On the Almost Everywhere Convergence of Rectangular Partial Sums of Multiple Fourier Series

Sci. Rep. Kanazawa Univ.
Vol. 22. No. 2, pp. 163—177
December 1977

On the Almost Everywhere Convergence of Rectangular Partial Sums of Multiple Fourier Series

Michitaka KOJIMA

Department of Mathematics, Faculty of Science, Kanazawa University

(Received October 4, 1977)

Abstract. In this paper we shall give three theorems on the almost everywhere convergence of the rectangular partial sums of L^2-multiple Fourier series. The first is a result about the summation process taken by the individually fixed monotone increasing path. The second is a general result on the lacunary partial sums and the third result is concerned with sufficient conditions on the general rectangular partial sums. All of them are extensions of theorems known in two-dimension to higher dimension.

§0. Introduction.

The main purpose of this paper is to obtain some sufficient conditions ensuring the almost everywhere convergence of the rectangular partial sums of L^2-multiple Fourier series.

Let R^n be the n-dimensional Euclidean space, Z^n be the set of all lattice points in R^n, and T^n be the n-dimensional torus. For any $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$ in R^n, we denote $\alpha x+\beta y=(\alpha x_1+\beta y_1,\dots,\alpha x_n+\beta y_n)$ where $\alpha, \beta \in R^1$, $(x, y)=x_1y_1+\cdots+x_ny_n$ and $|x|=(x_1^2+\cdots+x_n^2)^{\frac{1}{2}}$.

For a function $f\in L^1(T^n)$ we consider the problem of the almost everywhere convergence of the rectangular partial sums of the Fourier series of f,

$$S_{m_1,\dots,m_n}(f, x)=\sum_{|j_1|\leqq m_1}\cdots\sum_{|j_n|\leqq m_n}\hat{f}(j)e^{i(j,x)}$$

where

$$\hat{f}(j)=\frac{1}{(2\pi)^n}\int_{T^n}f(x)e^{-i(j,x)}\,dx \qquad (j\in Z^n)$$

as n-tuples of non-negative integers $(m_1,\dots,m_n)$ tend to infinity.

In the case of $n=1$, the following decisive result was proved by Carleson and Hunt; for any $f\in L^p(T^1)$ $(1<p<\infty)$,

$$\| \sup_{m \geqq 0} | S_m(f) | \|_p \leqq C_p \| f \|_p$$

is valid with some constant C_p independent of f, and so $S_m(f, x)$ converges to $f(x)$ for almost every $x \in T^1$ as $m \to \infty$.

On the other hand in the case of $n \geqq 2$, C.Fefferman [3] proved that there exists a $f \in C(T^n)$ such that

$$\varlimsup_{m_1, \ldots, m_n \to \infty} | S_{m_1, \ldots, m_n}(f, x) | = \infty \qquad \text{for all } x \in \mathbf{T}^n.$$

Hence it is of interest to consider the problems whether $S_{m_1, \ldots, m_n}(f, x)$ converges almost everywhere in T^n or not, if for general $f \in L^p(T^n)$ indices $(m_1, \ldots, m_n)$ run with some restrictions, or if for f satisfying some conditions indices $(m_1, \ldots, m_n)$ run without any restriction.

For the former problem, when indices run with mutual dependence the following results are known.

C.Fefferman's result [3] ; There exist $\delta_1, \delta_2 > 0$ and $f \in C(T^2)$ such that

$$\varlimsup_{\substack{m_1, m_2 \to \infty \\ \delta_1 \leqq \frac{m_1}{m_2} \leqq \delta_2}} | S_{m_1, m_2}(f, x) | = \infty \qquad \text{for all } x \in \mathbf{T}^2.$$

C.Fefferman's result [4] ; For any fixed $\delta_1, \ldots, \delta_n > 0$, if $f \in L^p(T^n)$ $(1 < p < \infty)$ then

$$\| \sup_{m \geqq 0} | S_{\delta_1 m, \ldots, \delta_n m}(f) | \|_p \leqq C_p \| f \|_p$$

is valid with some constant C_p independent of f, and so $S_{\delta_1 m, \ldots, \delta_n m}(f, x)$ converges to $f(x)$ almost everywhere in T^n as $m \to \infty$.

N.R.Tevzadze's result [9] ; For any given two sequences of non-negative integers $\{m_k^{(l)}\}$ $(l = 1, 2)$ increasing to ∞, if $f \in L^2(T^2)$ then $S_{m_k^{(1)}, m_k^{(2)}}(f, x)$ converges to $f(x)$ almost everywhere in T^2 as $k \to \infty$. In §1 we shall prove an extension of this result to $n \geqq 3$ (Theorem 1).

