Sci. Rep. Kanazawa Univ.
Vol. 33 No. 2 pp. 27~36
June 1989

An Asymptotic Behavior of $\{f(n_k t)\}$

Shigeru TAKAHASHI

Department of Mathematics, Faculty of Science, Kanazawa University

(Received October 30, 1988)

Abstract Let $f(t)\in \text{Lip}\ \delta$ $(\delta>1/2)$ and $f(t+1)=f(t)$. Then if $\{n_k\}$ satisfies $n_{k+1}/n_k>1+ck^{-\alpha}(c>0$ and $0\leqq\alpha<1/2)$, the law of the iterated logarithms for $\{f(n_k t)\}$ is studied.

1. Introduction Let $f(t)$ be a real valued Lebesgue measurable function on $(-\infty, +\infty)$ satisfying the conditions

$$f(t+1)=f(t), \quad \int_0^1 f(t)dt=0 \quad \text{and} \quad \int_0^1 f^2(t)dt<+\infty,$$

and $\{n_k\}$ be an increasing sequence of positive integers. Then it is well known that the sequence of functions $\{f(n_k t)\}$, although themselves not independent, exhibits the properties of independent random variables.

In [3] we proved that if $f\in \text{Lip}\ \delta$ $(\delta>0)$ and

$$n_{k+1}/n_k>1+c \quad (c>0 \quad \text{and} \quad k\geqq 1), \tag{1. 1}$$

then we have

$$\overline{\lim_{N\to\infty}}\,(N \log\log N)^{-1/2}\sum_{k=1}^{N} f(n_k t)\leqq C, \quad \text{a. e. } t, \tag{1. 2}$$

where C is a constant depending on f and c in (1. 1).

Recently, Dhompongsa [1] showed that if $f\in \text{Lip}\ \delta$ $(\delta>1/2)$ and $\{n_k\}$ satisfies the gap condition

$$\begin{cases} n_{k+1}/n_k>1+ck^{-\alpha},\ k\geqq 1, \\ \text{for some } c>0 \text{ and } 0<\alpha<1/2, \end{cases} \tag{1. 3}$$

then (1. 2) holds for some constant $C>0$.

The purpose of the present note is to prove the

THEOREM. *If* $f\in \text{Lip}\ \delta$ $(\delta>1/2)$ *and* $\{n_k\}$ *satisfies* (1. 3), then we have

$$\overline{\lim_{N\to\infty}}\,(N \log\log N)^{-1/2}\sum_{k=1}^{N} f(n_k t)\leqq \|f\|, \quad a.\ e.\ t,$$

where $f \sim \sum_{h=1}^{\infty} a_h \cos 2\pi h(t+\alpha_h)$, $a_h \geqq 0$, *and* $\| f \| = \sum_{h=1}^{\infty} a_h$.

2. Some Lemmas From now on let us assume that f belongs to the class Lip δ ($\delta > 1/2$) and $\{n_k\}$ satisfies the gap condition (1. 3).

i. If $f(t) \sim \sum_{h=1}^{\infty} a_h \cos 2\pi h(t+\alpha_h)$, $a_h \geqq 0$, then the following facts are well known (c f. [5], Vol. 1);

$$\sum_{h=1}^{\infty} a_h \log(h+1) < +\infty \tag{2. 1}$$

and

$$|f(t) - \sum_{h=1}^{n} a_h \cos 2\pi h(t+\alpha_h)| = O(n^{-\delta} \log n) \tag{2. 2}$$
$$= o(n^{-1/2}), \qquad \text{uniformly in } t, \quad \text{as } n \to +\infty.$$

For simplicity of writing the formulas we consider only cosine series. The general case follows the same lines.

ii. Let us put

$$p(0) = 0 \quad \text{and} \quad p(k) = \max\{m\,;\, n_m < 2^k\} \qquad \text{for } k \geqq 1.$$

If $p(k)+1 < p(k+1)$, then we have

$$2 > n_{p(k+1)}/n_{p(k)+1} > \prod_{m=p(k)+1}^{p(k+1)-1} (1+cm^{-\alpha})$$
$$> 1 + c\{p(k+1) - p(k) - 1\}\, p^{-\alpha}(k+1).$$

Therefore, we have

$$p(k+1) - p(k) = O(p^{\alpha}(k)), \qquad \text{as } k \to \infty. \tag{2. 3}$$

Further, the following lemmas are proved (cf. [4]).

