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Note on a Littlewood-Paley operator in higher dimensions

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Abstract. We give a simple proof of a Littlewood-Paley inequality for arbitrary rectangles in $\mathbb{R}^n$, $n \geq 3$.

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§1. Introduction. For a sequence $\{R_k\}$ of disjoint rectangles in $\mathbb{R}^n$ with sides parallel to the axes, let $\Delta f = \left(\sum |S_{R_k} f|^2\right)^{1/2}$, where S_{R_k} is a Fourier multiplier operator defined by $(S_{R_k} f)^\wedge = \chi_{R_k} \hat{f}$. Then, the following is known:

Theorem (a Littlewood-Paley inequality for arbitrary rectangles). For $p \in [2, \infty)$, there exists a constant c_p such that

$$\|\Delta f\|_p \le c_p \|f\|_p \qquad (f \in L^p(\mathbb{R}^n)).$$

This was proved by Rubio de Francia [4] for $n = 1$ and by Journé [1] for $n \ge 2$. Soria [5] gave a simple proof of the theorem for $n = 2$, by applying Journé's covering lemma of [2].

On the other hand, Journé's covering lemma was extended to higher dimensions by Pipher [3]. In this note we give a simple proof of the theorem for $n \ge 3$, by using Pipher's covering lemma. As in [1], we prove the L^∞-BMO boundedness of a certain decomposition operator (Lemma 8). We prove the lemma by induction on the dimension n, and then the covering lemma and Fubini's theorem are used effectively.

In §2, we review Pipher's covering lemma and apply it in §3 to show the theorem.

§2. Covering lemmas. For an open set U in $\mathbb{R}^n$, let $D(U)$ denote the collection of dyadic rectangles in U. Here, a dyadic rectangle is a rectangle of the form $\pi_{1\le i\le n}(\ell_i 2^{k_i}, (\ell_i + 1)2^{k_i})$ with ℓ_i, $k_i \in \mathbb{Z}$ (the set of integers). (For convenience, we consider open dyadic rectangles.) Put $D_n = D(\mathbb{R}^n)$. For a bounded open set Ω in $\mathbb{R}^n$ $(n \ge 2)$, let $M_n(\Omega)$ denote the collection of dyadic rectangles $R \subset \Omega$ which are maximal in Ω in the x_n-direction. Here, the maximality in Ω in the x_n-direction of R means that if $R = I_1\times\ldots\times I_{n-1}\times I_n$ and if $R' = I_1\times\ldots\times I_{n-1}\times I_n'$ is a dyadic rectangle such that $R' \subset \Omega$, $R \subset R'$, then $R = R'$. (For each i, $1 \le i \le n$, the maximality in Ω in the x_i-direction is defined in the same way.) When $\Omega \subset \mathbb{R}$, $M_1(\Omega)$ $(= M(\Omega))$ denotes the collection of maximal dyadic intervals in Ω.

Let Ω be a bounded open set in $\mathbb{R}^n$ $(n \ge 2)$. For $I \in D_1$ and $S \in D_{n-1}$, we define $\mathcal{I}(I,S;\Omega)$ to be the maximum element of the set:

$$\{I' \in D_1 : I' \supset I,\ |I'\times S \cap \Omega| > \tfrac{1}{2}|I'\times S|\}$$

if this set is not empty; otherwise, let $\mathcal{I}(I,S;\Omega) = I$. Next, for $I \in D_1$ and $k \in \mathbb{N}$ (the set of positive integers), put

$$G(I,k;\Omega) = \cup\,\{S \in D_{n-1} : I\times S \subset \Omega,\ \mathcal{I}(I,S;\Omega) = I(k-1)\},$$

where $I(k-1)$ denotes the dyadic interval containing I of length $2^{k-1}|I|$.

The following lemma is essentially due to Pipher [3].