On the other hand when indices run mutually independently, the following result was proved by P.Sjölin [7]. For any given lacunary sequence $\{m_k\}$, if $f \in L^p(T^2)$ $(1 < p < \infty)$ then,

$$\| \sup_{k, m} | S_{m_k, m}(f) | \|_p \leqq C_p \| f \|_p$$

is valid with some constant C_p independent of f, and so $S_{m_k, m}(f, x)$ converges to $f(x)$ almost everywhere in T^2 as $k, m \to \infty$. In §2 we shall prove an extension of this result to $n \geqq 3$ (Theorem 2).

For the latter problem above mentioned, in the theory of 1-dimensional Fourier series the following result is well-known as the theorem of Kolmogoroff-Seliverstoff-Plessner. If $f \in L^2(T^1)$ satisfies the condition

$$\sum_{j=-\infty}^{\infty} |\hat{f}(j)|^2 \log(|j|+2) < \infty ,$$

then $S_m(f, x)$ converges to $f(x)$ almost everywhere in T^1 as $m \to \infty$. Moreover since this condition is equivalent to the condition

$$\int_0^1 \| \Delta_s(f) \|_2^2 \frac{1}{s} ds < \infty$$

where $\Delta_s(f, x) = f(x+s) - f(x)$, therefore if $f \in L^2(T^1)$ satisfies the condition

$$\sup_{|s| \leqq \delta} \| \Delta_s(f) \|_2 = O\left(\frac{1}{(\log \frac{1}{\delta})^{\frac{1}{2}+\varepsilon}}\right) \text{ as } \delta \to +0 \text{ for some } \varepsilon > 0,$$

then $S_m(f, x)$ converges to $f(x)$ almost everywhere in T^1 as $m \to \infty$.

Our main purpose is to get its analogy for $n \geqq 2$.

When $n=2$, there is a following classical result of S. Kaczmarz [5]. If $f \in L^2(T^2)$ satisfies the condition

$$\sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{\infty} |\hat{f}(j_1, j_2)|^2 \log(|j_1|+2) \log(|j_2|+2) < \infty ,$$

then $S_{m_1, m_2}(f, x)$ converges to $f(x)$ almost everywhere in T^2 as $m_1, m_2 \to \infty$.

Up to now, it is known that this result can be improved in the best form by the use of the Carleson and Hunt theorem as follows.

P.Sjölin's result [7] ; If

$$\sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{\infty} |\hat{f}(j_1, j_2)|^2 [\log \min(|j_1|+2, |j_2|+2)]^2 < \infty$$

then, $S_{m_1, m_2}(f, x)$ converges to $f(x)$ almost everywhere in T^2 as $m_1, m_2 \to \infty$

E.M.Nikishin's result [6] ; The factor $[\log \min(|j_1|+2, |j_2|+2)]^2$ in the above result cannot be more improved.

Some sufficient conditions on the modulus of continuity of a function are known.

P. Sjölin's result [8] and M. Bakhbukh's result [1] ; The condition

$$\sum_{j_1=-\infty}^{\infty} \sum_{j_n=-\infty}^{\infty} |\hat{f}(j_1, j_2)|^2 [\log \min(|j_1|+2, |j_2|+2)]^2 < \infty$$

is equivalent to the condition

$$\int_{|s| \leqq 1} \| \Delta_s(f) \|_2^2 \frac{1}{|s|^2} \log \frac{1}{|s|} ds < \infty$$

where $\Delta_{s_1, s_2}(f, x) = f(x_1+s_1, x_2+s_2) - f(x_1+s_1, x_2) - f(x_1, x_2+s_2) + f(x_1, x_2)$. So if

$$\sup_{|s| \leqq \delta} \| \Delta_s(f) \|_2 = O\left(\frac{1}{(\log \frac{1}{\delta})^{1+\varepsilon}}\right) \text{ as } \delta \to +0 \text{ for some } \varepsilon > 0,$$

then $S_{m_1, m_2}(f, x)$ converges to $f(x)$ almost everywhere in T^2 as $m_1, m_2 \to \infty$.

M. Bakhbukh and E.M.Nikishin's result [2] ; In the above result, it is impossible to let $\varepsilon = 0$.

In §3 We shall show that for $n\geqq 3$, if $f\in L^2(T^n)$ satisfies the condition

$$\sum_{j_1=-\infty}^{\infty}\cdots\sum_{j_n=-\infty}^{\infty}|\hat{f}(j_1,\cdots,j_n)|^2\log(|j_1|+2)\cdots\log(|j_n|+2)\quad<\quad\infty$$

then $S_{m_1,\dots,m_n}(f,\ x)$ converges to $f(x)$ almost everywhere in T^n as $m_1,\cdots,m_n\to\infty$ (Theorem 3). This assertion is given in the synthetic monograph of L.V.Zhizhiashvili [10]. However for the proof there is nothing to be referred to. Recently this assertion is reported also by J.Chen and N.Shieh in Notice Amer. Math. Soc., 24(1977), A-241.*) Our proof is analogous to the case of $n=2$, and to do it we may use some result from Theorem 2. However Theorem 2 needs not necessarily for our proof because the used fact is valid easily under our conditions, but it seems to be of interest in itself. Moreover we shall deduce from Theorem 3 some sufficient conditions on the modulus of continuity of a function for the almost everywhere convergence given without proof in the paper of L.V.Zhizhiashvili [11].