LEMMA 1. *For any given integers k, j, q and h satisfying*

$$p(j) + 1 < h \leqq p(j+1) < p(k) + 1 < q \leqq p(k+1),$$

the number of solutions (n_r, n_i) of the equation

$$n_q - n_r = n_h - n_i,$$

where $p(j) < i < h$ and $p(k) < r < q$, is at most $C 2^{j-k} p^{\alpha}(k)$, where C is a positive constant independent of k, j, q and h.

LEMMA 2. *For any given integers k, j, q and h satisfying*

$$p(j+1) < h \leqq p(j+2) < p(k+1) < q \leqq p(k+2),$$

the number of solutions (n_r, n_i) of the equation

$$n_q - n_r = n_h - n_i,$$

where $p(j)<i\leqq p(j+1)$ *and* $p(k)<r\leqq p(k+1)$, *is at most* $C2^{j-k}p^{\alpha}(k)$, *where* C *is a positive constant independent of* k, j, q *and* h.

iii. Let β be a positive constant satisfying

(2. 4) $\quad 0 < 1/(1-\alpha) < \beta < 2,$

and $\{q(k)\}$ be a sequence of integers such that

(2. 5) $\quad p(q(k)-1) \leqq k^{\beta} < p(q(k)).$

Further, we put

$$(2.6)\qquad \begin{cases} \Delta_m(t)=\displaystyle\sum_{j=p(m)+1}^{p(m+1)} \cos 2\pi n_j t, \\ Q_k(t)=\displaystyle\sum_{m=q(k-1)}^{q(k)-2} \Delta_m(t). \end{cases}$$

Then we have, by (2. 4) and (2. 5),

$$(2.7)\qquad k^{\beta}-p(q(k)-1)\leqq p(q(k))-p(q(k)-1)$$

$$=O(p^{\alpha}(q(k))=O(k^{\alpha\beta})=o(k^{\beta-1}), \qquad \text{as } k\to+\infty.$$

LEMMA 3. *We have*

$$\|\sum_{k=1}^{N}(Q_k^2-\|Q_k\|_2^2)\|_2^2=O(N^{3\beta-2}), \qquad \text{as } N\to+\infty.$$

PROOF. We have

$$Q_k^2-\|Q_k\|_2^2=\sum_{m=q(k-1)}^{q(k)-2}(\Delta_m^2-\|\Delta_m\|_2^2)$$

$$+\ 2\sum_{m=q(k-1)+3}^{q(k)-2}\Delta_m\sum_{j=q(k-1)}^{m-2}\Delta_j\ +\ 2\sum_{m=q(k-1)+1}^{q(k)-2}\Delta_m\Delta_{m-1}.$$

Since for each r, $0\leqq r\leqq 2$, the functions $(\Delta_{3m+r}\sum_{j=q(k-1)}^{3m+r-2}\Delta_j)$, $q(k-1)+2<3m+r<q(k)-1$ and $k=1, 2, \cdots$, are orthogonal and by (2. 5) and (2. 7),

$$(2.8)\qquad \|\sum_{j=q(k-1)}^{q(k)-2}|\Delta_j|\ \|_{\infty}\leqq p(q(k)-1)-p(q(k-1))$$

