \beta)}.$$

Proof. Let $-\operatorname{Re}(\alpha) - n < \operatorname{Re}(\zeta) < -\operatorname{Re}(\alpha) - n/2$ and $-n/2 - 1 < \operatorname{Re}(\alpha) < -n/2$, $\operatorname{Re}(\beta) > 0$. Then by Fubini's theorem we have

$$\begin{aligned}\widehat{K}_{\alpha,\beta}\left(\frac{\zeta}{-2\pi i}\right) &= \int_{-\infty}^{\infty} e^{\zeta u} K_{\alpha,\beta}(u) du \\ &= -2\beta(2\pi)^{n/2} \int_0^{\infty} t^{n+\alpha+\zeta-1} \left(\int_0^1 r^{n+\alpha+1} (1-r^2)^{\beta-1} V_{n/2}(2\pi r t) dr \right) dt \\ &= -2\beta(2\pi)^{n/2} \int_0^1 r^{n+\alpha+1} (1-r^2)^{\beta-1} \left(\int_0^{\infty} t^{n+\alpha+\zeta-1} V_{n/2}(2\pi r t) dt \right) dr.\end{aligned}$$

Using Lemma 2.3, we see that

$$\begin{aligned}\int_0^{\infty} t^{n+\alpha+\zeta-1} V_{n/2}(2\pi r t) dt &= (2\pi r)^{-n-\alpha-\zeta} \int_0^{\infty} t^{n+\alpha+\zeta-1} V_{n/2}(t) dt \\ &= (2\pi)^{-n-\alpha-\zeta} 2^{n/2+\alpha+\zeta-1} \frac{\Gamma((n+\alpha+\zeta)/2)}{\Gamma(-\alpha/2-\zeta/2+1)} r^{-n-\alpha-\zeta}.\end{aligned}$$

Thus

$$\begin{aligned}\widehat{K}_{\alpha,\beta}\left(\frac{\zeta}{-2\pi i}\right) &= -2\beta(2\pi)^{-n/2-\alpha-\zeta} 2^{n/2+\alpha+\zeta-1} \frac{\Gamma((n+\alpha+\zeta)/2)}{\Gamma(-\alpha/2-\zeta/2+1)} \int_0^1 r^{1-\zeta} (1-r^2)^{\beta-1} dr \\ &= -\beta \pi^{-n/2-\alpha-\zeta} \frac{\Gamma((n+\alpha+\zeta)/2)}{\Gamma(-\alpha/2-\zeta/2+1)} 2^{-1} \int_0^1 t^{-\zeta/2} (1-t)^{\beta-1} dt \\ &= -2^{-1} \beta \pi^{-n/2-\alpha-\zeta} \frac{\Gamma((n+\alpha+\zeta)/2)}{\Gamma(-\alpha/2-\zeta/2+1)} \frac{\Gamma(-\zeta/2+1)\Gamma(\beta)}{\Gamma(-\zeta/2+1+\beta)}.\end{aligned}$$

Putting $\zeta = -2\pi i\xi$, $\xi \in \mathbb{R}$, we reach the conclusion. $\square$

In proving Proposition 2.10 we also need the following.

Lemma 2.8. *Let $\Psi_{\alpha,\beta}$ be as in Lemma 2.7 and*

$$H(\alpha, \beta) = \int_{-\infty}^{\infty} \Psi_{\alpha,\beta}(\xi) \widehat{\Theta}_{\alpha}(\xi) \widehat{g}(-\xi) d\xi.$$

Then $H(\alpha, \beta)$ is analytic in α and β if $-n < \operatorname{Re}(\alpha) < 2$, $\operatorname{Re}(\beta) > 0$.

To prove this we apply the following.

Lemma 2.9 (asymptotic formula for the gamma function). *Let $a, \xi \in \mathbb{R}$, $a > 0$. Then we have*

$$\lim_{|\xi| \rightarrow \infty} \frac{|\Gamma(a + i\xi)|}{\sqrt{2\pi} e^{-\pi|\xi|/2} |\xi|^{a-1/2}} = 1.$$

This is well-known. In Section 4, we shall give a proof for completeness based on the formula

$$(2.9) \quad \lim_{\operatorname{Re}(z) \geq c > 0, |z| \rightarrow \infty} \frac{\Gamma(z)}{\sqrt{2\pi} e^{-z} z^{z-1/2}} = 1.$$

Proof of Lemma 2.8. We can see that $\Psi_{\alpha,\beta}(\xi)$ is analytic in α, β for $-n < \operatorname{Re}(\alpha) < 2$, $\operatorname{Re}(\beta) > 0$, if ξ is fixed. By Lemma 2.9, $|\Psi_{\alpha,\beta}(\xi)|$ behaves like $|\xi|^{\operatorname{Re}(\alpha) - \operatorname{Re}(\beta) + n/2 - 1}$

if $|\xi|$ is sufficiently large. Also we note that $\Psi_{\alpha,\beta}(\xi)$ is continuous and does not vanish in α, β, ξ with $-n < \operatorname{Re}(\alpha) < 2, \operatorname{Re}(\beta) > 0, \xi \in \mathbb{R}^n$. Thus we have

$$(2.10) \quad A(1 + |\xi|)^{\operatorname{Re}(\alpha) - \operatorname{Re}(\beta) + n/2 - 1} \leq |\Psi_{\alpha,\beta}(\xi)| \leq B(1 + |\xi|)^{\operatorname{Re}(\alpha) - \operatorname{Re}(\beta) + n/2 - 1}$$

with some positive numbers A and B independent of ξ .

