An intrinsic characterization of the unit polydisc

Michigan Math. J. 56 (2008)

An Intrinsic Characterization of the Unit Polydisc

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1. Introduction

Let M be a connected complex manifold and $\mathrm{Aut}(M)$ the group of all biholomorphic automorphisms of M. Then, equipped with the compact-open topology, $\mathrm{Aut}(M)$ is a topological group acting continuously on M.

In 1907 it was shown by Poincaré [15] that the Riemann mapping theorem does not hold in the higher-dimensional case. In fact, he proved that *there exists no biholomorphic mapping from the unit polydisc* Δ^2 *onto the unit ball* B^2 *in* $\mathbb{C}^2$ by comparing carefully the topological structures of the isotropy subgroups of $\mathrm{Aut}(\Delta^2)$ and $\mathrm{Aut}(B^2)$ at the origin o of $\mathbb{C}^2$. In view of this fact, for a given complex manifold M it is an interesting problem to bring out some complex analytic nature of M under some topological conditions on $\mathrm{Aut}(M)$.

In connection with this problem, in this paper we would like to study the following question.

Question. Let M and N be connected complex manifolds and assume that their holomorphic automorphism groups $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$ are isomorphic as topological groups. Then, is M biholomorphically equivalent to N?

Recall that there exist relatively compact strictly pseudoconvex domains D_t $(t \in \mathbb{R})$ in a complex manifold X such that D_s is not biholomorphically equivalent to D_t unless $s = t$, and further, the only holomorphic automorphism of D_t is the identity for every t (see [3]). Thus, the answer to our question is negative, in general. However, there already exist several articles solving this question affirmatively in the case where the manifolds M or N are some special domains in $\mathbb{C}^n$ (see e.g. [4; 5; 6; 10; 11]). In particular, as an application of the classification theorem obtained by Isaev and Kruzhilin [6] for complex manifolds of dimension n admitting effective actions of the unitary group $U(n)$, Isaev [5] showed that *if the holomorphic automorphism group* $\mathrm{Aut}(M)$ *of a connected complex manifold* M *of dimension* n *is isomorphic to the holomorphic automorhism group* $\mathrm{Aut}(B^n)$ *of the unit ball* B^n *in* $\mathbb{C}^n$ *as topological groups, then* M *is biholomorphically equivalent to* B^n. In view of this, it would naturally be expected that exactly the same conclusion is

Received April 2, 2007. Revision received July 10, 2007.

The authors are partially supported by the Grant-in-Aid for Scientific Research (C) no. 17540153 and (C) no. 18540154, the Ministry of Education, Science, Sports and Culture, Japan.

valid also for the unit polydisc Δ^n in $\mathbb{C}^n$. This cannot be clarified in full generality at the moment. However, under some suitable condition on the manifold M, we can establish the following intrinsic characterization of the unit polydisc as our main result in this paper.

Theorem. *Let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(\Delta^n)$ *as topological groups. Then M is biholomorphically equivalent to Δ^n.*

Let D be an arbitrary domain in $\mathbb{C}^n$. Then it is well known that D admits a smooth envelope of holomorphy (cf. [13, Chaps. 6 and 7]). Hence, as an immediate consequence of the theorem, we obtain the following.

Corollary. *Let M be a connected Stein manifold of dimension n or a domain in $\mathbb{C}^n$. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(\Delta^n)$ *as topological groups. Then M is biholomorphically equivalent to Δ^n.*

Our proof of the theorem is based on three main facts: a well-known fact (due to Barrett, Bedford, and Dadok [1]) concerning torus actions on complex manifolds; an important fact (observed by Nakajima [12]) regarding homogeneous hyperbolic manifolds; and a fact (due to Kodama [9]) about the relationship between boundedness and hyperbolicity in the category of Reinhardt (more generally, circular) domains in $\mathbb{C}^n$. After recalling these facts as well as the structure of $\mathrm{Aut}(\Delta^n)$ in Section 2, we prove our theorem in Section 3.

Acknowledgment. The authors would like to thank the referee for many useful comments and suggestions that led to several improvements in the manuscript.

2. Preliminaries

For later purposes we collect some known facts in this section.