Lemma 1. (a) Let $\Omega^* = \{x \in \mathbb{R}^n: M_S(\chi_\Omega)(x) > 1/2\}$, where M_S denotes the strong maximal operator. Then

$$I(k - 1)\times G(I,k;\Omega) \subset \Omega^*.$$

(b) Let $w:[0,\infty) \to [0,\infty)$ be increasing and such that $\sum_{k=1}^{\infty} kw(2^{-k}) < \infty$. Then

$$\sum_{I\in D_1} \sum_{k=1}^{\infty} |I|w(2^{-k})|G(I,k;\Omega)| \leq c|\Omega|.$$

Proof. By the definition of $G(I,k;\Omega)$, (a) is obvious. Next, for an open set U and $I \in D_1$, let $E_I(U) = \cup\{S \in D_{n-1}: I\times S \subset U\}$. Then, as in [3], we have

$$|G(I,k;\Omega)| \leq c|E_I(\Omega)\setminus E_{I(k)}(\Omega)|.$$

Thus, (b) follows from the inequality:

$$\sum_{I\in D_1} \sum_{k=1}^{\infty} |I|w(2^{-k})|E_I(\Omega)\setminus E_{I(k)}(\Omega)| \leq c|\Omega|,$$

which was proved in [3]. This completes the proof of Lemma 1.

For $R = I_1 \times \ldots \times I_n \in D_n$, we define $k(R,\Omega)$ ($\in \mathbb{N}$) by

$$\mathcal{I}(I_1, I_{[2,n]};\Omega) = I_1(k(R,\Omega) - 1) \qquad (I_{[2,n]} = I_2 \times \ldots \times I_n).$$

Then, we easily see the following:

Lemma 2. If $R \in D(\Omega)$, then $I_{[2,n]} \subset G(I_1, k(R,\Omega);\Omega)$ and if $R \in M_n(\Omega)$, then $I_{[2,n]} \in M_{n-1}(G(I_1,k(R,\Omega);\Omega))$.

We have defined an open set $G(I,k;\Omega)$ and a positive integer $k(R,\Omega)$. In the following, we will consider G and k in different dimensions to make definitions.

When $n \geq 3$, for $I_1, \ldots, I_{n-1} \in D_1$ and $k_1, \ldots, k_{n-1} \in \mathbb{N}$, we define open sets $G(I_{[1,i]}, k_{[1,i]};\Omega)$ $(k_{[1,i]} = (k_1, \ldots, k_i))$ in $\mathbb{R}^{n-i}$ $(2 \leq i \leq n-1)$ by the relation:

$$G(I_{[1,i+1]}, k_{[1,i+1]};\Omega) = G(I_{i+1}, k_{i+1}; G(I_{[1,i]}, k_{[1,i]};\Omega))$$

$$(i = 1, \ldots, n-2)$$

Then, by Lemma 1 we have the following:

Lemma 3. Let $n \geq 3$. (a) For $i = 1, \ldots, n-2$,

$$I_{i+1}(k_{i+1} - 1) \times G(I_{[1,i+1]}, k_{[1,i+1]}; \Omega) \subset G^*(I_{[1,i]}, k_{[1,i]}; \Omega).$$

(b) Let w be as in Lemma 1. Then, for $i = 1, \ldots, n-2$,

$$\Sigma_{I_{i+1} \in D_1} \Sigma_{k_{i+1}=1}^{\infty} |I_{i+1}| w(2^{-k_{i+1}}) |G(I_{[1,i+1]}, k_{[1,i+1]}; \Omega)| \leq$$

$$c|G(I_{[1,i]}, k_{[1,i]}; \Omega)|.$$

Let $R = I_1 \times \ldots \times I_n \in D_n$. We define $k_i(R) = k_i(R,\Omega)$ $(\in \mathbb{N})$ for $i = 1, \ldots, n-1$. First, let $k_1(R,\Omega) = k(R,\Omega)$. Then, for $i \geq 2$, define k_i one after another by

$$k_i(R,\Omega) = k(I_{[i,n]}, G(I_{[1,i-1]}, k_{[1,i-1]}(R,\Omega); \Omega))$$

$$(k_{[1,i-1]}(R,\Omega) = (k_1(R,\Omega), \ldots, k_{i-1}(R,\Omega))).$$

Then, by Lemma 2 we obtain the following:

Lemma 4. If $R \in D(\Omega)$, then $I_{[i+1,n]} \subset G(I_{[1,i]}, k_{[1,i]}(R,\Omega); \Omega)$ and if $R \in M_n(\Omega)$, then, $I_{[i+1,n]} \in M_{n-i}(G(I_{[1,i]}, k_{[1,i]}(R,\Omega); \Omega))$ $(i = 1, \ldots, n-1)$.