§1. On the almost everywhere convergence of certain partial sum.

In this section we shall prove the following extension of the theorem by N.R.Tevzadze [9] to general $n\geqq 2$.

THEOREM 1. *If $f\in L^2(T^n)$, then for each given n-sequences of non-negative integers $\{m_k{}^{(l)}\}_{k=0}^{\infty}$ $(l=1,\cdots,n)$ increasing to ∞,*

$$\|\sup_{k\geqq 0}|S_{m_k^{(1)},\dots,m_k^{(n)}}(f)|\|_2\quad\leqq\quad C\ \|f\|_2$$

is valid with some constant C independent of f, and $S_{m_k^{(1)},\dots,m_k^{(n)}}(f,\ x)$ converges to $f(x)$ almost everywhere in T^n as $k\to\infty$.

PROOF. We shall use the same methods with the case of $n=2$. For any given sequences $\{m_k{}^{(l)}\}_{k=0}^{\infty}$ with $m_0{}^{(l)}=0$, $m_{-1}^{(l)}=-1$ $(l=1,\cdots,n)$, if we set

$$A_k=\ \{j=(j_1,\cdots,j_n)\in Z^n\,;\,|j_\nu|\leqq m_k{}^{(\nu)}\ (\nu=1,\cdots,n)\}$$

then we can devide A_k as following for each $k=1,2,\dots$. For each $l=1,\dots,n$ and for any given $j_l\in Z^1$, let $\lambda=\lambda_l\geqq 0$ be such the integer as $m_{\lambda-1}^{(l)}<|j_l|\leqq m_\lambda^{(l)}$ and let

$$\alpha_\nu^{(l)}(j_l)=\begin{cases} m_{\lambda-1}^{(\nu)} & \text{for}\quad 1\leqq\nu\leqq l-1\\ m_\lambda{}^{(\nu)} & \text{for}\quad l+1\leqq\nu\leqq n\end{cases}$$

If we set

$$A_k^{(l)}=\ \{j=(j_1,\cdots,j_n)\in Z^n\,;\ |j_\nu|\leqq\alpha_\nu^{(l)}(j_l)\ (\nu=1,\dots,n)\}$$

where $\alpha_l^{(l)}(j_l)\ =\ m_k^{(l)}$, then we obtain that

*) This work is performed independently of them.

(i) $A_k^{(l)} \cap A_k^{(l')} = \phi$ if $l \neq l'$

(ii) $A_k = \bigcup_{l=1}^{n} A_k^{(l)}$

(i) can be shown as following. We suppose that there exists a $j=(j_1,\dots,j_n)\epsilon A_k^{(l)} \cap A_k^{(l')}$ $(l<l')$. Then $j \epsilon A_k^{(l)}$ implies $|j_{l'}| \leqq \alpha_{l'}^{(l)}(j_l) = m_{\lambda_l}^{(l')}$ and so we have $\lambda_{l'} \leqq \lambda_l$. Similarly $j \epsilon A_k^{(l')}$ implies $|j_l| \leqq \alpha_l^{(l')}(j_{l'}) = m_{\lambda_{l'}-1}$ and so we have $\lambda_l < \lambda_{l'}$. Therefore we get a contradiction and (i) is valid. To show (ii) if we give any $j=(j_1,\dots,j_n)\epsilon A_k$, then when $\lambda_l \leqq \lambda_1$ for all $l=1,\dots,n$ we have $j \epsilon A_k^{(1)}$, and when there exists some l_0 $(2 \leqq l_0 \leqq n)$ such that $\lambda_l \leqq \lambda_{l_0}$ for all $l=1,\dots,n$ and $\lambda_l < \lambda_{l_0}$ for $l=1,\dots,l_0-1$ we have $j\epsilon A_k^{(l_0)}$. So we get (ii).

