On Three Magnetic Relativistic Schrödinger Operators and Imaginary-Time Path Integrals

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Dedicated to my friend Michael Demuth on the occasion of his sixty-fifth birthday

Abstract

Three magnetic relativistic Schrödinger operators are considered corresponding to the classical relativistic Hamiltonian symbol with magnetic vector and electric scalar potentials. We discuss their difference in general and their coincidence in the case of constant magnetic fields, as well as whether they are covariant under gauge transformation. Then results are surveyed on path integral representations for their respective imaginary-time relativistic Schrödinger equations, i.e. heat equations, by means of the probability path space measure coming from the Lévy process concerned.

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1 Introduction

In this note, we consider the relativistic Schrödinger operators corresponding to the classical relativistic Hamiltonian symbol

$$\sqrt{(\xi - A(x))^2 + m^2} + V(x)\,, \qquad (\xi, x) \in \mathbf{R}^d \times \mathbf{R}^d\,, \tag{1.1}$$

which is the sum of the kinetic energy term involving magnetic vector potential $A(x)$ and the potential energy term of electric scalar potential $V(x)$. There are in the literature three kinds of quantum relativistic Hamiltonians depending on how to quantize the kinetic energy term $\sqrt{(\xi - A(x))^2 + m^2}$.

We note first that they are in general different from one another, next observe that they coincide when the vector potential $A(x)$ is linear in x, so in particular, in the case of constant magnetic fields, and finally discuss whether they are gauge-covariant. Then, on this occasion, we would like to make survey on the results on path integral representations for their respective imaginary-time unitary groups, i.e. real-time semigroups. It will be of some interest to collect them in one place to observe how they look like and different, though all the three are essentially connected with the Lévy process.

2 Three magnetic relativistic Schrödinger operators

We consider the quantized operator $H := H_A + V$ corresponding to the classical Hamiltonian

$$\sqrt{(\xi - A(x))^2 + m^2} + V(x)\,, \qquad (\xi, x) \in \mathbf{R}^d \times \mathbf{R}^d, \tag{2.1}$$

for a *relativistic* particle of mass m under *magnetic* vector potential $A(x)$ and electric scalar potential $V(x)$. This H is used for a spinless particle in electromagnetic fields in the situation where we may ignore quantum-field theoretic effect like particles creation and annihilation but should take relativistic effect into consideration.

In this note, we pay attention to the following three quantized operators $H^{(1)}$, $H^{(2)}$ and $H^{(3)}$ corresponding to the classical relativistic Hamiltonian symbol (2.1). Their difference is in how to define the first term on the right, H_A, corresponding to the symbol $\sqrt{(\xi - A(x))^2 + m^2}$.

For simplicity, it is assumed here and throughout this note that $A(x)$ is a *smooth* $\mathbf{R}^d$-valued function and that $V(x)$ is a real-valued function *bounded below*, and at the same time the *operator sum* $H_A + V$ of H_A and V is selfadjoint on their common domain.

Definition 2.1 The first $H^{(1)} := H_A^{(1)} + V$ is defined with the first term on the right $H_A^{(1)}$ being the Weyl pseudo-differential operator through *mid-point* prescription (e.g. Ichinose–Tamura [13], Ichinose [6, 8]):

$$\begin{aligned}(H_A^{(1)} f)(x) &:= \frac{1}{(2\pi)^d} \int\!\!\int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y)\cdot\xi} \sqrt{\Big(\xi - A(\frac{x+y}{2})\Big)^2 + m^2} f(y) dy d\xi \\ &= \frac{1}{(2\pi)^d} \int\!\!\int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y)\cdot(\xi + A(\frac{x+y}{2}))} \sqrt{\xi^2 + m^2} f(y) dy d\xi \end{aligned} \tag{2.2}$$

Definition 2.2 The second $H^{(2)} := H_A^{(2)} + V$ is defined with term $H_A^{(2)}$ being the pseudo-differential operator modified by Iftimie–Măntoiu–Purice [13, 14, 15]:

$$(H_A^{(2)} f)(x) := \frac{1}{(2\pi)^d} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y)\cdot(\xi + \int_0^1 A((1-\theta)x+\theta y)d\theta)} \sqrt{\xi^2 + m^2} f(y) dy d\xi. \quad (2.3)$$

Here the integrals in (2.2), (2.3) on the right-hand side are oscillatory integrals with f being a function in $C_0^\infty(\mathbf{R}^d)$ or in $\mathcal{S}(\mathbf{R}^d)$.

Definition 2.3 The third $H^{(3)} := H_A^{(3)} + V$ is defined with term $H_A^{(3)}$ being the square root of the nonnegative selfadjoint operator $(-i\nabla - A(x))^2 + m^2$:

$$H_A^{(3)} := \sqrt{(-i\nabla - A(x))^2 + m^2}. \quad (2.4)$$

This $H_A^{(3)}$ does not seem to be defined as a pseudo-differential operator corresponding to a certain tractable symbol. So long as it is defined through Fourier and inverse-Fourier tansforms, the candiadte of its symbol will not be $\sqrt{(\xi - A(x))^2 + m^2}$.

The last $H^{(3)}$ is used, for instance, to study "stability of matter" in relativistic quantum mechanics in Lieb–Seiringer [18].

Needles to say, we can show these three relativistic Schrödinger operators $H^{(1)}$, $H^{(2)}$ and $H^{(3)}$ define *selfadjoint* operators in $L^2(\mathbf{R}^d)$. They are bounded from below, and, in general, different from one another but coincide with one another if $A(x)$ is linear in x. Let us observe these facts in the following.

Proposition 2.4 *$H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$ are in general different.*

Proof. That $H_A^{(1)} \neq H_A^{(3)}$ was shown by Umeda–Nagase [21, Lemma 7.1, p.851] to watch through pseudo-differential calculus that $(H_A^{(1)})^2 \neq (H_A^{(3)})^2 = (-i\nabla - A(x))^2 + m^2$. By the same method we can show that $H_A^{(2)} \neq H_A^{(3)}$.