For $R = I_1\times\ldots\times I_n \in D(\Omega)$ $(n \geq 2)$, define $\hat{I}_j = \hat{I}_j(R) = \hat{I}_j(R,\Omega) \in D_1$ $(j = 1, \ldots, n-1)$ by $\hat{I}_j = I_j(k_j(R,\Omega) - 1)$. Then, we have the following (see Pipher [3] for $n \geq 3$ and Journé [2] for $n = 2$):

Lemma 5. (a) $|\cup_{R\in D(\Omega)} \hat{R}| \leq c|\Omega|$, where $\hat{R} = \hat{R}(\Omega) = \hat{I}_1\times\ldots\times\hat{I}_{n-1}\times I_n$;

(b) $\Sigma_{R\in M_n(\Omega)} |R|w(|I_1|/|\hat{I}_1|)\ldots w(|I_{n-1}|/|\hat{I}_{n-1}|) \leq c|\Omega|$, where w is as in Lemma 1.

We give a proof of Lemma 5, for completeness. Using (a) of Lemma 1 and (a) of Lemma 3, we see that $\hat{I}_1\times\ldots\times\hat{I}_{n-1}\times I_n \subset \Omega^{*(n-1)}$, where $\Omega^{*(1)} = \Omega^*$ and $\Omega^{*(k+1)} = (\Omega^{*(k)})^*$. Thus (a) holds.

To prove (b), we rewrite the sum as follows:

$$\Sigma_{R\in M_n(\Omega)} |R| \pi_{i\in[1,n-1]} w(|I_i|/|\hat{I}_i|) =$$

$$\Sigma_{I_1\in D_1} \Sigma_{k_1=1}^{\infty} |I_1| w(2^{-k_1+1}) \ldots \Sigma_{I_{n-1}\in D_1} \Sigma_{k_{n-1}=1}^{\infty} |I_{n-1}| w(2^{-k_{n-1}+1}) \Sigma_{I_n} |I_n|,$$

where in the last sum Σ_{I_n}, the dyadic interval I_n runs over the set $\{I_n \in D_1 : R \in M_n(\Omega),\ k_i(R,\Omega) = k_i,\ 1 \leq i \leq n-1\}$. By Lemma 4, such an I_n belongs to $M_1(G(I_{[1,n-1]}, k_{[1,n-1]}; \Omega))$, which implies that

$$\Sigma_{I_n} |I_n| \le |G(I_{[1,n-1]}, k_{[1,n-1]}; \Omega)|.$$

Thus by successive applications of (b) of Lemma 3 and (b) of Lemma 1 in the rewrited sum, we obtain the desired inequality. This completes the proof.

Next, we enlarge the rectangle $\hat{R} \in D(\Omega^{*(n-1)})$, by applying the same procedure as above but changing the enlargement order of intervals. By continuing enlargement of rectangles in this way, we can obtain the following lemma, which will be used in the proof of the theorem.

Lemma 6. Let Ω be a bounded open set in $\mathbb{R}^n$ $(n \ge 2)$. Then for $R = I_1 \times \ldots \times I_n \in M_n(\Omega)$, there exist dyadic intervals $\tilde{I}_i = \tilde{I}_i(R)$, $1 \le i \le n$, satisfying the following properties:

(a) $\tilde{R} \supset R$, where $\tilde{R} = \tilde{I}_1 \times \ldots \times \tilde{I}_n$.

(b) $|\cup_{R \in M_n(\Omega)} \tilde{R}| \le c|\Omega|$.

(c) For every permutation σ of the set $\{1, 2, \ldots, n\}$, there exist a bounded open set $\Omega_\sigma \supset \Omega$ and a mapping T_σ: $M_n(\Omega) \to D(\Omega_\sigma)$ such that $|\Omega_\sigma| \le c|\Omega|$; $T_\sigma(R) \supset R$; and $S_\sigma^{-1}[\{S_\sigma T_\sigma(R)\}^{\wedge}(S_\sigma \Omega_\sigma)] \subset \tilde{R}$, where S_σ: $\mathbb{R}^n \to \mathbb{R}^n$ is defined by $S_\sigma x = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$.