Let us start with recalling the structure of $\mathrm{Aut}(\Delta^n)$. We fix a coordinate system $z = (z_1, \dots, z_n)$ in $\mathbb{C}^n$ and set

$$\Delta_j = \{z_j \in \mathbb{C} \mid |z_j| < 1\} \;\; (1 \le j \le n) \quad \text{and} \quad \Delta^n = \Delta_1 \times \cdots \times \Delta_n.$$

Then $\mathrm{Aut}(\Delta_j)$ is a connected, real simple Lie group of dimension 3 with trivial center and $\mathrm{Aut}(\Delta^n)$ is a real semi-simple Lie group of dimension $3n$. Since each element of $\mathrm{Aut}(\Delta_j)$ can be uniquely extended to an element of $\mathrm{Aut}(\Delta^n)$ in a trivial manner, we shall often regard $\mathrm{Aut}(\Delta_j)$ as a closed Lie subgroup of $\mathrm{Aut}(\Delta^n)$. Moreover, if we denote by $\mathrm{Aut}^o(\Delta^n)$ the identity component of $\mathrm{Aut}(\Delta^n)$, then we know that $\mathrm{Aut}^o(\Delta^n)$ can be identified with the direct product of $\mathrm{Aut}(\Delta_j)$:

$$\mathrm{Aut}^o(\Delta^n) = \mathrm{Aut}(\Delta_1) \times \cdots \times \mathrm{Aut}(\Delta_n). \tag{2.1}$$

Let $\mathfrak{g}(\Delta_j)$ and $\mathfrak{g}(\Delta^n)$ be the real Lie algebras consisting of all complete holomorphic vector fields on Δ_j and on Δ^n, respectively. Then it is well known that the

Lie algebras $\mathfrak{g}(\Delta_j)$ and $\mathfrak{g}(\Delta^n)$ are canonically identified with the Lie algebras of $\mathrm{Aut}(\Delta_j)$ and $\mathrm{Aut}(\Delta^n)$, respectively. This combined with (2.1) yields that

$$\mathfrak{g}(\Delta^n) = \mathfrak{g}(\Delta_1) \oplus \cdots \oplus \mathfrak{g}(\Delta_n),$$
$$[\mathfrak{g}(\Delta_i), \mathfrak{g}(\Delta_j)] = \{0\} \quad \text{for } 1 \le i, j \le n,\ i \ne j. \tag{2.2}$$

Now let us consider the 1-parameter subgroups $\{\phi_t^j\}_{t\in\mathbb{R}}$ and $\{\psi_t^j\}_{t\in\mathbb{R}}$ of $\mathrm{Aut}(\Delta_j)$ for $1 \le j \le n$ given by

$$\phi_t^j : z_j \longmapsto \left(\exp\sqrt{-1}t\right) z_j \qquad \text{for } t \in \mathbb{R},$$
$$\psi_t^j : z_j \longmapsto \frac{(\cosh t) z_j + \sinh t}{(\sinh t) z_j + \cosh t} \qquad \text{for } t \in \mathbb{R}.$$

It is easily seen that these 1-parameter groups induce the complete holomorphic vector fields

$$H_j := \sqrt{-1} z_j \frac{\partial}{\partial z_j} \quad \text{and} \quad V_j := (1 - z_j^2)\frac{\partial}{\partial z_j}$$

on Δ_j (and hence on Δ^n), respectively. Put $W_j = [H_j, V_j]$. Then, elementary calculations show that

$$\mathfrak{g}(\Delta_j) = \mathbb{R}\{H_j, V_j, W_j\} \quad \text{and} \quad [H_j, [H_j, V_j]] = -V_j, \quad [W_j, V_j] = 4H_j \tag{2.3}$$

for $1 \le j \le n$. These bracket relations will be important in the next section.