Finally, one has $H_A^{(1)} \neq H_A^{(2)}$ for general A, because we have

$$A(\frac{x+y}{2}) \neq \int_0^1 A(x + \theta(y-x))d\theta,$$

for instance, for $d = 3$, taking $A(x) \equiv (A_1(x), A_2(x), A_3(x)) = (0, 0, x_3^2)$, so that

$$\int_0^1 A_3(x + \theta(y-x))d\theta = \int_0^1 (x_3 + \theta(y_3 - x_3))^2 d\theta = \frac{y_3^2 + y_3 x_3 + x_3^2}{3}$$
$$\neq (\frac{x_3 + y_3}{2})^2 = A_3(\frac{x+y}{2}). \qquad \square$$

Theorem 2.5 *If $A(x)$ is linear in x, i.e. if $A(x) = \dot{A} \cdot x$ with $\dot{A}$ being any $d \times d$ real symmetric constant matrix, then $H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$ coincide. In particular, this holds for uniform magnetic fields for $d = 3$.*

Proof. Suppose $A(x) = \dot{A} \cdot x$. First, that $H_A^{(1)} = H_A^{(2)}$ can seen through

$$\begin{aligned}\int_0^1 A(\theta x + (1-\theta)y)d\theta &= \int_0^1 \dot{A} \cdot (\theta x + (1-\theta)y)d\theta = \int_0^1 \dot{A} \cdot (y + \theta(x-y))d\theta \\ &= \dot{A} \cdot \frac{x+y}{2} = A(\frac{x+y}{2}),\end{aligned}$$

which turns out to be midpoint prescription to yield the Weyl quantization.

To see that they also coincide with $H_A^{(3)}$, we need to show that $(H_A^{(1)})^2 = (-i\nabla - A(x))^2 + m^2$. To do so, let $f \in C_0^\infty(\mathbf{R}^d)$ or $\in \mathcal{S}(\mathbf{R}^d)$ and note ${}^t\dot{A} = \dot{A}$, then we have, with integrals in the sense of oscillatory integrals,

$$\begin{aligned}&((H_A^{(1)})^2 f)(x) \\ &= \frac{1}{(2\pi)^{2d}} \int\int\int\int e^{i(x-y)\cdot(\xi + \dot{A}\frac{x+y}{2}) + i(y-z)\cdot(\eta + \dot{A}\frac{y+z}{2})} \sqrt{\xi^2+m^2}\sqrt{\eta^2+m^2} f(z) dy d\xi dz d\eta \\ &= \frac{1}{(2\pi)^{2d}} \int\int\int\int e^{iy\cdot(-\xi+\eta)} e^{i[x\cdot(\xi + \dot{A}\frac{x}{2}) - z\cdot(\eta + \dot{A}\frac{z}{2})]} \sqrt{\xi^2+m^2}\sqrt{\eta^2+m^2} f(z) dy d\xi dz d\eta \\ &= \frac{1}{(2\pi)^{d}} \int\int\int \delta(-\xi+\eta) e^{i[x\cdot(\xi + \dot{A}\frac{x}{2}) - z\cdot(\eta + \dot{A}\frac{z}{2})]} \sqrt{\xi^2+m^2}\sqrt{\eta^2+m^2} f(z)\, d\xi dz d\eta \\ &= \frac{1}{(2\pi)^{d}} \int\int e^{i(x-z)\cdot\eta} e^{i\frac{1}{2}(x\cdot\dot{A}x - z\cdot\dot{A}z)} (\eta^2+m^2) f(z)\, dz d\eta \\ &= \frac{1}{(2\pi)^{d}} \int\int e^{i(x-z)\cdot(\eta + \dot{A}\frac{x+z}{2})} (\eta^2+m^2) f(z)\, dz d\eta \\ &= \frac{1}{(2\pi)^{d}} \int\int e^{i(x-z)\cdot(\eta + A(\frac{x+z}{2}))} (\eta^2+m^2) f(z)\, dz d\eta \\ &= \frac{1}{(2\pi)^{d}} \int\int e^{i(x-z)\cdot\eta} [(\eta - A(\frac{x+z}{2}))^2 + m^2] f(z)\, dz d\eta\,.\end{aligned}$$

The last equality is due to the fact that symbol $(\xi - A(x))^2 + m^2$ is polynomial of ξ, so that the corresponding Weyl pseudo-differential operator is equal to $(-i\nabla - A(x))^2 + m^2$. □

Finally, we are going to see the three magnetic relativistic Schrödinger operators $H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$ are bounded from below by the *same lower bound*, as in the following theorem.

Theorem 2.6

$$H_A^{(j)} \geq m\,, \qquad j = 1, 2, 3. \tag{2.5}$$

Proof. (Sketch) First, it is trivial for $H_A^{(3)}$. Next for $H_A^{(1)}$, we can show Kato's inequality, i.e. the following distributional inequality:

$$\mathrm{Re}[(\mathrm{sgn}f)H_A^{(1)}f] \geq \sqrt{-\Delta + m^2}\,|f| \tag{2.6}$$

for all $f \in L^2(\mathbf{R}^d)$ with $H_A^{(1)}f \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$, where $\mathrm{sgn}f$ is a bounded function in $\mathbf{R}^d$, dependent on f, defined by $(\mathrm{sgn}f)(x) = \overline{f(x)}/|f(x)|$ if $f(x) \neq 0$; $= 0$ if $f(x) = 0$. To show (2.6), one has to use the expression (4.4) for $H_A^{(1)}f$ in §4 instead of (2.2). Hence we can conclude (2.5). For the detail, see Ichinose [6, Theorems 4.1, 5.1]. Finally for $H_A^{(2)}$, it will be shown exactly by the same argument with the expression (4.13) for $H_A^{(2)}$ instead of (2.3). □

Though gilding the lily, we ask which of these three magnetic relativistic Schrödinger operators *Nature* will choose.