(d) $\Sigma_{R \in M_n(\Omega)} |R| \pi_{i=1}^{n-1} w(|I_i|/|\tilde{I}_i|) \le c|\Omega|$ (w is as in Lemma 1).

§3. A proof of a Littlewood-Paley inequality. Let $\psi \in \mathcal{S}(\mathbb{R})$ (the Schwartz space) be such that $\chi_{[-2,2]} \le \hat{\psi} \le \chi_{[-3,3]}$. For integers j, k, let

$$T_k^j f(x) = \int_{-\infty}^{\infty} K_k^j(x,y) f(y)\, dy,$$

where $K_k^j(x,y) = 2^k \psi(2^k(x - y)) e^{-2\pi i j 2^k y}$.

Let $\alpha\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \{0, 1\}$ be such that

$$\sum_{(j,k)\in\mathbb{Z}^{2n}} \alpha(j,k) \prod_{1\le i\le n} \chi_{[-3,3]}(2^{-k_i}\xi_i - j_i) \le c_\alpha$$

for some constant c_α, where $j = (j_i)$, $k = (k_i)$, $\xi_i \in \mathbb{R}$. Then, we define the bounded operator $F_\alpha\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n, \ell^2(\mathbb{Z}^{2n}))$ by

$$F_\alpha f(x) = \left(\alpha(j,k)[T_{1,k_1}^{j_1} \cdots T_{n,k_n}^{j_n}] f(x)\right)_{j,k},$$

where $T_{i,k_i}^{j_i}$ is the operator $T_{k_i}^{j_i}$ which acts only on the variable x_i, that is,

$$T_{i,k_i}^{j_i} f(x) = [T_{k_i}^{j_i} f(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)](x_i).$$

We also write $T_{i,k_i}^{j_i} = T_{k_i}^{j_i}$.

It is known that the theorem stated in section 1 follows from the boundedness of the operator F_α from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n, \ell^2(\mathbb{Z}^{2n}))$ for $2 \le p < \infty$ (see [1], [4]) and by interpolation this boundedness follows from that of F_α from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}\times\ldots\times\mathbb{R}, \ell^2(\mathbb{Z}^{2n}))$.

Let $Q_t = Q_{t_1}\ldots Q_{t_n}$ $(t = (t_1, \ldots, t_n), t_i > 0)$, where Q_{t_i} is an operator which acts only on the variable x_i by convolution with $q_{t_i}(x_i) = t_i^{-1}q(x_i/t_i)$. Here $q \in C_0^\infty(\mathbb{R})$ is even and such that $\mathrm{supp}(q) \subset (-1, 1)$, $\int q = 0$, $\int_0^\infty |\hat{q}(s)|^2\, ds/s = 1$.

Then the L^∞-BMO boundedness of F_α follows from the following:

Lemma 7. For $b \in L^\infty(\mathbb{R}^n)$, let

$$d\mu_b(x,t) = \sum_{(j,k)\in\mathbb{Z}^{2n}} \alpha(j,k)\,|Q_t T_k^j b(x)|^2\, dx\, \frac{dt}{t},$$

where $T_k^j = T_{k_1}^{j_1}\ldots T_{k_n}^{j_n}$, $dt/t = dt_1/t_1\ldots dt_n/t_n$. Then μ_b is a Carleson measure on $(\mathbb{R}_+^2)^n$, that is, $\mu_b(S(\Omega)) \le c\|b\|_\infty^2|\Omega|$ for every bounded open set Ω in $\mathbb{R}^n$, where $S(\Omega)$ denotes the set:

$$\{(x,t) = (x_1,t_1;\ldots;x_n,t_n) \in (\mathbb{R}_+^2)^n : \prod_{1\le i\le n}(x_i - t_i, x_i + t_i) \subset \Omega\}.$$

Lemma 7 is an immediate consequence of the following:

Lemma 8. Let Ω be a bounded open set in $\mathbb{R}^n$ and let $b \in L^\infty(\mathbb{R}^n)$. Then there exists a non-negative function $g \in L^1(\mathbb{R}^n)$ depending only on Ω such that $\|g\|_1 \le |\Omega|$ and

$$\mu_b(S(\Omega)) \le c\int_{\mathbb{R}^n} |b(z)|^2 g(z)\, dz,$$

where c is a constant depending only on c_α and n.