Using (2.10), since $\hat{g} \in \mathcal{S}(\mathbb{R})$ and $\hat{\Theta}_\alpha$ is bounded and analytic for $\operatorname{Re}(\alpha) < 2$ by part (1) of Lemma 2.5, we can see $H(\alpha, \beta)$ is analytic for $-n < \operatorname{Re}(\alpha) < 2, \operatorname{Re}(\beta) > 0$. $\square$

Now we are able to prove the following.

Proposition 2.10. *Let $-n < \operatorname{Re}(\alpha) < 2, \operatorname{Re}(\beta) > 0$. Then*

$$\sigma_\beta(I^\alpha f)(x)^2 = \int_{-\infty}^{\infty} |K_{\alpha,\beta} * \Theta_\alpha(u)|^2 du = \int_{-\infty}^{\infty} |\Psi_{\alpha,\beta}(\xi)|^2 |\hat{\Theta}_\alpha(\xi)|^2 d\xi.$$

Proof. We recall that $G(\alpha, \beta)$ in (2.8) is analytic in α and β for $\operatorname{Re}(\alpha) > -n - 2, \operatorname{Re}(\beta) > 0$ and that $H(\alpha, \beta)$ is analytic for $-n < \operatorname{Re}(\alpha) < 2, \operatorname{Re}(\beta) > 0$ (Lemma 2.8). Further, by Lemma 2.7 we have $G(\alpha, \beta) = H(\alpha, \beta)$ if $-n/2 - 1 < \operatorname{Re}(\alpha) < -n/2$ and $\operatorname{Re}(\beta) > 0$. Thus by analytic continuation we see that $G(\alpha, \beta) = H(\alpha, \beta)$ if $-n < \operatorname{Re}(\alpha) < 2, \operatorname{Re}(\beta) > 0$. So, we have $|G(\alpha, \beta)| = |H(\alpha, \beta)|$ if $-n < \operatorname{Re}(\alpha) < 2, \operatorname{Re}(\beta) > 0$. Thus, taking the supremum over g in the unit ball of $L^2(\mathbb{R})$ and recalling (2.7), we get the conclusion. $\square$

On the other hand, for $D^\alpha(f)$ in (1.2) we have the following result.

Proposition 2.11. *Let Θ_α be as in Lemma 2.5. Suppose that $0 < \operatorname{Re}(\alpha) < 2$. Then*

$$D^\alpha(f)(x)^2 = \int_{-\infty}^{\infty} |\alpha - 2\pi i \xi|^{-2} |\hat{\Theta}_\alpha(\xi)|^2 d\xi.$$

Proof. We note that

$$\int_{S^{n-1}} (f(x - t\theta) - f(x)) d\sigma(\theta) = \varphi(t; x, f) - \varphi(0; x, f) = \int_0^1 \theta(tr; x, f) \frac{dr}{r}$$

and

$$t^{-\alpha}(\varphi(t; x, f) - \varphi(0; x, f)) = \int_0^1 r^\alpha \theta_\alpha(tr; x, f) \frac{dr}{r}.$$

Thus

$$D^\alpha(f)(x) = \left(\int_0^\infty \left| \int_0^1 r^\alpha \theta_\alpha(tr; x, f) \frac{dr}{r} \right|^2 \frac{dt}{t} \right)^{1/2}.$$

By the change of variables $r = e^v, t = e^{-u}$, we have

$$D^\alpha(f)(x)^2 = \int_{-\infty}^{\infty} \left| \int_{-\infty}^0 e^{\alpha v} \Theta_\alpha(u - v) dv \right|^2 du = \int_{-\infty}^{\infty} |\hat{\psi}_\alpha(\xi)|^2 |\hat{\Theta}_\alpha(\xi)|^2 d\xi,$$

where $\psi_\alpha(u) = e^{\alpha u} \chi_{\{u \leq 0\}}(u)$ and hence

$$\hat{\psi}_\alpha(\xi) = \int_{-\infty}^0 e^{\alpha u} e^{-2\pi i u \xi} du = \frac{1}{\alpha - 2\pi i \xi}.$$

Here we note that ψ_α and Θ_α are in $L^1(\mathbb{R})$ and in $L^2(\mathbb{R})$ if $0 < \operatorname{Re}(\alpha) < 2$ (see Lemma 2.5 for Θ_α). This completes the proof. $\square$