3 Gauge covariance for magnetic relativistic Schrödinger operators

Among these three magnetic relativistic Schrödinger operators $H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$, the Weyl quantized one like $H_A^{(1)}$ (in general, the Weyl pseudo-differential operator) is *compatible well with path integral*. But the pity is that, for general vector potential $A(x)$ $H_A^{(1)}$ (and so $H^{(1)}$) is in general not covariant under gauge transformation, namely, there exists a real-valued function $\varphi(x)$ for which it fails to hold that $H^{(1)}_{A+\nabla\varphi} = e^{i\varphi}H_A^{(1)}e^{-i\varphi}$.

However, $H_A^{(2)}$ (and so $H^{(2)}$) and $H_A^{(3)}$ (and so $H^{(3)}$) are gauge-covariant, though these three are not in general equal as seen in Proposition 2.4. The gauge-covariance of the modified $H_A^{(2)}$ in contrast to $H_A^{(1)}$ in Ichinose–Tamura [13] was emphasized in Iftimie–Măntoiu–Purice [14, 15, 16] . Let us observe some of these facts in the following.

Proposition 3.1 *$H_A^{(2)}$ and $H_A^{(3)}$ are covariant under gauge transformation, i.e. it holds for $j = 2, 3$ that for $\forall\varphi \in \mathcal{S}(\mathbf{R}^d)$ $H^{(j)}_{A+\nabla\varphi} = e^{i\varphi}H_A^{(j)}e^{-i\varphi}$. But $H^{(1)}$ is in general not covariant under gauge transformation.*

Poof. First, we see why $H_A^{(3)} = \sqrt{(-i\nabla - A(x))^2 + m^2}$ is gauge-covariant, because the selfadjoint operator $(-i\nabla - A(x))^2 + m^2$ inside $\sqrt{\cdots}$ is gauge-covariant.

Next, for $H_A^{(2)}$, by mean-value theorem

$$\begin{aligned}\varphi(y)-\varphi(x) &= \int_0^1 (y-x)\cdot(\nabla\varphi)(x+\theta(y-x))d\theta \\ &= -\int_0^1 (x-y)\cdot(\nabla\varphi)((1-\theta)x+\theta y)d\theta.\end{aligned}$$

Hence

$$\begin{aligned}&(H_A^{(2)}e^{-i\varphi}f)(x)\\ =\ & \frac{1}{(2\pi)^d}\int\int_{\mathbf{R}^d\times\mathbf{R}^d} e^{i(x-y)\cdot(\xi+\int_0^1 A((1-\theta)x+\theta y)d\theta)} \\ &\qquad \times\sqrt{\xi^2+m^2}\, e^{-i\varphi(x)+i\int_0^1 (x-y)\cdot(\nabla\varphi)((1-\theta)x+\theta y)d\theta} f(y)\,dyd\xi \\ =\ & \frac{1}{(2\pi)^d}e^{-i\varphi(x)}\int\int_{\mathbf{R}^d\times\mathbf{R}^d} e^{i(x-y)\cdot(\xi+\int_0^1 (A+\nabla\varphi)((1-\theta)x+\theta y)d\theta)}\sqrt{\xi^2+m^2}\, f(y)\,dyd\xi \\ =\ & e^{-i\varphi(x)}(H_{A+\nabla\varphi}^{(2)}f)(x).\end{aligned}$$

Finally, to see non-gauge-invariance of $H_A^{(1)}$, we are going to use a second expression for $H_A^{(1)}$ as an integral operator to be given in the next section, (4.4). Then we show that it does not hold for all φ that $H_{A+\nabla\varphi}^{(1)} = e^{i\varphi}H_A^{(1)}e^{-i\varphi}$ or that, taking $A\equiv 0$,

$$\begin{aligned}&\text{p.v.}\int_{|y|>0}[e^{-iy\cdot(\nabla\varphi)(x+\frac{y}{2})}f(x+y)-f(x)]\,n(dy)\\ =\ & e^{i\varphi(x)}\text{p.v.}\int_{|y|>0}[(e^{-i\varphi}f)(x+y)-(e^{-i\varphi}f)(x)]\,n(dy)\\ =\ & \text{p.v.}\int_{|y|>0}[e^{-i(\varphi(x+y)-\varphi(x))}f(x+y)-f(x)]\,n(dy).\end{aligned}$$

Indeed, this cannot hold, because it does not hold for all φ that $\varphi(x+y)-\varphi(x) = y\cdot(\nabla\varphi)(x+\frac{y}{2})$. □

4 Imaginary-time path integrals for magnetic relativistic Schrödinger operators

Now, let H be one of the magnetic relativistic Schrödinger operators $H^{(1)}$, $H^{(2)}$, $H^{(3)}$ in Definitions 2.1, 2.2, 2.3. In the same way as in the nonrelativistic case, start from (real-time) relativistic Schrödinger equation $i\frac{\partial}{\partial t}\psi(t,x) = H\psi(t,x)$. Rotate it by -90^o

from real time t to imaginary time $-it$ in complex t-plane (cf. Ichinose [7, sect.4, p.23]), we arrive at the *imaginary-time relativistic Schrödinger equation*, i.e. *heat equation* for $H-m$ [formally putting $u(t,x):=\psi(-it,x)$]:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) &= -[H-m]u(t,x), \quad t>0, \\ u(0,x) &= g(x), \qquad\qquad\qquad x\in\mathbf{R}^d. \end{cases} \tag{4.1}$$

The semigroup $u(t,x)=(e^{-t[H-m]}g)(x)$ gives the solution of this Cauchy problem. We want to deal with path integral representation for each $e^{-[H^{(j)}-m]}g\,(j=1,2,3)$. The relevant path integral is connected with the *Lévy process* in Ikeda–Watanabe [17], Sato [19], Applebaum [1] on the space $D_x:=D_x([0,\infty)\to\mathbf{R}^d)$ of the *"càdlag* paths", i.e. right-continuous paths $X:[0,\infty)\to\mathbf{R}^d$ having left-hand limits, and with $X(0)=x$. The associated path space measure is a probability measure λ_x, for each $x\in\mathbf{R}^d$, on $D_x([0,\infty)\to\mathbf{R}^d)$ whose characteristic function is given by

$$e^{-t[\sqrt{\xi^2+m^2}-m]}=\int_{D_x([0,\infty)\to\mathbf{R}^d)}e^{i(X(t)-x)\cdot\xi}d\lambda_x(X), \quad t\geq 0, \quad \xi\in\mathbf{R}^d. \tag{4.2}$$

We are going to start on task of representing the semigroup $e^{-t[H-m]}g$ by path integral. Before that, let us note that when the vector potential $A(x)$ is absent, we can represent $u(t,x)$ by a formula looking the same as the Feynman–Kac formula for the nonrelativistic Schrödinger equation:

$$u(t,x)=(e^{-t[\sqrt{-\Delta+m^2}+V-m]}g)(x)=\int_{D_x([0,\infty)\to\mathbf{R}^d)}e^{-\int_0^t V(X(s))ds}g(X(t))d\lambda(X). \tag{4.3}$$

Now we turn to come to the case where the vector potential $A(x)$ is present.