To prove the theorem, thus it only remains to show Lemma 8.

Proof of Lemma 8. We prove it by induction. Let $A(n)$ ($n \ge 1$) denote the assertion of Lemma 8 for $\mathbb{R}^n$.

First we prove $A(1)$. If I is an interval in $\mathbb{R}$, rI ($r > 0$) denotes, as usual, the interval of length $r|I|$ and with the same center as I. For a bounded open set Ω in $\mathbb{R}$, we put $\tilde{\Omega} = \cup_{I \in M(\Omega)} 100I$.

Let $b_1 = b\chi_{\tilde{\Omega}}$, $b_2 = b - b_1$. Then by the L^2-boundedness of F_α we have

$$\mu_{b_1}(S(\Omega)) \le c\|b_1\|_2^2 = c\int |b(z)|^2 \chi_{\tilde{\Omega}}(z)\, dz.$$

In the following, we show the existence of a sequence $\{g_k\}$ of non-negative functions on $\mathbb{R}$ such that $\mu_{b_2}(S(\Omega)) \le c\sum_k \int |b|^2 g_k\, dz$, $\sum_k \|g_k\|_1 \le c|\Omega|$. Then the function g of Lemma 8 is obtained by normalizing $\chi_{\tilde{\Omega}} + \Sigma_k g_k$.

We use the following result of Journé [1].

Lemma 9. Let $(x,t) \in (\mathbb{R}_+^2)^n$ $(n \geq 1)$ and let $b_{x,t} \in L^\infty(\mathbb{R}^n)$ be such that $\mathrm{supp}(b_{x,t}) \subset \{z \in \mathbb{R}^n: |x_i - z_i| \geq 2t_i,\ 1 \leq i \leq n\}$. Then

$$\Sigma_{(j,k)\in\mathbb{Z}^{2n}} |Q_t T_k^j b_{x,t}(x)|^2 \leq c \int |b_{x,t}(z)|^2 \prod_{1\leq i\leq n} a(z_i,x_i,t_i)\, dz,$$

where $a(z_i,x_i,t_i) = t_i^\varepsilon/|x_i - z_i|^{1+\varepsilon}$, $0 < \varepsilon < 1/2$.

For an interval I, let $e(I,x) = |c(I) - x|^{-1-\varepsilon}$ $(x \in \mathbb{R},\ x \neq c(I))$, where $c(I)$ denotes the center of I. Then, since $S(\Omega) \subset \cup_{I\in M(\Omega)} S(\bar{I})$, where $\bar{I} = 5I$, by using Lemma 9 for $n = 1$ we have

$$\begin{aligned}
\mu_{b_2}(S(\Omega)) &\leq \mu_{b_2}(\cup_{I\in M(\Omega)} S(\bar{I})) \leq \Sigma_{I\in M(\Omega)} \mu_{b_2}(S(\bar{I})) \\
&= \Sigma_I \int_{S(\bar{I})} \Sigma\, \alpha(j,k) |Q_t T_k^j b_2(x)|^2\, dx\, \frac{dt}{t} \\
&\leq c\Sigma_I \int_{S(\bar{I})} \int |b_2(z)|^2 a(z,x,t)\, dz\, dx\, \frac{dt}{t} \\
&\leq c\Sigma_I \int |b(z)|^2\, |I|^{1+\varepsilon} \chi_{(100I)^c}(z) e(I,z)\, dz \\
&= c\Sigma_I \int |b(z)|^2 g_I(z)\, dz, \quad \text{say.}
\end{aligned}$$

We easily see that $\sum_{I\in M(\Omega)}\|g_I\|_1 \le c\sum_I |I| \le c|\Omega|$. This is what we need. Thus the proof of A(1) is complete.

