On the holomorphic automorphism group of a generalized complex ellipsoid

Akio KODAMA

Abstract

In this paper, we completely determine the structure of the holomorphic automorphism group of a generalized complex ellipsoid. This is a natural generalization of a result due to Landucci. Also this gives an affirmative answer to an open problem posed by Jarnicki and Pflug.

Keywords: Generalized complex ellipsoids; Holomorphic automorphisms

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1 Introduction

In this paper we study the structure of the holomorphic automorphism group of a *generalized complex ellipsoid*

$$E(n_0,\dots,n_K;p_0,\dots,p_K) := \left\{ (z_0,\dots,z_K) \in \mathbf{C}^{n_0} \times \cdots \times \mathbf{C}^{n_K} \ ; \ \sum_{k=0}^{K} \|z_k\|^{2p_k} < 1 \right\}$$

in $\mathbf{C}^N = \mathbf{C}^{n_0} \times \cdots \times \mathbf{C}^{n_K}$, where $n_0,\dots,n_K$ are positive integers and $p_0,\dots,p_K$ are positive real numbers, and $N = n_0 + \cdots + n_K$. In general this domain is not geometrically convex and its boundary is not smooth. In the special case where all the $p_k = 1$, this domain reduces to the unit ball B^N in $\mathbf{C}^N$ and the structure of its holomorphic automorphism group $\mathrm{Aut}(B^N)$ is well-known (cf. [7]). Also, it is known that $E(n_0,\dots,n_K;p_0,\dots,p_K)$ is homogeneous if and only if $p_k = 1$ for all k (cf. [3], [6], [8]).

For convenience and with no loss of generality, in the following we will always assume that $p_0 = 1$, $p_1,\dots,p_K \neq 1$, $n_1,\dots,n_K > 0$. Moreover, after relabeling the indices, if necessary, we may assume that there exist positive integers $k_1,\dots,k_s$ such that

$$\begin{aligned} &k_1 + \cdots + k_s = K, \\ &n_{k_1+\cdots+k_{j-1}+1} = \cdots = n_{k_1+\cdots+k_j}, \quad 1 \le j \le s, \\ &n_{k_1+\cdots+k_j} < n_{k_1+\cdots+k_j+1}, \quad 1 \le j \le s-1, \end{aligned}$$

where we put $p_0 = 0$.

Now let us choose an arbitrary generalized complex ellipsoid $\mathcal{E}$ in $\mathbf{C}^N$ and write it in the form

$$(*) \qquad \mathcal{E} = E(n_0, n_1, \dots, n_K; 1, p_1, \dots, p_K).$$

Here it is understood that 1 does not appear if $n_0 = 0$, and also this domain is the unit ball B^{n_0} in $\mathbf{C}^{n_0} = \mathbf{C}^N$ if $K = 0$.

The purpose of this paper is to establish the following theorem that gives a full description of the holomorphic automorphism group of generalized complex ellipsoids:

THEOREM *Let $\mathcal{E}$ be the generalized complex ellipsoid appearing in $(*)$. Then the holomorphic automorphism group $\mathrm{Aut}(\mathcal{E})$ of $\mathcal{E}$ consists of all transformations*

$$\varphi : (z_0, z_1, \dots, z_K) \longmapsto (\tilde{z}_0, \tilde{z}_1, \dots, \tilde{z}_K)$$

of the form

$$\tilde{z}_0 = H(z_0), \quad \tilde{z}_k = \gamma_k(z_0) U_k z_{\sigma(k)}, \quad 1 \le k \le K$$

(think of z_k as column vectors), where

(1) $H \in \mathrm{Aut}(B^{n_0})$,

(2) $\gamma_k(z_0)$ *are nowhere vanishing holomorphic functions on B^{n_0} defined by*

$$\gamma_k(z_0) = \left(\frac{1 - \|a\|^2}{\big(1 - \langle z_0, a \rangle\big)^2} \right)^{1/2p_k}, \quad a = H^{-1}(o) \in B^{n_0},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian inner product on $\mathbf{C}^{n_0}$ and $o \in B^{n_0}$ is the origin of $\mathbf{C}^{n_0}$,

(3) $U_k \in U(n_k)$, *the unitary group of degree n_k, and*

(4) σ *is a permutation of $\{1, \dots, K\}$ satisfying the following:*

$$\big\{\sigma(k_1 + \cdots + k_{j-1} + 1), \dots, \sigma(k_1 + \cdots + k_j)\big\} = \big\{k_1 + \cdots + k_{j-1} + 1, \dots, k_1 + \cdots + k_j\big\}, \quad 1 \le j \le s,$$

and $\sigma(\mu) = \nu$ can only happen when $p_\mu = p_\nu$.