(1) First consider the case for the Weyl pseudo-differential operator $H^{(1)}=H_A^{(1)}+V$ in Definition 2.1. The part $H_A^{(1)}$ can be rewritten as the integral operator:

$$\begin{aligned} ([H_A^{(1)}-m]f)(x) &= -\int_{|y|>0}[e^{-iy\cdot A(x+\frac{y}{2})}f(x+y)-f(x) \\ &\qquad\qquad -I_{\{|y|<1\}}y\cdot(\nabla-iA(x))\,f(x)]n(dy) \\ &= -\lim_{r\downarrow 0}\int_{|y|\geq r}[e^{-iy\cdot A(x+\frac{y}{2})}f(x+y)-f(x)]\,n(dy) \\ &= -\,\mathrm{p.v.}\int_{|y|>0}[e^{-iy\cdot A(x+\frac{y}{2})}f(x+y)-f(x)]\,n(dy). \end{aligned} \tag{4.4}$$

Here $n(dy) = n(y)dy$ is an m-dependent measure on $\mathbf{R}^d \setminus \{0\}$, called *Lévy measure* with density

$$n(y) = \begin{cases} 2(\frac{m}{2\pi})^{(d+1)/2} \frac{K_{(d+1)/2}(m|y|)}{|y|^{(d+1)/2}}, & m > 0, \\ \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}} \frac{1}{|y|^{d+1}}, & m = 0, \end{cases} \tag{4.5}$$

and appears in the *Lévy–Khinchin formula*:

$$\sqrt{\xi^2 + m^2} - m = -\int_{|y|>0} [e^{iy\cdot\xi} - 1 - i\xi\cdot y I_{\{|y|<1\}}] n(dy) = -\lim_{r\to 0+} \int_{|y|\geq r} [e^{iy\cdot\xi} - 1] n(dy). \tag{4.6}$$

Proof of (4.4). By the Lévy–Khinchin formula (4.6),

$$\begin{aligned} & (H_A^{(1)} f)(x) \\ = \; & (2\pi)^{-d} \int\int e^{i(x-y)\cdot(\xi + A(\frac{x+y}{2}))} \Big(m - \lim_{r\to 0+} \int_{|z|\geq r} [e^{iz\cdot\xi} - 1] n(dz) \Big) f(y) dy d\xi \\ = \; & (2\pi)^{-d} \Big[m \int\int e^{i(x-y)\cdot\xi} e^{i(x-y)\cdot A(\frac{x+y}{2})} \, dy d\xi \\ - \; & \lim_{r\to 0+} \int\int \int_{|z|\geq r} (e^{i(x-y+z)\cdot\xi} - e^{i(x-y)\cdot\xi}) n(dz) e^{i(x-y)\cdot A(\frac{x+y}{2})} \, f(y) \, dy d\xi \Big] \\ = \; & m \int \delta(x-y) e^{i(x-y)\cdot A(\frac{x+y}{2})} f(y) \, dy \\ - \; & \lim_{r\to 0+} \int \int_{|z|\geq r} (\delta(x-y+z) - \delta(x-y)) n(dz) e^{i(x-y)\cdot A(\frac{x+y}{2})} \, f(y) \, dy \\ = \; & mf(x) - \lim_{r\to 0+} \int \int_{|z|\geq r} (e^{-iz\cdot A(x+\frac{z}{2})} f(x+z) - f(x)) \, n(dz). \end{aligned}$$ □

To represent $e^{-t[H^{(1)}-m]}g$ by path integral, we need some further notations from Lévy process.

For each path $X(\cdot)$, $N_X(dsdy)$ denotes the *counting measure* on $[0,\infty) \times (\mathbf{R}^d \setminus \{0\})$ to count the number of discontinuities of $X(\cdot)$, i.e.

$$N_X((t,t'] \times U) := \#\{s \in (t,t']; \, 0 \neq X(s) - X(s-) \in U\} \tag{4.7}$$

with $0 < t < t'$ and $U \subset \mathbf{R}^d \setminus \{0\}$ being a Borel set. It satisfies $\int_{D_x} N_X(dsdy) \, d\lambda_x(X) = dsn(dy)$. Put $\widetilde{N}_X(dsdy) := N_X(dsdy) - dsn(dy)$, which may be thought of as a renormalization of $N_X(dsdy)$. Then any path $X \in D_x([0,\infty) \to \mathbf{R}^d)$ can be expressed with $N_x(\cdot)$ and $\widetilde{N}_X(\cdot)$ as

$$X(t) = x + \int_0^{t+} \int_{|y|\geq 1} y N_X(dsdy) + \int_0^{t+} \int_{0<|y|<1} y \widetilde{N}_X(dsdy). \tag{4.8}$$

Now we have the following path integral representation for $e^{-t[H^{(1)}-m]}g$.