Next we prove A(n) $(n \ge 2)$, assuming A(m) for every $m \le n - 1$. Let Ω be a bounded open set in $\mathbb{R}^n$ and let $b \in L^\infty(\mathbb{R}^n)$. We put $\tilde{\Omega} = \cup_{R\in M_n(\Omega)} 100\tilde{R}$, where for a rectangle $R = I_1\times\ldots\times I_n$ and $r > 0$, rR is defined by $rR = rI_1\times\ldots\times rI_n$. Recall that $|\tilde{\Omega}| \le c|\Omega|$ (Lemma 6).

We assume that $\operatorname{supp}(b) \subset \tilde{\Omega}^c$. Let $\Lambda = \{1, 2, \ldots, n\}$. Then

$$b(z) = \sum_{I\subset\Lambda,\ I\ne\phi} (-1)^{|I|-1} b(z) \prod_{i\in I} \chi((100\tilde{I}_i(R))^c, z_i) = \sum_I b_{I,R},$$

for all $R \in M_n(\Omega)$, where $|I|$ denotes the number of the elements of I and we write $\chi(E,z_i) = \chi_E(z_i)$. Note that $S(\Omega) \subset \cup_{R\in M_n(\Omega)} S(\bar{R})$, where $\bar{R} = 5R$. Thus if $\{\bar{S}(\bar{R})\}$ $(R \in M_n(\Omega))$ is a collection of disjoint sets such that $\bar{S}(\bar{R}) \subset S(\bar{R})$, $\cup\, \bar{S}(\bar{R}) = \cup\, S(\bar{R})$, then we have

$$\mu_b(S(\Omega)) \le \mu_b(\cup\, S(\bar{R})) = \sum_{R\in M_n(\Omega)} \mu_b(\bar{S}(\bar{R})) \le c\sum_I \sum_R \mu_{b_{I,R}}(\bar{S}(\bar{R})).$$

For each $I \subset \Lambda$, $I \ne \phi$, we prove below the existence of a sequence $\{g_{k,I}\}_k$ of non-negative functions such that

$$\sum_R \mu_{b_{I,R}}(\bar{S}(\bar{R})) \le c\sum_k \int |b|^2 g_{k,I}\, dz, \qquad \sum_k \|g_{k,I}\|_1 \le c|\Omega|.$$

By the same argument as in the proof of $A(1)$, this is sufficient for the proof of $A(n)$.

(I) Estimate for $\sum_R \mu_{b_{\Lambda,R}}(\bar{S}(\bar{R}))$. By Lemma 9, we have

$$\mu_{b_{\Lambda,R}}(\bar{S}(\bar{R})) \le c\int_{S(\bar{R})} \int |b_{\Lambda,R}(z)|^2 \prod_{1\le i\le n} a(z_i,x_i,t_i)\ dz\ dx\ \frac{dt}{t}$$

$$\le c\int |b(z)|^2 g_R(z)\ dz,$$

where $g_R(z) = |R|^{1+\varepsilon}\pi_i\chi((100\hat{r}_i(R))^c, z_i)e(I_i, z_i)$ $(R = I_1\times\ldots\times I_n)$. From (d) of Lemma 6 with $w(t) = t^{\varepsilon}$, it follows that $\sum_R \|g_R\|_1 \le c\sum |R|^{1+\varepsilon}|\hat{R}|^{-\varepsilon} \le c|\Omega|$. This is what we have to show.

(II) Estimate for $\sum_R \mu_{b_{I,R}}(\bar{S}(\bar{R}))$ in the case $|I| = q \le n - 1$. Let σ be the permutation of Λ such that $I = \{\sigma(1), \sigma(2), \ldots, \sigma(q)\}$ and $J = \Lambda - I = \{\sigma(q+1), \sigma(q+2), \ldots, \sigma(n)\}$ $(\sigma(i) < \sigma(i+1)$ if $i \ne q)$. Let $\Omega_\sigma \supset \Omega$, T_σ: $M_n(\Omega) \to D(\Omega_\sigma)$ be as in Lemma 6. Put $\{1,\ldots, q\} = K$. Then