$$=O(k^{\beta-1}), \qquad \text{as } k\to+\infty,$$

we have

$$\|\sum_{k=1}^{N}\sum_{m=q(k-1)+3}^{q(k)-2}\Delta_m\sum_{j=q(k-1)}^{m-2}\Delta_j\|_2^2$$

$$\leqq 3\sum_{k=1}^{N}\sum_{m=q(k-1)+3}^{q(k)-2}\|\Delta_m\sum_{j=q(k-1)}^{m-2}\Delta_j\|_2^2$$

$$=O(\sum_{k=1}^{N}k^{2\beta-2}\sum_{m=q(k-1)+3}^{q(k)-2}\|\Delta_m\|_2^2)$$

$$=O\Big(\sum_{k=1}^{N} k^{2\beta-2}\ \{p(q(k)-1)-p(q(k-1))\}\Big)=O(N^{3\beta-2}), \qquad \text{as } N\to+\infty.$$

On the other hand we have

$$\Delta_k^2-\|\Delta_k\|_2^2=U_k+V_k,$$

where

$$U_k=\sum_{q=p(k)+1}^{p(k+1)} \{(\cos 4\pi n_q t)/2 + \sum_{r=p(k)+1}^{q-1} \cos 2\pi(n_q+n_r)t\},$$

$$V_k=\sum_{q=p(k)+2}^{p(k+1)} \sum_{r=p(k)+1}^{q-1} \cos 2\pi(n_q-n_r)t.$$

Since $\{U_m\}$ is orthogonal, we have, by the Minkowski inequality and (2. 3) and (2. 5),

$$\Big\|\sum_{k=1}^{N} \sum_{m=q(k-1)}^{q(k)-2} U_m\Big\|_2^2=\sum_{k=1}^{N} \sum_{m=q(k-1)}^{q(k)-2} \|U_m\|_2^2$$

$$\leqq\sum_{k=1}^{N} \sum_{m=q(k-1)}^{q(k)-2} \{p(m+1)-p(m)\}^3=O(p^{2\alpha+1}(q(N)))$$

$$=O(N^{(2\alpha+1)\beta})=o(N^{3\beta-2}), \qquad \text{as } N\to+\infty.$$

In the same way we have

$$\sum_{k=1}^{N} \sum_{m=q(k-1)}^{q(k)-2} \|V_m\|_2^2=o(N^{3\beta-2}), \qquad \text{as } N\to+\infty.$$

On the other hand we have, by Lemma 1,

$$\sum_{j=1}^{k-1} \Big|\int_0^1 V_k(t)V_j(t)dt\Big|$$

$$\leqq Cp^{\alpha}(k)\{p(k+1)-p(k)\} \sum_{j=1}^{k-1} 2^{j-k}\ \{p(j+1)-p(j)\}$$

$$\leqq Cp^{\alpha}(k)\{p(k+1)-p(k)\} \sum_{j=1}^{k-1} 2^{j-k}p^{\alpha}(j).$$

Since $p(j+1)/p(j)\to1$, as $j\to+\infty$, we have

$$\sum_{j=1}^{k-1} 2^{j-k}p^{\alpha}(j)=O(p^{\alpha}(k)), \qquad \text{as } k\to+\infty.$$

Hence, we have, by (2. 3) and (2. 7),

$$\sum_{k=1}^{q(N)} \sum_{j=1}^{k-1} \Big|\int_0^1 V_k(t)V_j(t)dt\Big| =O\Big(\sum_{k=1}^{q(N)} p^{2\alpha}(k)\{p(k+1)-p(k)\}\Big)$$

$$=O(p^{2\alpha+1}(q(N)))=O(N^{(2\alpha+1)\beta})=o(N^{3\beta-2}), \qquad \text{as } N\to+\infty.$$

Therefore, we have

$$\| \sum_{k=1}^{N} \sum_{m=q(k-1)}^{q(k)-2} V_m \|_2^2 = o(N^{3\beta-2}), \qquad \text{as } N \to +\infty.$$

In the same way we have, by (2. 3) and Lemma 2,

$$\| \sum_{k=1}^{N} \sum_{m=q(k-1)+1}^{q(k)} \Delta_m \Delta_{m-1} \|_2^2 = O(N^{3\beta-2}), \qquad \text{as } N \to +\infty.$$

By the above relations we can complete the proof of the Lemma 3.