Theorem 4.1 (Ichinose–Tamura [13], Ichinose [8])

$$\begin{aligned}(e^{-t[H^{(1)}-m]}g)(x) &= \int_{D_x([0,\infty)\to\mathbf{R}^d)} e^{-S^{(1)}(t,X)}g(X(t))\,d\lambda_x(X),\\ S^{(1)}(t,X) &= i\int_0^{t+}\int_{|y|\geq 1} A(X(s-)+\frac{y}{2})\cdot y\,N_X(dsdy)\\ &\quad +i\int_0^{t+}\int_{0<|y|<1} A(X(s-)+\frac{y}{2})\cdot y\,\widetilde{N}_X(dsdy)r \qquad (4.9)\\ &\quad +i\int_0^t ds\,\text{p.v.}\int_{0<|y|<1} A(X(s)+\frac{y}{2})\cdot y\,n(dy)+\int_0^t V(X(s))ds.\end{aligned}$$

Proof. We only give a sketch. Put

$$(T(t)g)(x) := \int_{\mathbf{R}^d} k_0(t,x-y)e^{-iA\left(\frac{x+y}{2}\right)\cdot(y-x)-V\left(\frac{x+y}{2}\right)t}g(y)dy, \qquad (4.10)$$

where $k_0(t,x-y)$ is the integral kernel of $e^{-t(\sqrt{-\Delta+m^2}-m)}$. Then we can rewrite it as

$$(T(t)g)(x) = \int_{D_x} e^{-iA\left(\frac{x+X(t)}{2}\right)\cdot(X(t)-x)-V\left(\frac{x+X(t)}{2}\right)t}g(X(t))d\lambda_x(X)\,.$$

Do partition of $[0,t]$: $0=t_0<t_1<\cdots<t_n=t,\ t_j-t_{j-1}=t/n$, and put

$$S_n(x_0,\cdots,x_n) := i\sum_{j=1}^n A(\frac{x_{j-1}+x_j}{2})\cdot(x_j-x_{j-1})+\sum_{j=1}^n V(\frac{x_{j-1}+x_j}{2})\frac{t}{n}, \qquad (4.11)$$

where $x_j=X(t_j)(j=0,1,2,\ldots,n)$; $x=x_0=X(t_0)$, $y=x_n=X(t_n)\equiv X(t)$.

Substitute these $n+1$ points of path $x_j=X(t_j)$ into $S_n(x_0,\cdots,x_n)$ to get

$$\begin{aligned}S_n(X) &:= S_n(X(t_0),\cdots,X(t_n))\\ &= i\sum_{j=1}^n A\Big(\frac{X(t_{j-1})+X(t_j)}{2}\Big)\cdot(X(t_j)-X(t_{j-1}))+\sum_{j=1}^n V\Big(\frac{X(t_{j-1})+X(t_j)}{2}\Big)\frac{t}{n}\end{aligned}$$

Then

$$\begin{aligned}(T(t/n)^ng)(x) &= \overbrace{\int_{\mathbf{R}^d}\cdots\int_{\mathbf{R}^d}}^{n\text{ times}}\prod_{j=1}^n k_0(t/n,x_{j-1}-x_j)e^{-S_n(x_0,\cdots,x_n)}g(x_n)dx_1\cdots dx_n\\ &= \int_{D_x} e^{-S_n(X)}g(X(t))\,d\lambda_x(X). \qquad (4.12)\end{aligned}$$

We can show

Proposition 4.2 $T(t/n)^n g \to e^{-t[H^{(1)}-m]}g$ in $L^2(\mathbf{R}^d)$, $n \to \infty$.

(Proof is omitted. See [13].)

Now we are in a position to complete the proof of Theorem 5.1. By Proposition 4.2, we see the left-hand side of (4.12) converges to $e^{-t[H^{(1)}-m]}g$ as $n \to \infty$. On the other hand, we see by Itô's formula [see $*$) below] that the right-hand side converges to $\int_{D_x} e^{-S(X)} g(X(t))\, d\lambda_x(X)$ by Lebesgue convergence theorem. □

*) For instance, in $t_{j-1} \le s < t_j$, we have by Itô's formula,

$$
\begin{aligned}
& A\Big(\frac{X(t_{j-1})+X(t_j)}{2}\Big)\cdot\big(X(t_j)-X(t_{j-1})\big) \\
= \int_{t_j-1}^{t_j+}\int_{|y|>0}\Big[A\Big(\frac{X(s-)+X(t_{j-1})+yI_{|y|\ge 1}(y)}{2}\Big)\cdot\big(X(s-)-X(t_{j-1})+yI_{|y|\ge 1}(y)\big) \\
& \qquad -A\Big(\frac{X(s-)+X(t_{j-1})}{2}\Big)\cdot\big(X(s-)-X(t_{j-1})\big)\Big]\,N_X(dsdy) \\
& +\int_{t_j-1}^{t_j+}\int_{|y|>0}\Big[A\Big(\frac{X(s-)+X(t_{j-1})+yI_{|y|<1}(y)}{2}\Big)\cdot\big(X(s-)-X(t_{j-1})+yI_{|y|<1}(y)\big) \\
& \qquad -A\Big(\frac{X(s-)+X(t_{j-1})}{2}\Big)\cdot\big(X(s-)-X(t_{j-1})\big)\Big]\,\widetilde{N}(dsdy) \\
& +\int_{t_j-1}^{t_j}\int_{|y|>0}\Big[A\Big(\frac{X(s)+X(t_{j-1})+yI_{|y|<1}(y)}{2}\Big)\cdot\big(X(s)-X(t_{j-1})+yI_{|y|<1}(y)\big) \\
& \qquad -A\Big(\frac{X(s)+X(t_{j-1})}{2}\Big)\cdot\big(X(s)-X(t_{j-1})\big) \\
& \qquad\quad -I_{|y|<1}(y)\Big\{\Big(\frac{1}{2}(y\cdot\nabla)A\Big)\Big(\frac{X(s)+X(t_{j-1}}{2}\Big)\cdot\big(X(s)-X(t_{j-1})\big) \\
& \qquad\quad\quad +y\cdot A\Big(\frac{X(s)+X(t_{j-1})}{2}\Big)\Big\}\Big]\,dsn(dy).
\end{aligned}
$$