Next we consider an arbitrary connected complex manifold M and a Lie group G. When a continuous group homomorphism $\rho\colon G \to \mathrm{Aut}(M)$ of G into $\mathrm{Aut}(M)$ is given, the mapping

$$G \times M \ni (g, p) \longmapsto (\rho(g))(p) \in M$$

is necessarily of class C^ω by [2], and we say that *G acts on M as a Lie transformation group through ρ*. Also, the action of G on M is called *effective* if ρ is injective. Let $T^n = (U(1))^n$ be the n-dimensional torus, where $U(1)$ denotes the multiplicative group of complex numbers with absolute value 1. Then T^n acts as a group of holomorphic automorphisms on $\mathbb{C}^n$ by the standard rule

$$\alpha \cdot z = (\alpha_1 z_1, \ldots, \alpha_n z_n) \quad \text{for } \alpha = (\alpha_1, \ldots, \alpha_n) \in T^n,\ z = (z_1, \ldots, z_n) \in \mathbb{C}^n.$$

By definition, a *Reinhardt domain D* in $\mathbb{C}^n$ is a domain in $\mathbb{C}^n$ that is stable under this action of T^n. Moreover, it is said to be *complete* if $(z_1, \ldots, z_n) \in D$, $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, and $|w_j| \le |z_j|$ $(1 \le j \le n)$ imply that $w \in D$. Now let D be an arbitrary Reinhardt domain in $\mathbb{C}^n$. Then each element α of T^n induces an automorphism π_α of D given by $\pi_\alpha(z) = \alpha \cdot z$, and the mapping ρ_D sending α to π_α is an injective continuous group homomorphism of the torus T^n into the topological group $\mathrm{Aut}(D)$. The subgroup $\rho_D(T^n)$ of $\mathrm{Aut}(D)$ is denoted by $T(D)$.

Finally, we recall the following three theorems, which will play crucial roles in our proof of the theorem.

THEOREM A [1]. *Let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy. Assume* *that T^n acts effectively on M as a Lie transformation group through ρ. Then there exist a biholomorphic mapping F of M into $\mathbb{C}^n$ and a continuous group automorphism θ of the torus T^n such that*

$$F((\rho(\alpha))(p)) = \theta(\alpha) \cdot F(p) \quad \textit{for all } \alpha \in T^n \textit{ and all } p \in M.$$

Consequently, $D := F(M)$ is a Reinhardt domain in $\mathbb{C}^n$, and one has $F\rho(T^n)F^{-1} = T(D)$.

THEOREM B [12]. *Let M be a connected hyperbolic manifold in the sense of Kobayashi* [8] *of dimension n. Assume that M is homogeneous—that is, assume* $\mathrm{Aut}(M)$ *acts transitively on M. Then M is biholomorphically equivalent to a Siegel domain in $\mathbb{C}^n$. In particular, M is simply connected.*

THEOREM C ([9]; cf. [7, Thm. 7.1.2]). *Let M be a complete Reinhardt domain in $\mathbb{C}^n$. Then M is hyperbolic if and only if it is literally a bounded domain in $\mathbb{C}^n$.*

3. Proof of the Theorem

By Theorem A we may assume that M is a Reinhardt domain D in $\mathbb{C}^n$ and that there exists a topological group isomorphism $\Phi\colon \mathrm{Aut}(\Delta^n) \to \mathrm{Aut}(D)$ such that $\Phi(T(\Delta^n)) = T(D)$.

Now, the group $\mathrm{Aut}(D)$ can be turned into a Lie group simply by transferring the Lie group structure from $\mathrm{Aut}(\Delta^n)$ by means of Φ. We here assert that the Lie algebra of $\mathrm{Aut}(D)$ with respect to the Lie group structure defined in this way coincides with the algebra $\mathfrak{g}$ of all complete holomorphic vector fields on D. Indeed, the Lie group $\mathrm{Aut}(D)$ endowed with the compact-open topology acts continuously on D. Hence, by [2], the action is smooth with respect to the Lie group structure induced from $\mathrm{Aut}(\Delta^n)$. Furthermore, $\mathrm{Aut}(D)$ has only finitely many connected components, since $\mathrm{Aut}(\Delta^n)$ does. Then, by Theorem VI in [14, p. 101], the group $\mathrm{Aut}(D)$ is a Lie transformation group of D in the sense of Definition V in [14, p. 101]; consequently, the Lie algebra of $\mathrm{Aut}(D)$ coincides with the Lie algebra $\mathfrak{g}$ (cf. [14, p. 103, Thm. VII]), as asserted. We thus obtain the Lie algebra isomorphism $d\Phi\colon \mathfrak{g}(\Delta^n) \to \mathfrak{g}$ induced by Φ. Put