In the same way we can prove the following

LEMMA 4. *We have, for any* M *and* N,

$$\| \sum_{k=N}^{M} (\Delta_k^2 - \| \Delta_k \|_2^2) \|_2^2 \leqq Cp^{2\alpha}(M)\{p(M+1)-p(N)\},$$

where C *is a positive constant independent of* M *and* N.

iii. We have, by (2. 3) and (2. 7),

$$\text{(2. 9)} \qquad \| \sum_{k=1}^{N-1} \Delta_{q(k)-1} \|_2^2 \leqq \sum_{k=1}^{N-1} \{p(q(k))-p(q(k)-1)\}$$

$$= O(\sum_{k=1}^{N} p^{\alpha}(q(k))) = O(\sum_{k=1}^{N} k^{\alpha\beta}) = O(N^{\alpha\beta+1}), \qquad \text{as } N \to +\infty.$$

3. The Estimations of Probabilities.

i. If x is real and $|x| < 1/3$, then we have

$$\text{(3. 1)} \qquad \exp\{x - (x^2 + |x|^3)/2\} \leqq (1+x).$$

Hence, if $|\lambda| \max_{m \leqq N} \| Q_m \|_\infty < 1/3$, then

$$\exp(\sum_{m=1}^{N} \{\lambda Q_m(t) - (\lambda^2 Q_m^2(t) + |\lambda Q_m(t)|^3)/2\})$$

$$\leqq \prod_{m=1}^{N} \{1 + \lambda Q_m(t)\}.$$

Since $\{Q_m(t)\}$ is multiplicatively orthogonal, that is, if $0 \leqq m_1 < m_2 < \cdots < m_n$, then

$$\int_0^1 \prod_{j=1}^{n} Q_{m_j}(t)dt = 0,$$

we have

$$\int_0^1 \exp(\sum_{m=1}^{N} \{\lambda Q_m(t) - (\lambda^2 Q_m^2(t) + |\lambda Q_m(t)|^3)/2\})dt \leqq 1.$$

If we put, for $x>0$ and $\lambda>0$,

$$E(\lambda, N, x) = [t; t \in [0, 1], \sum_{m=1}^{N} Q_m(t) \geqq x + \sum_{m=1}^{N} \{\lambda Q_m^2(t) + \lambda^2 |Q_m(t)|^3\}/2],$$

then we have, by Tchebyschev's inequality,

(3. 2) $$|E(\lambda, N, x)| \leqq \exp(-\lambda x),$$

where $|E|$ denotes the Lebesgue measure of the set E.

ii. For any fixed $\theta>1$, we take an integer $M(k)$ and real numbers $\lambda(h, k)$ and $x(h, k)$, $1\leqq h\leqq \theta^k$, as follows:

(3. 3) $$\begin{cases} M(k)^\beta \leqq \theta^k < (M(k)+1)^\beta, \\ \lambda(h, k)=2\ [\{\log(h+1)+\log\log\theta^k\}/\theta^k]^{1/2}, \\ x(h, k)=(1+\eta)\ \{\log(h+1)+\log\log\theta^k\}/\lambda(h, k), \end{cases}$$

where η is any given positive number.