Then we can write as

$$S_{m_k^{(1)},\dots,m_k^{(n)}}(f, x) = \sum_{j\epsilon A_k} \hat{f}(j)e^{i(j,x)} = \sum_{l=1}^{n} \sum_{j\epsilon A_k^{(l)}} \hat{f}(j)e^{i(j,x)}$$
$$= \sum_{l=1}^{n} S_k^{(l)}(f, x) \text{ say.}$$

In order to prove the theorem it is enough to consider the only $S_k^{(1)}(f, x)$ since the rest terms can be treated similarly. Now

$$S_k^{(1)}(f, x) = \sum_{|j_1| \leqq m_k^{(1)}} \Big(\sum_{\substack{|j_\nu| \leqq \alpha_\nu^{(1)}(j_1) \\ \nu=2,\dots,n}} \hat{f}(j_1, j')e^{i(j',x')}\Big)e^{ij_1x_1}$$
$$= \sum_{|j_1| \leqq m_k^{(1)}} T_{j_1}(x')e^{ij_1x_1} \text{ say,}$$

where $j'=(j_2,\dots,j_n)$ and $x'=(x_2,\dots,x_n)$. Since

$$\int_{T^{n-1}} \sum_{|j_1| \leqq m_k^{(1)}} |T_{j_1}(x')|^2 dx' = \sum_{|j_1| \leqq m_k^{(1)}} \int_{T^{n-1}} \Big| \sum_{\substack{|j_\nu| \leqq \alpha_\nu^{(1)}(j_1) \\ \nu=2,\dots,n}} \hat{f}(j_1, j')e^{i(j',x')}\Big|^2 dx'$$
$$= (2\pi)^{n-1} \sum_{|j_1| \leqq m_k^{(1)}} \sum_{\substack{|j_\nu| \leqq \alpha_\nu^{(1)}(j_1) \\ \nu=2,\dots,n}} |\hat{f}(j_1, j')|^2 \leqq (2\pi)^{n-1} \sum_{j\epsilon Z^n} |\hat{f}(j)|^2 < \infty$$

so we have

$$\int_{T^{n-1}} \sum_{j_1=-\infty}^{\infty} |T_{j_1}(x')|^2 dx' \leqq (2\pi)^{n-1} \sum_{j\epsilon Z^n} |\hat{f}(j)|^2 < \infty .$$

Therefore, for almost every $x'\epsilon T^{n-1}$, it follow that $\sum_{j_1=-\infty}^{\infty} |T_{j_1}(x')|^2 < \infty$ and so, by the Riesz and Fischer Theorem and by the Carleson and Hunt result, there exists a function $g_{x'}(x_1)\epsilon L^2(T^1)$ such that $\sum_{j_1=-\infty}^{\infty} T_{j_1}(x')e^{ij_1x_1}$ is the Fourier series of $g_{x'}$, converges to $g_{x'}(x_1)$ for almost every $x_1 \epsilon T^1$ and

$$\| \sup_{k\geqq 0} | \sum_{|j_1| \leqq m_k^{(1)}} T_{j_1}(x')e^{ij_1x_1} | \|_{L^2(T^1)}^2 \leqq C \| g_{x'} \|_{L^2(T^1)}^2 = C \sum_{j_1=-\infty}^{\infty} |T_{j_1}(x')|^2$$

is valid. Hence we obtain that $S_k^{(1)}(f, x)$ converges almost everywhere in T^n as $k\to\infty$ by Fubini's Theorem and that

$$\| \sup_{k \geqq 0} | S_k^{(1)}(f) | \|_{L^2(T^n)}^2 \leqq C \int_{T^{n-1}} \sum_{j_1=-\infty}^{\infty} | T_{j_1}(x') |^2 dx'$$

$$\leqq C' \sum_{j \in Z^n} | \hat{f}(j) |^2 = C'' \| f \|_{L^2(T^n)}^2$$

So the theorem is proved.

§2. On the almost everywhere convergence of lacunary partial sum.

In this section we shall prove the following extension of a P.Sjölin's result [7] to $n \geqq 3$.

THEOREM 2. (i) *If* $f \in L^p(T^n)$ $(1<p<\infty)$, *then for any given* $(n-1)$-*lacunary sequences* $\{m_k^{(j)}\}$ $(j=1,\dots,n-1)$,

$$\| \sup_{k_1,\dots,k_{n-1},m} | S_{m_{k_1}^{(1)},\dots,m_{k_{n-1}}^{(n-1)},m}(f) | \|_p \leqq C_p \| f \|_p$$

is valid with some constant C_p *independent of* f, *and so* $S_{m_{k_1}^{(1)},\dots,m_{k_{n-1}}^{(n-1)},m}(f, x)$ *converges to* $f(x)$ *almost everywhere in* T^n *as* $k_1,\dots,k_{n-1},m \to \infty$.

(ii) *The result* (i) *cannot be more improved in the following sense; for any given* $m'=(m_3,\dots,m_n) \to \infty$, *there exists a* $f \in C(T^n)$ *such that*

$$\overline{\lim_{m_1, m_2 m' \to \infty}} | S_{m_1, m_2, m'}(f, x) | = \infty \quad \textit{for all } x \in T^n.$$

PROOF. (ii) follows easily by the use of the result of C. Fefferman [3].