In particular, considering the special case where $n_k = 1$ and $2 \le p_k \in \mathbf{N}$ for all k, we obtain a natural generalization of Landucci [4; Corollary to Theorem]. This also gives an affirmative answer to an open problem posed in Jarnicki and Pflug [2; Remark 2.5.11].

In the next Section 2 we prove the Theorem and, in Section 3, we give a concrete example illustrating our result.

2 Proof of the Theorem

As mentioned in the introduction, the structure of the holomorphic automorphism group of the unit ball B^N in $\mathbf{C}^N$ is well-known. So we prove the Theorem in the case where $K \geq 1$.

For the given generalized complex ellipsoid $\mathcal{E}$ in $\mathbf{C}^N = \mathbf{C}^{n_0} \times \cdots \times \mathbf{C}^{n_K}$, let us consider the subset G of $\mathrm{Aut}(\mathcal{E})$ consisting of all elements

$$\varphi : (z_0, z_1, \ldots, z_K) \longmapsto (\tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_K)$$

having the form

$$\tilde{z}_0 = H(z_0), \quad \tilde{z}_k = \gamma_k(z_0) U_k z_k, \quad 1 \leq k \leq K, \tag{2.1}$$

where $H \in \mathrm{Aut}(B^{n_0})$, $U_k \in U(n_k)$ and $\gamma_k(z_0)$ are the same objects appearing in the statement of the Theorem. Then one can see that G is a connected Lie subgroup of the Lie group $\mathrm{Aut}(\mathcal{E})$ of dimension

$$d(\mathcal{E}) := n_0^2 + 2n_0 + \sum_{k=1}^{K} n_k^2.$$

On the other hand, we know from Naruki [6] and Sunada [8] that $\mathrm{Aut}(\mathcal{E})$ is a real Lie group of dimension $d(\mathcal{E})$; hence, G is exactly the identity component of $\mathrm{Aut}(\mathcal{E})$. In particular, G is a normal subgroup of $\mathrm{Aut}(\mathcal{E})$.

By making use of the concrete description in (2.1) of elements of G, it is an easy matter to check that the G-orbit passing through the origin $o \in \mathcal{E} \subset \mathbf{C}^N$ is of lowest dimension in the set of all G-orbits, i.e.,

$$\dim(G \cdot o) < \dim(G \cdot p) \quad \text{for any point } \ p \in \mathcal{E} \setminus G \cdot o.$$

Hence, recalling the normality of G in $\mathrm{Aut}(\mathcal{E})$, we obtain that

$$g \cdot (G \cdot o) = G \cdot o = \left\{ (z_0, 0, \ldots, 0) \in \mathbf{C}^{n_0} \times \mathbf{C}^{n_1} \times \cdots \times \mathbf{C}^{n_K} \,;\, \|z_0\| < 1 \right\} \tag{2.2}$$

for each element $g \in \mathrm{Aut}(\mathcal{E})$. This combined with a well-known theorem of H. Cartan (cf. [5; p. 67] assures us that every element $g \in \mathrm{Aut}(\mathcal{E})$ can be expressed as $g = \psi_g \cdot \ell_g$, where $\psi_g \in G$ and ℓ_g is a linear automorphism of $\mathcal{E}$, that is, a non-singular linear transformation of $\mathbf{C}^N$ leaving $\mathcal{E}$ invariant. Hence, the proof of our Theorem is now reduced to showing the following:

LEMMA *Every linear automorphism $L : (z_0, z_1, \ldots, z_K) \mapsto (\tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_K)$ of $\mathcal{E}$ can be written in the form*

$$\tilde{z}_0 = A z_0, \ \ \tilde{z}_k = U_k z_{\sigma(k)}, \ \ 1 \leq k \leq K, \tag{2.3}$$

where $A \in U(n_0)$, $U_k \in U(n_k)$ and σ is a permutation of $\{1, \ldots, K\}$ satisfying the same condition (4) *as in the Theorem.*