$$\begin{aligned} G &= \Phi(\mathrm{Aut}^o(\Delta^n)), \quad & G_j &= \Phi(\mathrm{Aut}(\Delta_j)), \quad & \mathfrak{g}_j &= d\Phi(\mathfrak{g}(\Delta_j)), \\ I_j &= d\Phi(H_j), & X_j &= d\Phi(V_j), & Y_j &= d\Phi(W_j) \end{aligned}$$

for $1 \le j \le n$. Then $G = \mathrm{Aut}^o(D)$, the identity component of $\mathrm{Aut}(D)$, and G_j is a 3-dimensional simple Lie group with Lie algebra $\mathfrak{g}_j$ for each j. Moreover, by (2.1)–(2.3) we have

$$G = G_1 \times \cdots \times G_n; \tag{3.1}$$

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n, \quad [\mathfrak{g}_i, \mathfrak{g}_j] = \{0\} \text{ for } 1 \le i, j \le n,\ i \ne j; \tag{3.2}$$

$$\mathfrak{g}_j = \mathbb{R}\{I_j, X_j, Y_j\} \quad \text{and} \quad [I_j, [I_j, X_j]] = -X_j,\ [Y_j, X_j] = 4I_j \tag{3.3}$$

for every $1 \le j \le n$.

Now we identify the tori $T(\Delta^n)$ and $T(D)$ naturally with T^n. Then, since the Lie group isomorphism $\Phi\colon \mathrm{Aut}(\Delta^n) \to \mathrm{Aut}(D)$ satisfies $\Phi(T^n) = T^n$, there exists an element (p_{ij}) of $\mathrm{GL}(n, \mathbb{Z})$ such that

$$\begin{aligned}\Phi\big((\exp 2\pi\sqrt{-1}\theta_1, \dots, \exp 2\pi\sqrt{-1}\theta_n)\big) \\ = \left(\exp 2\pi\sqrt{-1}\left(\sum_{j=1}^{n} p_{1j}\theta_j\right), \dots, \exp 2\pi\sqrt{-1}\left(\sum_{j=1}^{n} p_{nj}\theta_j\right)\right)\end{aligned}$$

for all $\theta_1, \dots, \theta_n \in \mathbb{R}$. Accordingly, after noting that the complete holomorphic vector field I_j is induced by the 1-parameter subgroup $\{\Phi(\phi_t^j)\}_{t\in\mathbb{R}}$ of $T^n \subset \mathrm{Aut}(D)$, we can see that I_j has the form

$$I_j = \sqrt{-1}\sum_{i=1}^{n} (p_{ij} z_i)\frac{\partial}{\partial z_i} \quad \text{for } 1 \le j \le n.$$

From now on, we set

$$D^* = \{(z_1, \dots, z_n) \in D \mid z_1 \cdots z_n \neq 0\} = D \cap (\mathbb{C}^*)^n.$$

Then we have the following lemma.

LEMMA 1. *For every point $p \in D^*$, there exists a local holomorphic coordinate system $(U, \varphi) = (U, w_1, \dots, w_n)$ on D^*, centered at p, such that $I_j = \partial/\partial w_j$ on U for every $1 \le j \le n$.*

Proof. Consider the holomorphic mapping

$$\varpi : \mathbb{C}^n \ni (w_1, \dots, w_n) \longmapsto (z_1, \dots, z_n) \in (\mathbb{C}^*)^n$$

defined by

$$z_i = \exp\sqrt{-1}\left(\sum_{j=1}^{n} p_{ij} w_j\right) \quad \text{for } 1 \le i \le n.$$

Then ϖ is a local biholomorphic (in fact, the universal covering) mapping from $\mathbb{C}^n$ onto $(\mathbb{C}^*)^n$, and each vector field I_j restricted to D^* can be locally expressed as $I_j = \partial/\partial w_j$ with respect to $(w_1, \dots, w_n)$. From this we obtain the assertion of the lemma. □

Without loss of generality, we may assume that $\varphi(U)$ is a polydisc.