To prove (i), we may restrict ourself to the case of $n=3$. For the case of $n \geqq 4$, it can be shown by induction. We shall use the result and the method of P. Sjölin [7]. Simplifing notations, we write that

$$\{m_k\} = \{m_{k_1}^{(1)}\}$$

$$\{m'\} = \{(m_{k_2}^{(2)}, m)\} .$$

Then, $\{m_k\}$ is a lacunary sequence and for $\{m'\}$ we can use the result of P. Sjölin [7].

For any $g \in L^1(T^1)$, we define

$$\Delta_k(g, x_1) = \begin{cases} S_0(g, x_1) & \text{for } k=0 \\ S_{m_k}(g, x_1) - S_{m_{k-1}}(g, x_1) & \text{for } k=1,2,3,\dots \end{cases}$$

where $m_0=0$, $m_1=1$. Then there are two functions $f'(x_1, x')$, $f''(x_1, x') \in L^p(T^3)$, where $x'=(x_2, x_3) \in T^2$, such that

$$f'(x_1, x') \sim \sum_{k=0}^{\infty} \Delta_{2k+1}(f(\cdot, x'); x_1)$$

$$f''(x_1, x') \sim \sum_{k=0}^{\infty} \Delta_{2k}(f(\cdot, x'); x_1).$$

$$f(x_1, x')=f'(x_1, x')+f''(x_1, x') \qquad \text{a.e. } (x_1, x')\in T^3$$

$$\| f' \|_p \leqq C_p \| f \|_p, \quad \| f'' \|_p \leqq C_p \| f \|_p$$

Here we used the theorem (4.11) from A. Zygmund [12], vol. 2, p. 231. If we put

$$G_{m'}(x_1, x') = S_{m'}(f(x_1, \cdot\,); x')$$

$$G'_{m'}(x_1, x') = S_{m'}(f'(x_1, \cdot\,); x')$$

$$G''_{m'}(x_1, x') = S_{m'}(f''(x_1, \cdot\,); x')$$

then we have

$$S_{m_k, m'}(f; x_1, x') = S_{m_k}(G_{m'}(\cdot\,, x'); x_1)$$

$$= S_{m_k}(G'_{m'}(\cdot\,, x'); x_1)+S_{m_k}(G''_{m'}(\cdot\,, x'); x_1).$$

Since for each fixed $x'\in T^2$ the Fourier series $S(G'_{m'}(\cdot\,, x'); x_1)$ of x_1-variable function $G'_{m'}(x_1, x')$ has infinitely many gaps $m_{2k-1}< |\mu| \leqq m_{2k}$, with $\sup_k \frac{m_{2k}}{m_{2k-1}}>1$, therefore we have by the inequality (1.20) from A.Zygmund [12], vol. 2, p. 164,

$$\sup_k | S_{m_k}(G'_{m'}(\cdot\,, x'); x_1) | \leqq C \sup_l \int_{T^1} K_l(x_1-t) \, | G'_{m'} (t, x') | \, dt$$

$$= C \sup_l \int_{T^1} K_l(x_1-t) \, | S_{m'}(f'(t, \cdot\,); x') | \, dt$$

$$\leqq C \sup_l \int_{T^1} K_l(x_1-t) \sup_{m'} | S_{m'}(f'(t, \cdot\,); x') | \, dt = CP^*(f'; x_1, x') \text{ say.}$$

So we have

$$\sup_{k,m'} | S_{m_k}(G'_{m'}(\cdot\,, x'); x_1) | \quad \leqq \quad CP^*(f'; x_1, x')$$

and by the use of the result of P. Sjölin [7] and the theorem (7.8) from A. Zygmund [12], vol. 1, p. 156,

$$\| P^*(f') \|_p \quad \leqq \quad C_p \| f' \|_p \quad \leqq \quad C'_p \| f \|_p$$

Similarily we have

$$\sup_{k,m'} | S_{m_k}(G''_{m'}(\cdot\,, x'); x_1) | \quad \leqq \quad CP^*(f''; x_1, x')$$

and

$$\| P^*(f'') \|_p \quad \leqq \quad C_p \| f'' \|_p \quad \leqq \quad C''_p \| f \|_p$$

Hence we get

$$\| \sup_{k,m'} | S_{m_k, m'}(f) | \, \|_p \quad \leqq \quad C_p \| f \|_p.$$

§3. On sufficient conditions for the almost everywhere convergence of rectangular partial sum.

In this section we shall prove the following theorem.

THEOREM 3. *If $f \in L^2(T^n)$ satisfies the condition* $\sum_{j_1=-\infty}^{\infty} \cdots \sum_{j_n=-\infty}^{\infty} |\hat{f}(j_1,\ldots,j_n)|^2 \log(|j_1|+2)\cdots\log(|j_n|+2) < \infty$ *then* $S_{m_1,\ldots,m_n}(f, x)$ *converges to* $f(x)$ *almost everywhere in* T^n *as* $m_1,\ldots,m_n \to \infty$.