Then we have, by (2. 4) and (2. 8),

(3. 4) $$\max\ \{\lambda(h, k)\ \| Q_m \|_\infty ;\ m\leqq M(k),\ h\leqq \theta^k\}$$
$$=O(k\theta^{k(2^{-1}-\beta^{-1})})=o(1), \qquad \text{as } k\to+\infty.$$

Hence, we have, by (3. 2),

$$|E(\lambda(h, k), M(k), x(h, k))| \leqq \exp\{-(1+\eta)\ (\log(h+1)+\log\log\theta^k)\},$$

and this implies that

$$\sum_{k=1}^{\infty} \sum_{h\leqq\theta^k} |E(\lambda(h, k), \mathrm{M}(k), x(h, k))| < +\infty.$$

Therefore, by Borel-Cantelli's lemma, for a. e. t, there exists an integer $k_o(t)$ such that $k \geqq k_o(t)$ implies

$$\Big|\sum_{h\leqq\theta^k} a_h \sum_{m=1}^{M(k)} Q_m(ht)\Big|$$

$$\leqq \sum_{h\leqq\theta^k} a_h\ [x(h, k)+\lambda(h, k) \sum_{m=1}^{M(k)} \{Q_m^2(ht)+\lambda(h, k)\ |Q_m(ht)|^3\}/2].$$

On the other hand we have, by (2. 1) and (3. 3),

$$\overline{\lim_{k\to\infty}} \sum_{h\leqq\theta^k} a_h x(h, k)/(\theta^k \log\log\theta^k)^{1/2} \leqq (1+\eta) \sum_{h=1}^{\infty} a_h/2.$$

By (2. 1) and Lemma 3, we have

$$\Big\| \sum_{h\leqq\theta^k} a_h\lambda(h, k) \sum_{m=1}^{M(k)} \{Q_m^2(h\cdot)-\|Q_m\|_2^2\} \Big\|_2$$

$$\leqq \sum_{h\leqq\theta^k} a_h\lambda(h, k) \Big\| \sum_{m=1}^{M(k)} (Q_m^2-\|Q_m\|_2^2) \Big\|_2$$

$$=O(kM(k)^{3\beta-2}\ \theta^{-k})^{1/2}=O(k\theta^{2k(1-\beta^{-1})})^{1/2}, \qquad \text{as } k\to+\infty.$$

Hence, we have

$$\sum_{k=1}^{\infty} \Big\| \sum_{h\leqq\theta^k} a_h\lambda(h, k) \sum_{m=1}^{M(k)} \{Q_m^2(h\cdot)-\|Q_m\|_2^2\} \Big\|_2^2/(\theta^k\log\log\theta^k) < +\infty.$$

By (2. 1), (2. 9) and (3. 3), this shows that, for a. e. t,

$$\sum_{h\leq\theta^{k'}} a_h\lambda(h, k) \sum_{m=1}^{M(k)} Q_m^2(ht)/(\theta^k\log\log\theta^k)^{1/2}$$

$$\sim\sum_{h\leq\theta^k} a_h\lambda(h, k) \sum_{m=1}^{M(k)} \| Q_m \|_2^2/(\theta^k\log\log\theta^k)^{1/2}$$

$$\sim\sum_{h\leq\theta^k} a_h, \qquad \text{as } k\to+\infty,$$

and thus, by (3. 4),

$$\sum_{h\leq\theta^k} a_h\lambda^2(h, k) \sum_{m=1}^{M(k)} | Q_m^3(ht) | / \{\theta^k\log\log\theta^k\}^{1/2}\to 0, \qquad \text{as } k\to+\infty.$$

Further, we have, by (2. 9), (3. 3) and (2.4),

$$\sum_{k=1}^{\infty} \| \sum_{h\leq\theta^k} a_h \sum_{m=1}^{M(k)} \Delta_{q(m)-1}(h\,\cdot) \|_2^2/(\theta^k\log\log\theta^k)<+\infty.$$