(2) Next we come to the case for the pseudo-differential operator modified by Iftimie–Măntoiu–Purice : $H^{(2)} := H_A^{(2)} + V$ in Definition 2.2. By exactly the same argument as used to show (4.4), we can show that

$$
\begin{aligned}
([H_A^{(2)}-m]f)(x) &= -\int_{|y|>0}[e^{-iy\cdot\int_0^1 A(x+\theta y)d\theta}f(x+y)-f(x) \\
&\qquad -I_{\{|y|<1\}}y\cdot(\nabla - iA(x))\,f(x)]n(dy) \\
&= -\lim_{r\downarrow 0}\int_{|y|\ge r}[e^{-iy\cdot\int_0^1 A(x+\theta y)d\theta}f(x+y)-f(x)]\,n(dy) \\
&= -\,\mathrm{p.v.}\int_{|y|>0}[e^{-iy\cdot\int_0^1 A(x+\theta y)d\theta}f(x+y)-f(x)]\,n(dy). \quad (4.13)
\end{aligned}
$$

Theorem 4.3 (Iftimie–Măntoiu–Purice [14, 15, 16])

$$
\begin{aligned}
(e^{-t[H^{(2)}-m]}g)(x) &= \int_{D_x([0,\infty)\to\mathbf{R}^d)} e^{-S^{(2)}(t,X)} g(X(t))\, d\lambda_x(X), \\
S^{(2)}(t,X) &= i\int_0^{t+}\int_{|y|\geq 1}\Big(\int_0^1 A(X(s-)+\theta y)\cdot y\, d\theta\Big)\, N_X(dsdy) \\
&\quad + i\int_0^{t+}\int_{0<|y|<1}\Big(\int_0^1 A(X(s-)+\theta y)\cdot y\, d\theta\Big)\, \widetilde{N}_X(dsdy) \qquad (4.14) \\
&\quad + i\int_0^t ds\ \mathrm{p.v.}\int_{0<|y|<1}\Big(\int_0^1 A(X(s)+\theta y)\cdot y\, d\theta\Big) n(dy) + \int_0^t V(X(s))ds.
\end{aligned}
$$

The proof of Theorem 4.3 will be done in exactly the same way as that of Theorem 4.1. Indeed, we have only to replace $A(X(s-)+\frac{y}{2})\cdot y$ by $\int_0^1 A(X(s-)+\theta y)\cdot y\, d\theta$.

(3) Finally, we consider the case for the operator defined, in Definition 2.3, with the square root of a nonnegative selfadjoint operator, $H^{(3)} := H_A^{(3)} + V$.

On the one hand, we can determine by functional analysis, namely, by theory of fractional powers (e.g. Yosida [22, Chap.IX,11, pp.259–261]) $e^{-t[H_A^{(3)}-m]}$ from the nonnegative selfadjoint operator $S := (-i\nabla - A(x))^2 + m^2 =: 2mH_A^{NR} + m^2$ where H_A^{NR} stands for the magnetic nonrelativistic Schrödinger operator $\frac{1}{2m}(-i\nabla - A(x))^2$ without scalar potential. Indeed, we have

$$
\begin{aligned}
e^{-t[H_A^{(3)}-m]}g &= \begin{cases} e^{mt}\int_0^\infty f_t(\lambda)e^{-\lambda S} g\, d\lambda, & t>0, \\ 0, & t=0 \end{cases} \\
f_t(\lambda) &= \begin{cases} (2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\lambda - tz^{1/2}} dz, & \lambda \geq 0, \\ 0, & \lambda < 0 \quad (\sigma>0). \end{cases} \qquad (4.15)
\end{aligned}
$$

Here we quickly insert the *Feynman–Kac–Itô formula* (e.g. Demuth–van Casteren [4], Simon [20]) for the magnetic nonrelativistic Schrödinger operator $H^{NR} := H_A^{NR} + V := \frac{1}{2m}(-i\nabla - A(x))^2 + V(x)$ $(m>0)$, a more general formula than the Feynman–Kac formula:

$$
\begin{aligned}
&(e^{-tH^{NR}}g)(x) \\
&= \int_{C_x([0,\infty)\to\mathbf{R}^d)} e^{-[i\int_0^t A(B(s))dB(s) + \frac{i}{2}\int_0^t \mathrm{div}A(B(s))ds + \int_0^t V(B(s))ds]} g(B(t)) d\mu_x(B) \\
&\equiv \int_{C_x([0,\infty)\to\mathbf{R}^d)} e^{-[i\int_0^t A(B(s))\circ dB(s) + \int_0^t V(B(s))ds]} g(B(t))\, d\mu_x(B). \qquad (4.16)
\end{aligned}
$$

This can provide a kind of path integral representation for $e^{-t[H_A^{(3)}-m]}g$ with the Wiener measure, by substituting the Feynman–Kac–Itô formula (4.16) for $V=0$ with $t = 2m\lambda$

into $e^{-\lambda(S-m^2)} = e^{-2m\lambda H_A^{NR}}$ in the integrand of equation (4.15) for $e^{-t[H_A^{(3)}-m]}g$. Then, to represent $e^{-t[H^{(3)}-m]}g$ for $V \neq 0$, we might apply the Trotter–Kato product formula

$$e^{-t[H^{(3)}-m]} = \text{s-}\lim_{n\to\infty} (e^{-(t/n)[H_A^{(3)}-m]}e^{-(t/n)V})^n, \tag{4.17}$$

to the sum $H^{(3)} - m = (H_A^{(3)} - m) + V$ to express the semigroup $e^{-t[H^{(3)}-m]}$ as a "limit", where convergence of the right-hand side usually takes place in strong sense as indicated, but now even, *in operator norm*, by the recent results on operator norm convergence in Ichinose–Tamura [9], Ichinose–Tamura–Tamura–Zagrebnov [12] (also [10, 11]). However it is not clear whether this procedure could further yield a path integral representation for $e^{-t[H^{(3)}-m]}g$.

On the other hand, it does not seem possible to represent $e^{-t[H^{(3)]}-m]}g$ by path integral through directly applying Lévy process, as we saw in the cases for $e^{-t(H^{(1)}-m)}g$ and $e^{-t(H^{(2)}-m)}g$, because $H_A^{(3)}$ does not seem to be explicitly expressed by a pseudo-differential operator of a certain tractable symbol. It was in this situation that the problem of path integral representation for $e^{-t[H^{(3)]}-m]}g$ was studied first by DeAngelis–Serva [3] and DeAngelis–Rinaldi–Serva [2] with use of *subordination /time-change* of Brownnian motion, and recently more extensively in Hiroshima–Ichinose–Lőrinczi [5] not only for the magnetic relativistic Schrödinger operator $H_A^{(3)}$ but also for Bernstein functions of the magnetic nonrelativistic Schrödinger operator even with spin.