Proof. We will show this Lemma by generalizing the argument used in the proofs of [4; Proposition 2.1] and [1; Lemma 8.5.3]. It is clear that the linear transformation L of $\mathbf{C}^N$ written in the form (2.3) induces a linear automorphism of $\mathcal{E}$. So, taking an arbitrary linear automorphism L of $\mathcal{E}$, we would like to show that L can be described as in (2.3). To this end, we define the coordinate vector subspaces V_k, W_k of $\mathbf{C}^N$ by setting

$$\begin{aligned} V_k &= \big\{(z_0, z_1, \ldots, z_K) \in \mathbf{C}^N \,;\, z_j = 0,\, j \neq k\,\big\}, \\ W_k &= \big\{(z_0, z_1, \ldots, z_K) \in \mathbf{C}^N \,;\, z_k = 0\,\big\} \end{aligned}$$

for $0 \leq k \leq K$; accordingly $\bigcap_{j \neq k} W_j = V_k$ for $0 \leq k \leq K$. Here, recalling our assumption that $K \geq 1$ and all the $p_k \neq 1$, we put

$$W = \big\{(z_0, z_1, \ldots, z_K) \in \mathbf{C}^N \,;\, \|z_1\| \cdots \|z_K\| = 0\,\big\} \quad \text{and} \quad \mathcal{W} = W \cap \partial\mathcal{E},$$

where $\partial\mathcal{E}$ stands for the boundary of $\mathcal{E}$. Then, by routine computations it follows that $\partial\mathcal{E} \setminus \mathcal{W}$ is just the set consisting of all C^ω-smooth strongly pseudoconvex boundary points of $\mathcal{E}$; consequently, $L(\mathcal{W}) = \mathcal{W}$. This, combined with the facts that W is invariant under the dilations $\delta_r : z \mapsto rz$ $(r > 0)$ on $\mathbf{C}^N$ and $L(\delta_r(z)) = \delta_r(L(z))$ on $\mathbf{C}^N$, yields at once that $L(W) = W$.

With respect to the coordinate system $(z_0, z_1, \ldots, z_K)$ in $\mathbf{C}^N$, the linear automorphism L can be expressed as $L = (L_0, L_1, \ldots, L_K)$. Recall here the fact in (2.2). It then follows that

- each L_k $(1 \leq k \leq K)$ does not depend on the variable z_0,

and

- the restriction $L_{0|V_0} : V_0 \to V_0$ of L_0 to V_0 gives rise to a holomorphic automorphism of the unit ball B^{n_0}; and hence, it has to be a unitary transformation of $V_0 \equiv \mathbf{C}^{n_0}$.

Therefore, one may assume that L has the form:

$$L(z) = \big(z_0 + A(z_1, \ldots, z_K), L_1(z_1, \ldots, z_K), \ldots, L_K(z_1, \ldots, z_K)\big)$$

for $z = (z_0, z_1, \ldots, z_K) \in \mathbf{C}^N$, where A, L_k $(1 \leq k \leq K)$ are all linear mappings.

Now we will proceed in steps.

1) *There exists a permutation τ of $\{1, \ldots, K\}$ such that $L_{\tau(k)}(W_k) = \{0\}$ for every $1 \leq k \leq K$. In particular, we have $L(W_k) \subset W_{\tau(k)}$ for $1 \leq k \leq K$.* Indeed, let $1 \leq k \leq K$ and assume that $L_j(W_k) \neq \{0\}$ for all j, $1 \leq j \leq K$. Then, considering the proper complex analytic subset $\mathcal{A}$ of W_k consisting of all points $z \in W_k$ with $L_j(z) = 0$ for some j, $1 \leq j \leq K$, we have

$$\|L_1(z^o)\| \cdots \|L_K(z^o)\| > 0 \quad \text{for any point } z^o \in W_k \setminus \mathcal{A}.$$