Proof of Theorem 1.1. Let $0 < \alpha < 2$. Then we note that

$$(\alpha/2)(1 + |\xi|) \leq |\alpha + 2\pi i\xi| \leq 2\pi(1 + |\xi|).$$

By this and (2.10) it follows that $|\alpha - 2\pi i\xi|^{-1}$ and $|\Psi_{\alpha,\beta}(\xi)|$ are pointwise equivalent as functions of ξ if $\beta = \alpha + n/2$. Using the pointwise equivalence and the formulas of Propositions 2.10 and 2.11, we have $\sigma_\beta(I^\alpha f)(x) \approx D^\alpha(f)(x)$ for $f \in \mathcal{S}_0(\mathbb{R}^n)$. Substituting $I_\alpha f$ for f and recalling (1.4), (2.3), we reach the conclusion. $\square$

3. APPLICATIONS

Let D^α , I_α and σ_β be as in (1.2), (1.4) and (1.3), respectively. Define

$$(3.1) \quad S_\alpha(f)(x) = D^\alpha(I_\alpha f)(x)$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Then, some L^p estimates for S_α , $0 < \alpha < 2$, are shown in [7] with weights for $1/2 \leq \alpha < 2$, which are useful in characterizing the Sobolev spaces of order α (see [1] for relevant results). When $1 < \alpha < 2$, the result is proved by applying a theorem of [6] for the boundedness of Littlewood-Paley operators. When $0 < \alpha < 1$ it is shown by using Theorem 1.1 for $\alpha \in (0, 1)$, which is due to [4], and applying known properties of σ_β . The result for the case $\alpha = 1$ is due to [3]. Here, we focus on the case $1/2 \leq \alpha < 2$. Then, more precisely, we can find the following result in [7].

Theorem A. *Suppose that $1/2 \leq \alpha < 2$, $w \in A_p$, $1 < p < \infty$. Let S_α be as in (3.1). Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\|S_\alpha(f)\|_{p,w} \simeq \|f\|_{p,w}.$$

We recall the weight class A_p of Muckenhoupt. A weight w belongs to A_p , $1 < p < \infty$, if

$$\sup_B \left(|B|^{-1} \int_B w(x) dx \right) \left(|B|^{-1} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and $|B|$ denotes the Lebesgue measure of B (see [2] for the A_p class). The weighted L^p space is defined as L_w^p with the norm

$$\|f\|_{L_w^p} = \|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}.$$

Theorem A is due to [3] when $\alpha = 1$, as mentioned above. We can now give a different proof of this by applying Theorem 1.1 as follows. Since it is known that $\sigma_{n/2+1}$ is bounded on L_w^p for all $p \in (1, \infty)$ and $w \in A_p$ (see [7]), by Theorem 1.1 with $\alpha = 1$ we have

$$\|S_1(f)\|_{p,w} \leq C\|f\|_{p,w}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. The reverse inequality follows from this by duality as in [7].

Similarly, we can give another proof of Theorem A for $\alpha \in (1, 2)$, which was proved in [7] from a result of [6], by applying Theorem 1.1 and the boundedness of $\sigma_{n/2+\alpha}$ on L_w^p with $w \in A_p$, $1 < p < \infty$.

4. PROOF OF LEMMA 2.9

By (2.9) to prove Lemma 2.9 it suffices to show

$$(4.1) \quad \exp\left(-\frac{a^3}{a^2 + \xi^2}\right) \left(1 + \frac{a^2}{\xi^2}\right)^{(a-1/2)/2} \leq \frac{|e^{-z} z^{z-1/2}|}{e^{-\pi|\xi|/2} |\xi|^{a-1/2}} \leq \left(1 + \frac{a^2}{\xi^2}\right)^{(a-1/2)/2},$$

where $z = a + i\xi$, $\xi \neq 0$.

To prove (4.1), we first note that

$$|z^{z-1/2}| = e^{(a-1/2) \log |z|} e^{-\xi \arg z} = (a^2 + \xi^2)^{(a-1/2)/2} e^{-|\xi| \arg(a+i|\xi|)},$$

where $-\pi/2 < \arg z < \pi/2$. We write $\arg(a + i|\xi|) = \arctan(|\xi|/a)$. Define a function F on $[0, \infty)$ by $F(x) = \arctan(1/x)$, $x > 0$, $F(0) = \pi/2$. Then by the mean value theorem, we have

$$\pi/2 - \arctan(|\xi|/a) = F(0) - F(a/|\xi|) = \frac{a}{|\xi|} \frac{1}{\eta^2 + 1}$$

for some $\eta \in (0, a/|\xi|)$. Thus

$$e^{-a} e^{|\xi|(\pi/2 - \arg(a+i|\xi|))} = e^{-a\eta^2/(\eta^2+1)}$$

and hence

$$(4.2) \quad |e^{-z} z^{z-1/2}| = e^{-a} (a^2 + \xi^2)^{(a-1/2)/2} e^{-\pi|\xi|/2} e^{|\xi|(\pi/2 - \arg(a+i|\xi|))} \\ = e^{-\pi|\xi|/2} |\xi|^{a-1/2} \left(1 + \frac{a^2}{\xi^2}\right)^{(a-1/2)/2} e^{-a\eta^2/(\eta^2+1)}.$$

Since

$$e^{-a^3/(a^2+\xi^2)} < e^{-a\eta^2/(\eta^2+1)} < 1,$$

from (4.2) we obtain (4.1). This completes the proof of Lemma 2.9.

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