To proceed, let us briefly explain about subordination (e.g. Sato [19, Chap.6, p.197], Applebaum [1, 1.3.2, p.52]). Subordination is a transformation, through random time change, of a stochastic process to a new one which is a non-decreasing Lévy process independent of the original one, what is called *subordinator*. The new process is said to be subordinate to the original one.

As the original process, take $B^1(t)$, the one-dimensional standard Brownian motion, so that $B^1 \equiv B^1(\cdot)$ is a function belonging to the space $C_0([0,\infty) \to \mathbf{R})$ of real-valued continuous functions on $[0,\infty)$ satisfying $B^1(0) = 0$, equipped with the Wiener measure μ_0^1 such that $e^{-t\frac{\xi^2}{2}} = \int_{C_0([0,\infty)\to\mathbf{R})} e^{i\xi B^1(t)} d\mu_0^1(B^1)$.

Let $m > 0$, and for each B^1 and $t \geq 0$, put

$$T(t) \equiv T(t, B^1) := \inf\{\, s > 0\,;\, B^1(s) + \sqrt{m}\, s = \sqrt{m}\, t\}. \tag{4.18}$$

Then $T \equiv T(\cdot)$ is a monotone, non-decreasing function on $[0,\infty)$ with $T(0) = 0$, belonging to $D_0([0,\infty)\to\mathbf{R})$ and so becoming a one-dimensional Lévy process, called *inverse Gaussian subordinator*. This correspondence defines a map $\hat{T}$ of $C_0([0,\infty) \to \mathbf{R})$ into $D_0([0,\infty) \to \mathbf{R})$ by $\hat{T}B^1(\cdot) = T(\cdot, B^1)$. Let ν_0 be the probability measure on $D_0([0,\infty) \to \mathbf{R})$ defined by $\nu_0(G) = \mu_0(\hat{T}^{-1}G)$ first for cylinder subsets $G \subset D_0([0,\infty)\to\mathbf{R})$ and then extended for more general subsets.

Proposition 4.4 (e.g. [1, p.54, Example 1.3.21], and Exercise 2.2.10, p.96; cf. Theorem 2.2.9, p.95])

$$e^{-t[\sqrt{2m\sigma+m^2}-m]} = \int_{D_0([0,\infty)\to\mathbf{R})} e^{-T(t)\sigma} d\nu_0(T), \qquad \sigma \geq 0. \tag{4.19}$$

This proposition implies that the characteristic function of the measure ν_0 is given by

$$\begin{aligned} e^{-t\phi(\rho)} &= \int_{D_0([0,\infty)\to\mathbf{R})} e^{iT(t)\rho} d\nu_0(T), \qquad \rho \in \mathbf{R}\,, \\ \phi(\rho) &= (\frac{m}{2})^{1/2} \frac{\sqrt{m^2+4\rho^2}-m}{(\sqrt{m^2+4\rho^2}+m)^{1/2}+\sqrt{2}m^{1/2}} - \frac{(2m)^{1/2}\rho}{(\sqrt{m^2+4\rho^2}+m)^{1/2}}\, i\,. \end{aligned} \tag{4.20}$$

To see this, first analytically extend $\sqrt{2m\sigma+m^2}$ to the right-half complex plane $z := \sigma + i\rho,\ \sigma > 0,\ \rho \in \mathbf{R}$, and then we have $\phi(\rho) = \lim_{\sigma\to+0} \sqrt{2m(\sigma+i\rho)+m^2} - m$, of which the right-hand side is calculated as in (4.20).

We are in a position to give a path integral representation for $e^{-t[H^{(3)}-m]}g$.

Theorem 4.5 [3, 2; 5]

$$\begin{aligned} (e^{-t[H^{(3)}]-m]}g)(x) &= \int \int_{\substack{C_x([0,\infty)\to\mathbf{R}^d) \\ \times D_0([0,\infty)\to\mathbf{R})}} e^{-S^{(3)}(t,B,T)} g(B(T(t)))\, d\mu_x(B) d\nu_0(T), \\ S^{(3)}(t,B,T) &= i\int_0^{T(t)} A(B(s))\, dB(s) + \frac{i}{2}\int_0^{T(t)} \mathrm{div} A(B(s)) ds + \int_0^t V(B(T(s)))ds, \\ &\equiv i\int_0^{T(t)} A(B(s)) \circ dB(s) + \int_0^t V(B(T(s)))ds, \end{aligned} \tag{4.21}$$

where μ_x is the Wiener measure on $C_x([0,\infty) \to \mathbf{R}^d)$.

Proof of Theorem 4.5. (Sketch) We use Proposition 4.4 and the Feynman–Kac–Itô formula (4.16). Note that $H_A^{(3)} = \sqrt{2mH_A^{NR}+m^2}$. By Spectral Theorem for the nonnegative selfadjoint operator H_A^{NR} , we have $H_A^{NR} = \int_{\mathrm{Spec}(H_A^{NR})} \sigma\, dE(\sigma)$, where $E(\cdot)$ is the spectral measure associated with H_A^{NR}. Then for $f,\, g \in L^2(\mathbf{R}^d)$

$$\langle f, e^{-t[H_A^{(3)}-m]} g\rangle = \int_{\mathrm{Spec}(H_A^{NR})} e^{-t[\sqrt{2m\sigma+m^2}-m]}\, \langle f, dE(\sigma) g\rangle,$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product of the Hilbert space $L^2(\mathbf{R}^d)$. By Propositopn 4.4 and again by Spectral Theorem,

$$\begin{aligned}\langle f, e^{-t[H_A^{(3)}-m]}g\rangle &= \int_{\mathrm{Spec}(H_A^{NR})}\int_{D_0([0,\infty)\to\mathbf{R})} e^{-T(t)\sigma}d\nu_0(T)\,\langle f, dE(\sigma)g\rangle \\ &= \int_{D_0([0,\infty)\to\mathbf{R})}\langle f, e^{-T(t)H_A^{NR}}g\rangle\, d\nu_0(T).\end{aligned}$$