$$(A) = \sum_{R\in M_n(\Omega)} \mu_{b_{I,R}}(\bar{S}(\bar{R})) = \sum_{Q\in D(\Omega_\sigma)} \sum_{R\in T_\sigma^{-1}(Q)} \mu_{b_{I,R}}(\bar{S}(\bar{R}))$$

$$= \sum_{H\in D_q} \sum_{m_K\in \mathbb{N}^q} \sum_{L\in \mathcal{L}(H,m_K)} \sum_{R\in T_\sigma^{-1}S_\sigma^{-1}(H\times L)} \mu_{b_{I,R}}(\bar{S}(\bar{R})),$$

where

$$\mathcal{L}(H, m_K) =$$

$$\{L \in D(G(H, m_K; S_\sigma \Omega_\sigma)):\ H \times L \in D(S_\sigma \Omega_\sigma),\ k_i(H \times L, S_\sigma \Omega_\sigma) = m_i,\ 1 \le i \le q\}$$

with $m_K = (m_1, \ldots, m_q)$ (see Lemma 4).

Fix H, m_K and let

$$(B) = \Sigma_{L \in \mathcal{L}(H, m_K)} \Sigma_{R \in T_\sigma^{-1} S_\sigma^{-1}(H \times L)} \mu_{b_{I,R}}(\bar{S}(\bar{R})).$$

Then by (c) of Lemma 6, we have

$$(B) = \Sigma_{\ell_K \ge m_K} \Sigma_{L \in \mathcal{L}(H, m_K)} \Sigma_{R \in \mathcal{R}(L, \ell_K)} \mu_{b_{I,R}}(\bar{S}(\bar{R}))$$

$$= \Sigma_{\ell_K \ge m_K} \Sigma_{L \in \mathcal{L}(H, m_K)} \Sigma_{R \in \mathcal{R}(L, \ell_K)} \mu_{b(H, \ell_K)}(\bar{S}(\bar{R})),$$

where

$$\mathcal{R}(L, \ell_K) = \{R \in T_\sigma^{-1} S_\sigma^{-1}(H \times L):\ \tilde{I}_{\sigma(i)}(R) = J_i(\ell_i - 1),\ 1 \le i \le q\}$$

with $H = J_1 \times \ldots \times J_q$,

$$b(H, \ell_K)(z) = b_{H, \ell_K}(z) = (-1)^{q-1} b(z) \pi_{i \in K}\ \chi((100 J_i(\ell_i - 1))^c, z_{\sigma(i)})$$

and $\ell_K \ge m_K$ means that $\ell_i \ge m_i$ for every $i \in \{1, \ldots, q\}$.

Since $T_\sigma(R) \supset R$ and $\{\bar{S}(\bar{R})\}$ is disjoint, we see that

$$\Sigma_{L\in\mathcal{L}(H,m_K)} \Sigma_{R\in T_\sigma^{-1}S_\sigma^{-1}(H\times L)} \chi_{\bar{S}(\bar{R})}(x,t) \le \chi_{S(\bar{H})}(x_I,t_I)\chi_{\cup S(\bar{L})}(x_J,t_J),$$

where $\cup S(\bar{L}) = \cup_{L\in\mathcal{L}(H,m_K)} S(\bar{L})$ and $x_I = (x_{\sigma(1)},\dots,x_{\sigma(q)})$, etc. Thus

$$(B) = \Sigma_{\ell_K\ge m_K} \Sigma_{L\in\mathcal{L}(H,m_K)} \Sigma_{R\in\mathcal{R}(L,\ell_K)} \int_{\bar{S}(\bar{R})} \Sigma\, \alpha(j,k)\,|Q_tT_k^jb_{H,\ell_K}(x)|^2\, dx\, \frac{dt}{t}$$

$$\le \Sigma_{\ell_K} \int_{S(\bar{H})} \Sigma_{j_I,k_I}\Big(\int_{\cup S(\bar{L})} \Sigma_{j_J,k_J}\alpha(j,k)\,|Q_tT_k^jb_{H,\ell_K}(x)|^2\, dx_J\, \frac{dt_J}{t_J}\Big)dx_I\, \frac{dt_I}{t_I},$$