We need the following lemmas for the proof.

LEMMA 1. *Let $g \in L^2(T^n)$ be given.*

(i) *If we put*

$$S^*(g, x) = \sup_{m_1,\cdots,m_n \geqq 2} \frac{|S_{m_1,\cdots,m_n}(g, x)|}{\sqrt{\log m_1}\cdots\sqrt{\log m_n}}$$

then we have $\|S^*(g)\|_2 \leqq C\|g\|_2$ *and so*

$$S^*(g, x) = O(\sqrt{\log m_1}\cdots\sqrt{\log m_n}) \ a.e.$$

(ii) *If we put*

$$\sigma^*(g, x) = \sup_{\substack{m_1,\cdots,m_{k_0}\geqq 2 \\ l_{k_0+1},\cdots,l_n \geqq 2}} \frac{1}{\sqrt{\log m_1}\cdots\sqrt{\log m_{k_0}}} \left| \frac{1}{(l_{k_0+1}+1)\cdots(l_n+1)} \sum_{\nu_{k_0+1}=0}^{l_{k_0+1}} \cdots \sum_{\nu_n=0}^{l_n} S_{m_1,\ldots,m_{k_0},\nu_{k_0+1},\ldots,\nu_n}(g, x) \right|$$

then we have $\|\sigma^*(g)\|_2 \leqq C\|g\|_2$ *and so*

$$\sum_{\nu_{k_0+1}=0}^{l_{k_0+1}} \cdots \sum_{\nu_n=0}^{l_n} S_{m_1,\ldots,m_{k_0},\nu_{k_0+1},\ldots,\nu_n}(g, x) = O(l_{k_0+1}\cdots l_n\sqrt{\log m_1}\cdots\sqrt{\log m_{k_0}}) \quad a.e.$$

(i) is shown in A. Zygmund [12], vol. 2, p. 167 when $n=1$, and was proved by S. Kaczmarz [5] when $n=2$. For $n \geqq 3$ it can be proved dy the same method. (ii) can be easily shown by the use of (i) and the theorem (7.8) from A. Zygmund [12], vol. 1, p. 156.

LEMMA 2. *If $f \in L^2(T^n)$ satisfies the same condition as in Theorem 3, then* $S_{2^{M_1^2},\ldots,2^{M_n^2}}(f,x)$ *converges to* $f(x)$ *almost everywhere in* T^n *as* $M_1,\ldots,M_n \to \infty$.

This conclusion follows immedately from Theorem 2, under the only assumption $f \in L^2(T^n)$. But we shall prove this result easily under our conditions. We may write

$$S_{2^{M_1^2},\ldots,2^{M_n^2}}(f, x) = \sum_{|j_1| \leqq 2^{M_1^2}} \cdots \sum_{|j_n| \leqq 2^{M_n^2}} \hat{f}(j)e^{i(j,x)} = \Sigma' + \Sigma''$$

where Σ' or Σ'' is the summation respectively over $j=(j_1,\dots,j_n)$ with $|j_k|\geqq 1$ for all k or over $j=(j_1,\dots,j_n)$ with $j_k=0$ for some k. Now

$$\Sigma'=\sum_{\mu_1=0}^{M_1}\cdots\sum_{\mu_n=0}^{M_n} u_{\mu_1,\dots,\mu_n}(x)$$

where

$$u_{\mu_1,\dots,\mu_n}(x)=\sum_{2^{(\mu_1-1)^2}<|j_1|\leqq 2^{\mu_1^2}}\cdots\sum_{2^{(\mu_n-1)^2}<|j_n|\leqq 2^{\mu_n^2}}\hat{f}(j)\,e^{i(j,x)}\ .$$

Then,

$$\int_{T^n}|u_{\mu_1,\dots,\mu_n}(x)|\,dx\leqq(2\pi)^{\frac{n}{2}}\left[\int_{T^n}|u_{\mu_1,\dots,\mu_n}(x)|^2\,dx\right]^{\frac{1}{2}}$$

$$=(2\pi)^n\left[\sum_{2^{(\mu_1-1)^2}<|j_1|\leqq 2^{\mu_1^2}}\cdots\sum_{2^{(\mu_n-1)^2}<|j_n|\leqq 2^{\mu_n^2}}|\hat{f}(j)|^2\right]^{\frac{1}{2}}$$