However, since $W_k \subset W$ for every $1 \leq k \leq K$ and $L(W) = W$, this is absurd. Therefore we have shown that, for every $1 \leq k \leq K$, there exists at least one integer j, $1 \leq j \leq K$, such that $L_j(W_k) = \{0\}$. Let us fix, once and for all, the correspondence $\tau : k \mapsto j$. Then this τ is injective. Indeed, assume contrarily that $\tau(k) = \tau(\ell) =: j_0$ for some k, ℓ with $1 \leq k \neq \ell \leq K$. Then, since $\mathbf{C}^N =$

$W_k + W_\ell$, the sum of the vector subspaces W_k and W_ℓ, and since $L : \mathbf{C}^N \to \mathbf{C}^N$ is a linear isomorphism, we obtain a contradiction: $\mathbf{C}^N = L(\mathbf{C}^N) \subset W_{j_0} \subsetneqq \mathbf{C}^N$. As a result, τ is a permutation of $\{1, \ldots, K\}$ satisfying the condition required in 1).

2) *Let τ be the permutation of $\{1, \ldots, K\}$ appearing in 1). Then we have*

$$\left\{\tau(k_1 + \cdots + k_{j-1} + 1), \ldots, \tau(k_1 + \cdots + k_j)\right\} = \\ \left\{k_1 + \cdots + k_{j-1} + 1, \ldots, k_1 + \cdots + k_j\right\}, \quad 1 \le j \le s,$$

where we put $k_0 = 0$. Indeed, for every $1 \le k \le K$, we have

$$L(V_k) = \bigcap_{0 \le j \le K, j \ne k} L(W_j) \subset L(W_0) \bigcap \left(\bigcap_{1 \le j \le K, j \ne k} W_{\tau(j)} \right)$$

by 1); consequently,

$$L_{\tau(k)}(V_k) \subset V_{\tau(k)} \quad \text{and} \quad L_{\tau(j)}(V_k) = \{0\}, \;\; 1 \le j \le K, \; j \ne k. \tag{2.4}$$

From now on, putting $M = n_1 + \cdots + n_K$, we identify in the obvious way $\mathbf{C}^M = \mathbf{C}^{n_1} \times \cdots \times \mathbf{C}^{n_K}$ with the coordinate vector subspace W_0 of $\mathbf{C}^N$. Then the linear transformation $\tilde{L} := (L_1, \ldots, L_K) : \mathbf{C}^M \to \mathbf{C}^M$ induced by L is non-singular; and hence, we see that $L_{\tau(k)}(V_k) = V_{\tau(k)}$ in (2.4) and $n_k = n_{\tau(k)}$. This, together with the ordering among the integers $n_1, \ldots, n_K$ as in the previous section, guarantees that τ has to satisfy the condition in 2), as desired.

Let $\sigma := \tau^{-1}$ be the inverse of τ in 1). Then, by (2.4) L can be written in the form

$$L(z) = \left(z_0 + A(z_1, \ldots, z_K), U_1 z_{\sigma(1)}, \ldots, U_K z_{\sigma(K)}\right) \tag{2.5}$$

for $z = (z_0, z_1, \ldots, z_K) \in \mathbf{C}^N$ (think of z_k as column vectors), where U_k are non-singular $n_k \times n_k$ matrices for $1 \le k \le K$. Here we wish to verify the following:

3) *For every $1 \le k \le K$, we have $U_k \in U(n_k)$.* To show this, we first assert that $A(z_1, \ldots, z_K) \equiv 0$ in (2.5). Indeed, the fact $L(\partial\mathcal{E}) = \partial\mathcal{E}$ yields that

$$\|z_0 + A(z_1, \ldots, z_K)\|^2 + \sum_{k=1}^{K} \|U_k z_{\sigma(k)}\|^{2p_k} = 1, \quad z \in \partial\mathcal{E}.$$

For any point $z = (z_0, z_1, \ldots, z_K) \in \partial\mathcal{E}$, write $z_0 = \left(z_0^1, \ldots, z_0^{n_0}\right)$. Then, by taking a suitable point $\hat{z}_0$ of the form

$$\hat{z}_0 = \left(\xi_1 z_0^1, \ldots, \xi_{n_0} z_0^{n_0}\right), \quad \xi_j \in \mathbf{C}, \; |\xi_j| = 1, \; 1 \le j \le n_0,$$

we see that $\mathrm{Re}\langle z_0, A(z_1, \ldots, z_K)\rangle = 0$; and hence,

$$-\sum_{k=1}^{K} \|z_k\|^{2p_k} + \|A(z_1, \ldots, z_K)\|^2 + \sum_{k=1}^{K} \|U_k z_{\sigma(k)}\|^{2p_k} = 0, \quad z \in \partial\mathcal{E}. \tag{2.6}$$