where $dt_I/t_I = dt_{\sigma(1)}/t_{\sigma(1)}\cdots dt_{\sigma(q)}/t_{\sigma(q)}$, $dx_I = dx_{\sigma(1)}\cdots dx_{\sigma(q)}$, etc. Observing that $\cup\, S(\bar{L}) \subset S(G^{*M}(H,m_K))$ for some $M \in \mathbb{N}$ (we omit $S_\sigma\Omega_\sigma$) and

$$\Sigma_{j_J,k_J}\, \alpha(j,k)\,\Pi_{i\in J}\, \chi_{[-3,3]}(2^{-k_i}\xi_i - j_i) \le c_\alpha\ ,$$

we apply to the inner integral the assertion $A(n-q)$, taking for Ω the open set $G^{*M}(H,m_K)$. Then it is majorized by

$$c \int |Q_{t_I}T_{k_I}^{j_I}b_{H,\ell_K}(S_\sigma^{-1}(x_I,z_J))|^2 g_{H,m_K}(z_J)\, dz_J,$$

where $Q_{t_I} = \pi_{i\in I} Q_{t_i}$, $T_{k_I}^{j_I} = \pi_{i\in I} T_{k_i}^{j_i}$ and the function g_{H,m_K} satisfies $\|g_{H,m_K}\|_1 \le c|G(H,m_K)|$.

Thus using Lemma 9, we have

$$(B) \le c\Sigma_{\ell_K \ge m_K} \int_{S(\bar{H})} \int |b_{H,\ell_K}(z)|^2 \pi_{i\in I} a(z_i,x_i,t_i) g_{H,m_K}(z_J)\ dz\ dx_I \frac{dt_I}{t_I}$$

$$\le c\Sigma_{\ell_K \ge m_K} \int |b(z)|^2\ g_{H,m_K,\ell_K}(z)\ dz,$$

where

$$g_{H,m_K,\ell_K}(z) =$$
$$|H|^{1+\varepsilon} g_{H,m_K}(z_J) \pi_{i\in K}\ \chi(((100J_i(\ell_i-1))^c, z_{\sigma(i)}) e(J_i, z_{\sigma(i)}).$$

Thus we obtain $(A) \le c\Sigma_H\ \Sigma_{m_K}\ \Sigma_{\ell_K \ge m_K} \int |b|^2\ g_{H,m_K,\ell_K}\ dz$. Furthermore, by (b) of Lemma 1 and (b) of Lemma 3 with $w(t) = t^{\varepsilon}$, we see that

$$\Sigma_H\ \Sigma_{m_K}\ \Sigma_{\ell_K \ge m_K} \|g_{H,m_K,\ell_K}\|_1 \le c\Sigma_H\ \Sigma_{m_K}\ \Sigma_{\ell_K \ge m_K} |H| 2^{-\varepsilon\ell_1} \cdots 2^{-\varepsilon\ell_q} \|g_{H,m_K}\|_1$$

$$\le c\Sigma_H\ \Sigma_{m_K}\ |H| 2^{-\varepsilon m_1} \cdots 2^{-\varepsilon m_q} |G(H,m_K)| \le c|S_\sigma \Omega_\sigma| \le c|\Omega|.$$

This gives a necessary estimate for the case (II). Thus the proof of A(n) is complete, which finishes the proof of Lemma 8.

References

[1] J. L. Journé, Calderón-Zygmund operators on product spaces, Revista Ibero-Americana 1 (1985), 55-91.

[2] J. L. Journé, A covering lemma for product spaces, Proc. Amer. Math. Soc. 96 (1986), 593-598.

[3] J. Pipher, Journé's covering lemma and its extension to higher dimensions, Duke Math. J. 53 (1986), 683-690.

[4] J. L. Rubio de Francia, A Littlewood-Paley inequality for arbitrary intervals, Mittag Leffler Report No.18, 1983.

[5] F. Soria, A note on a Littlewood-Paley inequality for arbitrary intervals in $\mathbb{R}^2$, J. London Math. Soc. 36 (1987), 137-142.

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