LEMMA 2. *With respect to the local coordinate system $(U, w_1, \dots, w_n)$ as in Lemma 1, the vector fields X_j, Y_j $(1 \le j \le n)$ can be written in the form*

$$X_j = \left\{a_j \exp\big(\sqrt{-1}w_j\big) + b_j \exp\big(-\sqrt{-1}w_j\big)\right\}\frac{\partial}{\partial w_j},$$

$$Y_j = \sqrt{-1}\left\{a_j \exp\big(\sqrt{-1}w_j\big) - b_j \exp\big(-\sqrt{-1}w_j\big)\right\}\frac{\partial}{\partial w_j}$$

on U, where a_j, b_j are some complex constants with $a_j b_j = 1$.

Proof. Let us write $X_j = \sum_{k=1}^n f_k^j(w)\partial/\partial w_k$ on U with holomorphic functions $f_k^j(w)$ on U. Then, since $[\mathfrak{g}_i, \mathfrak{g}_j] = \{0\}$ for all $1 \le i, j \le n$ with $i \ne j$, we have

$$\sum_{k=1}^n \frac{\partial f_k^j(w)}{\partial w_i} \frac{\partial}{\partial w_k} = [I_i, X_j] = 0 \text{ on } U \quad \text{for all } i \ne j.$$

Hence $f_k^j(w)$ does not depend on the variables w_i for all $1 \le i \le n$ with $i \ne j$, so $f_k^j(w)$ has the form $f_k^j(w) = f_k^j(w_j)$. It then follows from the first bracket relation in (3.3) that

$$\sum_{k=1}^n \frac{d^2 f_k^j(w_j)}{dw_j^2} \frac{\partial}{\partial w_k} = -\sum_{k=1}^n f_k^j(w_j) \frac{\partial}{\partial w_k} \text{ on } U.$$

Therefore, the holomorphic functions $f_k^j(w_j)$ can be expressed as

$$f_k^j(w_j) = a_k^j \exp\big(\sqrt{-1}w_j\big) + b_k^j \exp\big(-\sqrt{-1}w_j\big) \text{ on } U \tag{3.4}$$

with some complex constants a_k^j, b_k^j; accordingly, X_j, Y_j have the form

$$X_j = \sum_{k=1}^n \big\{a_k^j \exp\big(\sqrt{-1}w_j\big) + b_k^j \exp\big(-\sqrt{-1}w_j\big)\big\} \frac{\partial}{\partial w_k}, \tag{3.5}$$

$$Y_j = \sqrt{-1} \sum_{k=1}^n \big\{a_k^j \exp\big(\sqrt{-1}w_j\big) - b_k^j \exp\big(-\sqrt{-1}w_j\big)\big\} \frac{\partial}{\partial w_k} \tag{3.6}$$

for $1 \le j \le n$. By routine computations, it then follows that

$$[Y_j, X_j] = \sum_{k=1}^n 2(a_j^j b_k^j + b_j^j a_k^j) \frac{\partial}{\partial w_k} \text{ on } U.$$

This together with $[Y_j, X_j] = 4I_j$ from (3.3) shows that

$$a_j^j b_j^j = 1 \quad \text{and} \quad a_j^j b_k^j + b_j^j a_k^j = 0 \quad \text{for all } 1 \le j, k \le n, \ j \ne k. \tag{3.7}$$

Once it is shown that $a_k^j = 0$ for all $1 \le j, k \le n$ with $j \ne k$, then $b_k^j = 0$ by (3.7); hence X_j, Y_j have the form required in the lemma. Thus we need only show that $a_k^j = 0$ if $j \ne k$. Toward this end, observe that

$$\begin{aligned}
[X_j, X_k] = & \sum_{m \ne j,k} \left\{ f_k^j(w_j) \frac{df_m^k(w_k)}{dw_k} - f_j^k(w_k) \frac{df_m^j(w_j)}{dw_j} \right\} \frac{\partial}{\partial w_m} \\
& + \left\{ f_k^j(w_j) \frac{df_j^k(w_k)}{dw_k} - f_j^k(w_k) \frac{df_j^j(w_j)}{dw_j} \right\} \frac{\partial}{\partial w_j} \\
& - \left\{ f_j^k(w_k) \frac{df_k^j(w_j)}{dw_j} - f_k^j(w_j) \frac{df_k^k(w_k)}{dw_k} \right\} \frac{\partial}{\partial w_k}
\end{aligned}$$