Applying the Feynman–Kac–Itô formula (4.16) (with $V=0$) to $e^{-T(t)H_A^{NR}}g$ on the right, we have

$$\begin{aligned}&\langle f, e^{-t[H_A^{(3)}-m]}g\rangle \\ &= \int_{D_0([0,\infty)\to\mathbf{R})} d\nu_0(T)\int_{\mathbf{R}^d} dx\overline{f(B(0))}\int_{C_x([0,\infty)\to\mathbf{R}^d)} e^{-i\int_0^{T(t)}A(B(s))\circ dB(s)}g(B(T(t)))d\mu_x(B) \\ &= \int_{\mathbf{R}^d} dx\overline{f(x)}\int\int_{\substack{C_x([0,\infty)\to\mathbf{R}^d)\\ \times D_0([0,\infty)\to\mathbf{R})}} e^{-i\int_0^{T(t)}A(B(s))\circ dB(s)}g(B(T(t)))\,d\mu_x(B)d\nu_0(T),\end{aligned}$$

where note $B(0)=x$. This proves the assertion when $V=0$.

When $V\neq 0$, with partition of $[0,t]$: $0=t_0<t_1<\cdots<t_n=t$, $t_j-t_{j-1}=t/n$, we can express $e^{-t[H^{(3)}-m]}g = e^{-t[(H_A^{(3)}-m)+V]}$ by the Trotter–Kato formula (4.17). Rewrite the product of these n operators by path integral with respect to the product of two probability measures $\nu_0(T)\cdot\mu_x(B)$ and note that $T(0)=T(t_0)=0$, $B(0)=B(T(t_0))=x$, then we have

$$\begin{aligned}&\langle f, (e^{-(t/n)[H_A^{(3)}-m]}e^{-(t/n)V})^n g\rangle \\ &= \int_{\mathbf{R}^d} dx\int_{D_0([0,\infty)\to\mathbf{R})} d\nu_0(T)\int_{C_x([0,\infty)\to\mathbf{R}^d)}\overline{f(B(0))} \\ &\qquad\qquad \times e^{-i\sum_{j=1}^n\int_{T(t_{j-1})}^{T(t_j)}A(B(s))\circ dB(s)}e^{-\sum_{j=1}^n V(B(T(t_j))\frac{t}{n}}g(B(t_n))\,d\mu_x(B).\end{aligned}$$

We see, as $n\to\infty$, that the left-hand side converges to $\langle f, e^{-t[H_A^{(3)}-m]}g\rangle$, and the right-hand side also converges to the goal formula by the Lebesgue theorem, as integral by $dx\cdot\nu_0(T)\cdot\mu_x(B)$. Hence or similarly we can also get (4.21). □

Finally, as summary, we will collect the three path integral representation formulas in Theorems 4.1, 4.3, 4.5, below, so as to be able to easily see x-dependence. To do so, make change of space, probablity measure and paths by translation:

$D_x\to D_0$, $\lambda_x\to\lambda_0$, $X(s)\to X(s)+x$, $B(s)\to B(s)+x$, $B(T(s))\to B(T(s))+x$, then

$$(4.9)\quad :\quad (e^{-t[H^{(1)}-m]}g)(x)=\int_{D_0([0,\infty)\to\mathbf{R}^d)} e^{-S^{(1)}(t,X)}g(X(t)+x)\,d\lambda_0(X),$$

$$\begin{aligned}S^{(1)}(t,X)&=i\int_0^{t+}\int_{|y|\ge 1} A(X(s-)+x+\frac{y}{2})\cdot y\,N_X(dsdy)\\&\quad+i\int_0^{t+}\int_{0<|y|<1} A(X(s-)+x+\frac{y}{2})\cdot y\,\widetilde{N}_X(dsdy)\\&\quad+i\int_0^t ds\,\mathrm{p.v.}\int_{0<|y|<1} A(X(s)+x+\frac{y}{2})\cdot y\,n(dy)+\int_0^t V(X(s)+x)ds\,;\end{aligned}$$

$$(4.14)\quad :\quad (e^{-t[H^{(2)}-m]}g)(x)=\int_{D_0([0,\infty)\to\mathbf{R}^d)} e^{-S^{(2)}(t,X)}g(X(t)+x)\,d\lambda_0(X),$$

$$\begin{aligned}S^{(2)}(t,X)&=i\int_0^{t+}\int_{|y|\ge 1}\Big(\int_0^1 A(X(s-)+x+\theta y)\cdot y\,d\theta\Big)\,N_X(dsdy)\\&\quad+i\int_0^{t+}\int_{0<|y|<1}\Big(\int_0^1 A(X(s-)+x+\theta y)\cdot y\,d\theta\Big)\,\widetilde{N}_X(dsdy)\\&\quad+i\int_0^t ds\,\mathrm{p.v.}\int_{0<|y|<1}\Big(\int_0^1 A(X(s)+x+\theta y)\cdot y\,d\theta\Big)n(dy)+\int_0^t V(X(s)+x)ds\,;\end{aligned}$$

$$(4.21)\quad :\quad (e^{-t[H^{(3)]}-m]}g)(x)=\int\int_{\substack{C_0([0,\infty)\to\mathbf{R}^d)\\ \times D_0([0,\infty)\to\mathbf{R})}} e^{-S^{(3)}(t,B,T)}g(B(T(t))+x)\,d\mu_0(B)d\nu_0(T),$$

$$\begin{aligned}S^{(3)}(t,B,T)&=i\int_0^{T(t)} A(B(s)+x)\cdot dB(s)+\frac{i}{2}\int_0^{T(t)}\mathrm{div}A(B(s)+x)ds+\int_0^t V(B(T(s))+x)ds,\\&\equiv i\int_0^{T(t)} A(B(s)+x)\circ dB(s)+\int_0^t V(B(T(s))+x)ds\end{aligned}$$

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