Notice that this equality holds also for any point

$$(z_1, \ldots, z_K) \in \mathbf{C}^M \quad \text{with} \quad \sum_{k=1}^{K} \|z_k\|^{2p_k} \le 1,$$

because one can always find a point $z_0 \in \mathbf{C}^{n_0}$ such that $(z_0, z_1, \ldots, z_K) \in \partial\mathcal{E}$. Now, in order to prove that $A\,(z_1, \ldots, z_K) \equiv 0$, take an arbitrary point $z_1 \in \mathbf{C}^{n_1}$ with $\|z_1\| = 1$ and set $j = \sigma^{-1}(1)$, for simplicity. Then

$$-x^{2p_1} + x^2 \|A\,(z_1, 0, \ldots, 0)\,\|^2 + x^{2p_j} \|U_j z_1\|^{2p_j} = 0, \quad 0 \le x \le 1.$$

Since all the $p_k \ne 1$, this says that $A\,(z_1, 0, \ldots, 0) = 0$. Analogously, for every $2 \le k \le K$ one can show that $A\,(0, \ldots, 0, z_k, 0, \ldots, 0) = 0$ for $z_k \in \mathbf{C}^{n_k}$ with $\|z_k\| = 1$. Obviously this means that $A\,(z_1, \ldots, z_K) \equiv 0$ on $\mathbf{C}^M$, as asserted.

Next, put $j = \sigma(k)$ for a given k, $1 \le k \le K$. It then follows from (2.6) that

$$\|U_k z_j\| = 1 \quad \text{for all } \ z_j \in V_j,\ \|z_j\| = 1;$$

which implies that $U_k \in U(n_k)$ for every $1 \le k \le K$; verifying the assertion 3).

Summarizing the above, we have shown that L has the form

$$L(z) = \big(z_0, U_1 z_{\sigma(1)}, \ldots, U_K z_{\sigma(K)}\big), \quad z = (z_0, z_1, \ldots, z_K) \in \mathbf{C}^N,$$

where $U_k \in U(n_k)$, $1 \le k \le K$, and σ is a permutation of $\{1, \ldots, K\}$ satisfying the condition:

$$\begin{aligned} &\big\{\sigma(k_1 + \cdots + k_{j-1} + 1), \ldots, \sigma(k_1 + \cdots + k_j)\big\} \ = \\ &\qquad\qquad \big\{k_1 + \cdots + k_{j-1} + 1, \ldots, k_1 + \cdots + k_j\big\}, \ \ 1 \le j \le s. \end{aligned}$$

Therefore, in order to complete the proof of the Lemma, we have only to show the following assertion:

4) *Let* $k_1 + \cdots + k_{j-1} + 1 \le \mu,\ \nu \le k_1 + \cdots + k_j,\ 1 \le j \le s$. *Then* $\sigma(\mu) = \nu$ *can only happen when* $p_\mu = p_\nu$. We verify this only in the case where $j = 1$, since the verification in the general case is almost identical. Moreover, once the proof of 4) for $k_1 \ge 4$ is accomplished, then that for $1 \le k_1 \le 3$ follows by a simple modification of it. Taking these into account, we will carry out the proof of 4) in the case where $j = 1$ and $k_1 \ge 4$. Clearly $\sigma(\mu) = \nu$ is possible when $p_\mu = p_\nu$. So, assuming that $\sigma(\mu) = \nu$ for $1 \le \mu, \nu \le k_1$, $\mu \ne \nu$, we wish to prove that $p_\mu = p_\nu$. For this purpose, we first remark the following: Since $L(\partial\mathcal{E}) = \partial\mathcal{E}$, with exactly the same argument as in the proof of 3), we can see that

$$\text{(2.7)} \qquad \sum_{1 \le k \le k_1,\, k \ne \mu} \|z_{\sigma(k)}\|^{2p_k} + \left(1 - \sum_{1 \le j \le k_1,\, j \ne \nu} \|z_j\|^{2p_j}\right)^{p_\mu/p_\nu} = 1$$

for any point

$$(z_1, \ldots, z_{\nu-1}, z_{\nu+1}, \ldots, z_{k_1}) \quad \text{with} \quad \sum_{1\le j\le k_1,\, j\ne\nu} \|z_j\|^{2p_j} \le 1.$$