and $[X_j, X_k] = 0$ on U for all $j \ne k$ by (3.2). Thus, expressing the functions $f_\beta^\alpha(w_\alpha)$ as in (3.4) and comparing the coefficients of $\partial/\partial w_k$ in both sides of the equality $[X_j, X_k] = 0$, we obtain

$$
\begin{aligned}
&a_k^j(a_j^k - a_k^k)\exp\{\sqrt{-1}(w_k + w_j)\} \\
&\quad + (b_j^j)^2 a_k^j(a_j^k + a_k^k)\exp\{\sqrt{-1}(w_k - w_j)\} \\
&\quad - (b_k^k)^2 a_k^j(a_j^k - a_k^k)\exp\{\sqrt{-1}(w_j - w_k)\} \\
&\quad - (b_j^j b_k^k)^2 a_k^j(a_j^k + a_k^k)\exp\{-\sqrt{-1}(w_k + w_j)\} = 0 \ \text{ on } U.
\end{aligned}
$$

Combined with $a_k^k b_j^j b_k^k \neq 0$ from (3.7), this yields that

$$a_k^j(a_j^k - a_k^k) = 0, \qquad a_k^j(a_j^k + a_k^k) = 0$$

and, accordingly, $a_k^j = 0$ for all $1 \leq j, k \leq n$ with $j \neq k$, as desired. □

With the same notation as in Lemma 2, we define a subset $\mathcal{A}$ of U by setting

$$\mathcal{A} = \left\{ w \in U \;\middle|\; \prod_{j=1}^{n} \Im\{a_j \exp(\sqrt{-1}w_j) + b_j \exp(-\sqrt{-1}w_j)\} = 0 \right\},$$

where $\Im\{\cdot\}$ means the imaginary part of $\cdot$. Clearly $\mathcal{A}$ is a nowhere dense real analytic subset of U.

Choose a point $p \in U \setminus \mathcal{A}$ arbitrarily and let $(\mathfrak{g}_j)_p$ and $\mathfrak{g}_p$ be the subspaces in the tangent space to D at p that consist of the values of the elements of $\mathfrak{g}_j$ and $\mathfrak{g}$ (respectively) at p. Then Lemma 2 guarantees that, for every $1 \leq j \leq n$,

$$(\mathfrak{g}_j)_p = \mathbb{R}\{(I_j)_p, (X_j)_p, (Y_j)_p\} = \mathbb{C}\left\{\left(\frac{\partial}{\partial w_j}\right)_p\right\} \tag{3.8}$$

and consequently

$$\mathfrak{g}_p = \mathbb{C}\left\{\left(\frac{\partial}{\partial w_1}\right)_p\right\} \oplus \cdots \oplus \mathbb{C}\left\{\left(\frac{\partial}{\partial w_n}\right)_p\right\}. \tag{3.9}$$

Therefore, denoting by K, K_j the isotropy subgroups of G, G_j (respectively) at the point p and considering the orbits

$$D_p := G \cdot p = G/K, \qquad S_j := G_j \cdot p = G_j/K_j \quad (1 \leq j \leq n)$$

of G, G_j passing through p, one concludes that every S_j is a 1-dimensional complex submanifold of D and D_p is a nonempty open subset of D. Here it should be remarked that the S_j may a priori be nonclosed submanifolds of D and that the topology of S_j may a priori differ from that induced from D. Moreover, notice that D_p is a Reinhardt domain in $\mathbb{C}^n$ because G is connected and contains the torus $T(D) = T^n$.