The above relations show that

$$\overline{\lim_{k\to\infty}} \sum_{h\leq\theta^k} a_h \sum_{m=1}^{M(k)^\beta} \cos 2\pi n_m ht/(\theta^k\log\log\theta^k)^{1/2}$$

$$\leq (1+\eta) \sum_{h=1}^{\infty} a_h, \qquad \text{a. e. } t.$$

Since η is arbitrary, we have, by (2. 2), (2. 5) and (3. 3),

$$\overline{\lim_{k\to\infty}} \sum_{m\leq\theta^k} f(n_m t)/(\theta^k\log\log\theta^k)^{1/2}\leq \sum_{h=1}^{\infty} a_h, \quad \text{a. e. } t. \tag{3. 5}$$

4. The Maximal Theorems. By (2. 3) we can take, for large k, an increasing sequence of integers $\{m_{j,k}\}$, $0\leq j\leq k$, as follows:

$$p(m_{j,k})\leq\theta^k+\{(\theta^{k+1}-\theta^k)j/k\}<p(m_{j,k}+1). \tag{4. 1}$$

Then we have, for some constant C,

$$\max_{0\leq j\leq k} \{p(m_{j,k}+1)-p(m_{j,k})\}$$
$$=C(p^\alpha(m_{j,k}))=O(\theta^{k\alpha}), \text{ as } k\to+\infty. \tag{4. 2}$$

i. If we put, for $1\leq h\leq\theta^k$ and $k\geq 1$,

$$\begin{cases} \eta(h, k)= [\{\log(h+1)+\log\log\theta^k\}/(\theta^{k+1}-\theta^k)]^{1/2}, \\ y(h, k)=3\{\log(h+1)+\log\log\theta^k\}/\eta(h, k). \end{cases} \tag{4. 3}$$

Then we have, by (4. 2) and (4. 3),

$$\max\ \{\eta(h, k) \| \Delta_m \|_\infty ;\ 1\leq h\leq\theta^k,\ m_{o,k}\leq m\leq m_{k,k}\}$$

$$= O(k^{1/2}\theta^{k(\alpha-1/2)}) = o(1), \qquad \text{as } k\to+\infty$$

By (3. 1), we have, for large k,

$$\exp\{\eta(h,k)\sum_{m=m_{o,k}}^{m_{j,k}}\Delta_m-\eta^2(h,k)\sum_{m=m_{o,k}}^{m_{h,k}}\Delta_m^2\}$$

$$\leqq \exp\{\eta(h,k)\sum_{m=m_{o,k}}^{m_{j,k}}(\Delta_m-\eta(h,k)\Delta_m^2-2\,|\,\eta^2(h,k)\Delta_m^3\,|\,)\}$$

$$\leqq \sum_{m=m_{o,k}}^{m_{j,k}}\{1+2\eta(h,k)\Delta_m(t)\}^{1/2}.$$

Since both sequences $\{\Delta_{2m}(t)\}$ and $\{\Delta_{2m+1}(t)\}$ are multiplicatively orthogonal on the interval (0, 1), we have

$$\int_0^1\exp\{\eta(h,k)\sum_{m=m_{o,k}}^{m_{j,k}}\Delta_m-\eta^2(h,k)\sum_{m=m_{o,k}}^{m_{h,k}}\Delta_m^2\}\,dt$$

$$\leqq\int_0^1\prod_{m=m_{o,k}}^{m_{j,k}}\{1+2\eta(h,k)\Delta_m(t)\}^{1/2}dt$$

$$\leqq\ [\int_0^1\Pi_1\{1+2\eta(h,k)\Delta_{2m}(t)\}\,dt\int_0^1\Pi_2\{1+2\eta(h,k)\Delta_{2m+1}(t)\}\,dt]^{1/2}=1,$$

where Π_1 (or Π_2) is the product over all m such that

$m_{o,k}\leqq 2m\leqq m_{j,k}$ (or $m_{o,k}\leqq 2m+1\leqq m_{j,k}$).