Now, since $k_1 \ge 4$, we can always choose an integer m, $1 \le m \le k_1$, in such a way that

$$m \ne \mu, \nu \quad \text{and} \quad j := \sigma(m) \ne \mu, \nu.$$

Then, putting $z_\ell = 0$ for $\ell \ne j$ in (2.7), we obtain that

$$\|z_j\|^{2p_m} + \left(1 - \|z_j\|^{2p_j}\right)^{p_\mu/p_\nu} = 1, \quad \|z_j\| \le 1.$$

Accordingly, by taking the points xz_j^o with $0 \le x \le 1$, $\|z_j^o\| = 1$, we have

$$x^{2p_m} + \left(1 - x^{2p_j}\right)^{p_\mu/p_\nu} = 1, \quad 0 \le x \le 1.$$

A simple computation shows that this can only happen when $p_m = p_j$ and $p_\mu = p_\nu$; completing the proof of the Lemma. $\square$

Hence we have completed the proof of our Theorem.

3 An example

As a concrete example illustrating our result, we here give the following generalized complex ellipsoid $\mathcal{E}$ in $\mathbf{C}^{11}$ defined by

$$\mathcal{E} = \Big\{(z, w_1, w_2, w_3, w_4, w_5, w_6) \in \mathbf{C}\times\mathbf{C}\times\mathbf{C}\times\mathbf{C}\times\mathbf{C}^2\times\mathbf{C}^2\times\mathbf{C}^3\,; \\ |z|^2 + |w_1|^{2/3} + |w_2|^3 + |w_3|^{2/3} + \|w_4\|^3 + \|w_5\|^3 + \|w_6\|^3 \ < \ 1\Big\}.$$

So, with the notation of the introduction, we have:

$$K = 6,\ \ n_1 = n_2 = n_3 = 1 < n_4 = n_5 = 2 < n_6 = 3,\ \ k_1 = 3,\ k_2 = 2,\ k_3 = 1 \\ \text{and} \quad \mathcal{E} = E(1,1,1,1,2,2,3;\ 1, 1/3, 3/2, 1/3, 3/2, 3/2, 3/2).$$

And our Theorem tells us that every element φ of $\mathrm{Aut}(\mathcal{E})$ can be described as

$$\varphi(u) = \Bigg(\xi\frac{z-a}{1-\bar{a}z},\ \rho(z)^{3/2}\xi_1 w_{\sigma(1)},\ \rho(z)^{1/3}\xi_2 w_{\sigma(2)},\ \rho(z)^{3/2}\xi_3 w_{\sigma(3)}, \\ \rho(z)^{1/3}U_4 w_{\sigma(4)},\ \rho(z)^{1/3}U_5 w_{\sigma(5)},\ \rho(z)^{1/3}U_6 w_{\sigma(6)}\Bigg)$$

for $u = (z, w_1, w_2, w_3, w_4, w_5, w_6) \in \mathcal{E}$, where

$$a,\ \xi,\ \xi_1,\ \xi_2,\ \xi_3 \in \mathbf{C} \ \text{with}\ |a| < 1,\ |\xi| = |\xi_1| = |\xi_2| = |\xi_3| = 1, \\ U_4,\ U_5 \in U(2),\ \ U_6 \in U(3),\ \ \rho(z) = \frac{1-|a|^2}{\left(1-\bar{a}z\right)^2},\ |z| < 1,$$

and σ is a permutation of $\{1, \dots, 6\}$ such that

$$\{\sigma(1), \sigma(3)\} = \{1, 3\}, \ \{\sigma(4), \sigma(5)\} = \{4, 5\}, \ \sigma(2) = 2, \ \sigma(6) = 6.$$

Therefore we conclude that $\mathrm{Aut}(\mathcal{E})$ is a 23-dimensional Lie group with four connected components.

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