Therefore

$$\int_{T^n}\sum_{\mu_1=0}^{\infty}\cdots\sum_{\mu_n=0}^{\infty}|u_{\mu_1,\dots,\mu_n}(x)|\,dx$$

$$\leqq(2\pi)^n\left[\sum_{\mu_1=0}^{\infty}\cdots\sum_{\mu_n=0}^{\infty}\frac{1}{(\mu_1+1)^2\cdots(\mu_n+1)^2}\right]^{\frac{1}{2}}$$

$$\left[\sum_{\mu_1=0}^{\infty}\cdots\sum_{\mu_n=0}^{\infty}(\mu_1+1)^2\cdots(\mu_n+1)^2\sum_{2^{(\mu_1-1)^2}<|j_1|\leqq 2^{\mu_1^2}}\cdots\sum_{2^{(\mu_n-1)^2}<|j_n|\leqq 2^{\mu_n^2}}|\hat{f}(j)|^2\right]^{\frac{1}{2}}$$

$$\leqq A\left[\sum_{j_1=-\infty}^{\infty}\cdots\sum_{j_n=-\infty}^{\infty}|\hat{f}(j)|^2\log(|j_1|+2)\cdots\log(|j_n|+2)\right]^{\frac{1}{2}}<\infty$$

So Σ' is convergent a.e. Similarly but more simply Σ'' converges a.e. and thus the lemma is proved.

PROOF OF THEOREM 3. We shall prove it in the case of $n=3$. For $n\geqq 4$ we can do similarly. For any given large integers m_1, m_2, m_3, let M_k $(k=1,2,3)$ be such integers that $2^{M_k^2}\leqq m_k<2^{(M_k+1)^2}$. Then $M_k^2\sim\log m_k$. By Theorem 2 since the sequence $\{2^{M^2}\}_{M=1}^{\infty}$ is lacunary, or by Lemma 2, we see that $S_{2^{M_1^2},2^{M_2^2},2^{M_3^2}}(f,x)$ converges to $f(x)$ almost everywhere in T^3 as M_1, M_2, $M_3\to\infty$. So in order to prove Theorem 3, abbreviating simply as $S_{m_1,m_2,m_3}=S_{m_1,m_2,m_3}(f,x)$ and writing as

$$S_{m_1,m_2,m_3}=(S_{m_1,m_2,m_3}-S_{2^{M_1^2},m_2,m_3})+(S_{2^{M_1^2},m_2,m_3}-S_{2^{M_1^2},2^{M_2^2},m_3})$$

$$+(S_{2^{M_1^2},2^{M_2^2},m_3}-S_{2^{M_1^2},2^{M_2^2},2^{M_3^2}})+S_{2^{M_1^2},2^{M_2^2},2^{M_3^2}}\ ,$$

we need only to show that the first term in the riget hand side converges to 0 almost everywhere in T^3 as m_1, m_2, $m_3\to\infty$, since the rest terms can be treated similarly.

By our assumption there exists a sequence $\{p_j\}$ with $p_j>0$, $p_j=p_{-j}$, and increasing to $+\infty$ as slowly as we like as $j\to\infty$, such that

$$\sum_{j_1=-\infty}^{\infty}\sum_{j_2=-\infty}^{\infty}\sum_{j_3=-\infty}^{\infty}|\hat{f}(j)|^2\log(|j_1|+2)\log(|j_2|+2)\log(|j_3|+2)p_{j_1}p_{j_2}p_{j_3}<\infty$$

So there exists a function $g \in L^2(T^3)$ of which Fourier series is

$$g \sim \sum_{j \in Z^3} \hat{f}(j) \sqrt{p_{j_1} \log(|j_1|+2)} \sqrt{p_{j_2} \log(|j_2|+2)} \sqrt{p_{j_3} \log(|j_3|+2)}\, e^{i(j,x)}.$$

If we put

$$b_k = \frac{1}{\sqrt{p_k \log(|k|+2)}} \qquad (k=0, \pm 1, \pm 2, \ldots)$$

then, $b_k = b_{-k}$, b_k is decreasing to 0 as $k \to \infty$, and since we can take that the sequence $\{b_k\}_{k=0}^{\infty}$ is convex by the definition of $\{p_j\}$, so we may assume that $k \Delta b_k \to 0$ as $k \to \infty$, and $\sum_{k=1}^{\infty} k \Delta^2 b_k < \infty$, where $\Delta b_k = b_k - b_{k+1}$, and $\Delta^2 b_k = \Delta b_k - \Delta b_{k+1}$.