LEMMA 3. *Every S_j is biholomorphically equivalent to the unit disc Δ in $\mathbb{C}$.*

Proof. Once it is shown that the universal covering $\tilde{S}_j$ of S_j is the unit disc Δ, then S_j is a homogeneous hyperbolic Riemann surface and hence is biholomorphically equivalent to Δ. Thus we need only show that $\tilde{S}_j = \Delta$. Clearly S_j is noncompact in D; consequently, $\tilde{S}_j = \Delta$ or $\mathbb{C}$. Assume that $\tilde{S}_j = \mathbb{C}$. Since it is obvious that G_j acts effectively on S_j by biholomorphic transformations, it follows that $\dim \mathrm{Aut}(S_j) \geq 3$. Therefore, S_j itself must be biholomorphically equivalent to $\mathbb{C}$. On the other hand, every 3-dimensional subgroup of $\mathrm{Aut}(\mathbb{C})$ that acts transitively on $\mathbb{C}$ contains the group of translations and is therefore not simple. However, since the group G_j is simple, this is a contradiction. As a result, we have shown that $\tilde{S}_j = \Delta$ as desired. □

By Lemma 3 we see that the isotropy subgroup K_j of G_j at p is a maximal compact subgroup of G_j of dimension 1.

LEMMA 4. *The subdomain $D_p = G \cdot p$ of D is biholomorphically equivalent to the unit polydisc Δ^n in $\mathbb{C}^n$. In particular, D_p is a hyperbolic pseudoconvex Reinhardt domain in $\mathbb{C}^n$.*

Proof. Define the mapping

$$\pi : S_1 \times \cdots \times S_n \to D_p$$

by setting $\pi(z_1, \ldots, z_n) = g_1 \cdots g_n \cdot p$, where $z_j = g_j \cdot p = g_j K_j$ are arbitrary elements of $S_j = G_j \cdot p = G_j / K_j$ for $1 \leq j \leq n$. Observe that the identity component of K coincides with $K_1 \times \cdots \times K_n$. Then it can easily be seen that π is a well-defined holomorphic covering mapping. This combined with Lemma 3 implies that $D_p = G/K$ is a homogeneous hyperbolic manifold; therefore, by Theorem B, it must be simply connected. Thus π is now a biholomorphic mapping and our assertion in Lemma 4 is an immediate consequence of Lemma 3. □

By Lemma 4 we see that $K = K_1 \times \cdots \times K_n$ and that K is a maximal compact subgroup of G conjugate to $T(D) = T^n$.

We can now prove our main theorem from Section 1. First we claim that D_p is a bounded domain in $\mathbb{C}^n$ or, equivalently, that the topological closure $\bar{D}_p$ of D_p in $\mathbb{C}^n$ is a compact subset of $\mathbb{C}^n$. Indeed, since D_p is a contractible pseudoconvex Reinhardt domain by Lemma 4, we can see that

$$D_p \cap \{z_i = 0\} \neq \emptyset \quad \text{for every } 1 \leq i \leq n;$$

accordingly, it must be a complete Reinhardt domain. Moreover, by Lemma 4 we know that D_p is hyperbolic. Hence D_p is a bounded domain in $\mathbb{C}^n$ by Theorem C, as claimed.

Our next task is to show that $D^* \subset \bar{D}_p$. We argue by contradiction, so we assume that there exists a point $q \in D^* \setminus \bar{D}_p$. Then, by taking a suitable nearby point if necessary, we may assume that the point q satisfies the same conditions as in (3.8) and (3.9). By repeating exactly the same argument as before, it can be shown that the orbit $D_q = G \cdot q$ of G passing through q is a complete bounded Reinhardt domain in $\mathbf{C}^n$. In particular, both the domains D_p and D_q contain the origin o of $\mathbb{C}^n$ and hence $D_p \cap D_q \neq \emptyset$. However, since $q \notin D_p = G \cdot p$, it is clear that $D_p \cap D_q = \emptyset$—a contradiction. Thus we have shown that $D^* \subset \bar{D}_p$.

We shall complete the proof by showing that $D = D_p$. Since D^* is an open dense subset of D and since D^* is contained in the compact set $\bar{D}_p$ as before, D itself must be a bounded domain in $\mathbb{C}^n$. Consequently, $D = D_p$, because D is now a hyperbolic manifold and hence $\mathrm{Aut}(D)$, as well as $\mathrm{Aut}^o(D) = G$, acts on D with closed orbits (cf. [8; Chap. V]). Therefore, D is biholomorphically equivalent to the unit polydisc Δ^n by Lemma 4, completing the proof.

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