If we put

$$E'(h,k,j)$$

$$=\ \{t\,;\,t\in[0,1]\,,\ \sum_{m=m_{o,k}}^{m_{j,k}}\Delta_m(ht)\geqq y(h,k)+\eta(h,k)\sum_{m=m_{o,k}}^{m_{h,k}}\Delta_m^2(ht)\}\ .$$

Then we have, for any j,

$$|\,E'(h,k,j)\,|\leqq\exp\{-3(\log(h+1)+\log\log\theta^k)\}$$
$$\leqq C\{(h+1)k\}^{-3}, \qquad \text{for some } C>0.$$

Hence, we have

$$(4.\ 4)\qquad \sum_{k=1}^{\infty}\sum_{j=o}^{k}\sum_{h\leqq\theta^k}|\,E'(h,k,j)\,|<+\infty.$$

On the other hand we have, by Lemma 4 and (2. 1),

$$\sum_{k=1}^{\infty}\|\sum_{h\leqq\theta^k}a_h\eta(h,k)\sum_{m=m_{o,k}}^{m_{h,k}}\{\Delta_m^2(h\,\cdot\,)-\|\Delta_m\|_2^2\}\|_2/(\theta^k\log\log\theta^k)^{1/2}$$

$$\leqq\sum_{k=1}^{\infty}\sum_{h\leqq\theta^k}a_h\eta(h,k)\|\sum_{m=m_{o,k}}^{m_{h,k}}(\Delta_m^2-\|\Delta_m\|_2^2)\|_2/(\theta^k\log\log\theta^k)^{1/2}<+\infty.$$

Thus we have, by (2. 1) and (4. 1), for a. e. t,

$$2\sum_{h\leq\theta^k} a_h\eta(h,k)\sum_{m=m_{o,k}}^{m_{k,k}}\Delta_m^2(ht)/(\theta^k\log\log\theta^k)^{1/2}$$

$$\sim 2\sum_{h\leq\theta^k} a_h\eta(h,k)\sum_{m=m_{o,k}}^{m_{k,k}}\|\Delta_m\|_2^2/(\theta^k\log\log\theta^k)^{1/2}$$

$$\sim\sqrt{\theta-1}\sum_{h=1}^{\infty}a_h, \qquad \text{as } k\to+\infty.$$

Further, we have, by (2. 1),

$$\overline{\lim_{k\to\infty}}\sum_{h\leq\theta^k} a_h y(h,k)/(\theta^k\log\log\theta^k)^{1/2}\leqq 3\sqrt{\theta-1}\sum_{h=1}^{\infty}a_h.$$

From the above two relations and (4. 4), we obtain

$$\overline{\lim_{k\to\infty}}\sup_{o\leqq j\leqq k}\sum_{h\leq\theta^k} a_h\sum_{m=m_{o,k}}^{m_{j,k}}\Delta_m(ht)/(\theta^k\log\log\theta^k)^{1/2}$$

$$\leqq 4\sqrt{\theta-1}\sum_{h=1}^{\infty}a_h, \qquad \text{a. e. } t.$$

Therefore, we have, by (2. 2),

$$\text{(4. 5)}\quad \overline{\lim_{k\to\infty}}\sup_{o\leqq j\leqq k}\sum_{m=p(m_{o,k})}^{p(m_{j,k})} f(n_m t)/(\theta^k\log\log\theta^k)^{1/2}$$

$$\leqq 4\sqrt{\theta-1}\sum_{h=1}^{\infty}a_h, \qquad \text{a. e. } t.$$

ii. If we put

$$A(j,k,t)=\sum_{h\leq\theta^k}a_h\sup_{m_{j,k}\leqq m\leqq m_{j+1,k}}\sum_{r=m_{j,k}}^{m}\Delta_r(ht),$$

then we have, by (2. 1)

$$\|A(j,k.\cdot)\|_4\leqq\Big(\sum_{h\leq\theta^k}a_h\Big)\Big\|\sup_{m_{j,k}\leqq m\leqq m_{j+1,k}}\sum_{r=m_{j,k}}^{m}\Delta_r\Big\|_4.$$

By the theorems of trigonometric series (c f. [5] vol. II (4. 4) p. 231 and (4. 24) p. 233) and Lemma 4, we have, for some constants c_1, c_2 and c_3.

$$\Big\|\sup_{m_{j,k}\leqq m\leqq m_{j+1,k}}\sum_{r=m_{j,k}}^{m}\Delta_r\Big\|_4^4\leqq c_1\Big\|\sum_{r=m_{j,k}}^{m_{j+1,k}}\Delta_r\Big\|_4^4$$

$$\leqq c_2\Big\|\sum_{r=m_{j,k}}^{m_{j+1,k}}\Delta_r^2\Big\|_2^2\leqq 2c_2\Big(\Big\|\sum_{r=m_{j,k}}^{m_{j+1,k}}(\Delta_r^2-\|\Delta_r\|_2^2)\Big\|_2^2+\Big(\sum_{r=m_{j,k}}^{m_{j+1,k}}\|\Delta_r\|_2^2\Big)^2\Big)$$

$$\leqq c_3(\theta^{2k}k^{-2}), \qquad \text{for } j=0,1,\cdots,k, \text{ as } k\to+\infty.$$

Therefore, we have

$$\sum_{k=1}^{\infty}\int_0^1 \max_{0\leqq j\leqq k} | A(j, k, t) |^4/(\theta^k \log\log \theta^k)^2 dt$$

$$\leqq \sum_{k=1}^{\infty}\sum_{j=0}^{k}\int_0^1 | A(j, k, t) |^4/(\theta^k \log\log \theta^k)^2 dt < +\infty,$$

and this shows that

$$\lim_{k\to\infty}\max_{0\leqq j\leqq k} | A(j, k, t) | /(\theta^k \log\log \theta^k)^{1/2}=0, \qquad \text{a. e. } t.$$

By (2. 2), we have

$$\lim_{k\to\infty}\max_{0\leqq j\leqq k}\sup_{m_{j,k}\leqq m\leqq m_{j+1,k}}\sum_{r=p(m_{j,k})}^{p(m)} f(n_r t)/(\theta^k \log\log \theta^k)^{1/2}=0, \qquad \text{a. e. } t.$$

By the above relation and (4. 5) and (2. 3), we have

$$\text{(4. 6)}\qquad \overline{\lim_{k\to\infty}}\sup_{\theta^k\leqq m\leqq \theta^{k+1}}\sum_{r=\theta^k}^{m} f(n_r t)/(\theta^k \log\log \theta^k)^{1/2}$$

$$\leqq 4\sqrt{\theta-1}\sum_{h=1}^{\infty} a_h, \qquad \text{a. e. } t.$$

Since we can choose θ as close as 1, by (3. 5) and (4. 6), we can complete the proof of the theorem.

As a result in the opposite direction P. Erdös [2] proved that there exist a sequence $\{n_k\}$ of positive integers with *Hadamard's gap* and a function $f\in L^2(0, 1)$ such that

$$\overline{\lim_{N\to\infty}} N^{-1}\sum_{k=1}^{N} f(n_k t)=+\infty, \qquad \text{a. e. } t.$$

REFERENCES

[1] S. DHOMPONGSA, Uniform laws of the iterated logarithm for Lipschitz classes, *Acta Sci. Math.*, 50(1986), 105-124.

[2] P. ERDÖS, On the strong law of large numbers, *Trans. Amer. Math. Soc.*, 67(1949), 51-56.

[3] S. TAKAHASHI, An asymptotic property of a gap sequence, *Proc. Japan Acad.*, 38(1962), 101-104.

[4] S. TAKAHASHI, On the law of the iterated logarithm for lacunary trigonometric series, *Tôhoku Math. J.*, 24(1972), 319-329.

[5] A. ZYGMUND, *Trigonometric series*, Vol. I and II, Cambridge University Press, (1959).