Now we can write

$$\begin{aligned} S_{m_1,m_2,m_3} - S_{2M_1^2, m_2, m_3} &= \sum_{2M_1^2 < |j_1| \leqq m_1} \sum_{|j_2| \leqq m_2} \sum_{|j_3| \leqq m_3} \hat{f}(j) e^{i(j,x)} \\ &= \sum_{|j_2| \leqq m_2} \sum_{|j_3| \leqq m_3} A_{j_2,j_3} b_{j_2} b_{j_3} \end{aligned}$$

where

$$A_{j_2,j_3} = \sum_{2M_1^2 < |j_1| \leqq m_1} \hat{f}(j) \frac{1}{b_{j_2} b_{j_3}} e^{i(j,x)}.$$

Then, putting

$$B_{\mu_2,\mu_3} = \sum_{|j_2| \leqq \mu_2} \sum_{|j_3| \leqq \mu_3} A_{j_2,j_3}$$

we get by the use of Abel transformation

$$\begin{aligned} & S_{m_1,m_2,m_3} - S_{2M_1^2, m_2, m_3} \\ (1) \qquad & = \sum_{\mu_2=0}^{m_2} \sum_{\mu_3=0}^{m_3} B_{\mu_2,\mu_3} \Delta b_{\mu_2} \Delta b_{\mu_3} + \sum_{\mu_2=0}^{m_2} B_{\mu_2,m_3} \Delta b_{\mu_2} \cdot b_{m_3+1} \\ & \quad + \sum_{\mu_3=0}^{m_3} B_{m_2,\mu_3} \Delta b_{\mu_3} \cdot b_{m_2+1} + B_{m_2,m_3} b_{m_2+1} b_{m_3+1} . \end{aligned}$$

Here, if we let

$$T_{\mu_1,\mu_2,\mu_3} = \sum_{|j_1| \leqq \mu_1} \sum_{|j_2| \leqq \mu_2} \sum_{|j_3| \leqq \mu_3} \hat{f}(j) \frac{1}{b_{j_1} b_{j_2} b_{j_3}} e^{i(j,x)} = S_{\mu_1,\mu_2,\mu_3}(g, x)$$

then by the Abel transformation we have

$$B_{\mu_2,\mu_3} = \sum_{\mu_1 = 2M_1^2+1}^{m_1} T_{\mu_1,\mu_2,\mu_3} \Delta b_{\mu_1} + T_{m_1,\mu_2,\mu_3} b_{m_1+1} - T_{2M_1^2, \mu_2,\mu_3} b_{2M_1^2+1} ,$$

and we shall estimate (1) with this expression.

For the last term in (1), we know that from Lemma 1 (i),

$$T_{\mu_1,\mu_2,\mu_3} = O(\sqrt{\log \mu_1} \sqrt{\log \mu_2} \sqrt{\log \mu_3}) \quad \text{a.e.}$$

and so we get

$$B_{m_2,m_3}\, b_{m_2+1}\, b_{m_3+1}$$
$$=O(b_{m_2}\, b_{m_3} \sum_{\mu_1=2^{M_1^2}+1}^{m_1} \sqrt{\log \mu_1}\, \sqrt{\log m_2}\, \sqrt{\log m_3}\, \Delta b_{\mu_1})$$
$$+O(b_{m_1}\, b_{m_2}\, b_{m_3} \sqrt{\log m_1}\, \sqrt{\log m_2}\, \sqrt{\log m_3})$$
$$=O(b_{m_1}\, b_{m_2}\, b_{m_3} \sqrt{\log m_1}\, \sqrt{\log m_2}\, \sqrt{\log m_3})=O(\frac{1}{\sqrt{p_{m_1}}\sqrt{p_{m_2}}\sqrt{p_{m_3}}})=o(1) \text{ a.e.}$$

For the second term in (1), we may write

$$\sum_{\mu_2=0}^{m_2} B_{\mu_2 m_3}\, \Delta b_{\mu_2} \cdot b_{m_3+1}$$
$$=b_{m_3+1} \sum_{\mu_1=2^{M_1^2}+1}^{m_1} \sum_{\mu_2=0}^{m_2} T_{\mu_1,\mu_2,m_3}\, \Delta b_{\mu_2}\, \Delta b_{\mu_1}$$
$$+b_{m_1+1} b_{m_3+1} \sum_{\mu_2=0}^{m_2} T_{m_1,\mu_2,m_3}\, \Delta b_{\mu_2} - b_{2^{M_1^2}+1} b_{m_3+1} \sum_{\mu_2=0}^{m_2} T_{2^{M_1^2},\, \mu_2,m_3} \Delta b_{\mu_2}$$

Since, putting

$$\sigma_l^{(\mu_1,\mu_3)}=\sum_{\mu_2=0}^{l} T_{\mu_1,\mu_2,\mu_3}=\sum_{\mu_2=0}^{l} S_{\mu_1,\mu_2,\mu_3}(g,\, x)$$

we get from Lemma 1 (ii),

$$\sigma_l^{(\mu_1,\mu_3)}=O(l\sqrt{\log \mu_1}\, \sqrt{\log \mu_3}) \quad \text{a.e.}$$

so it follow that

$$\sum_{\mu_2=0}^{m_2} T_{\mu_1,\mu_2,\mu_3} \Delta b_{\mu_2}=\sum_{\mu_2=0}^{m_2} \sigma_{\mu_2}^{(\mu_1,\mu_3)} \Delta^2 b_{\mu_2}+\sigma_{m_2}^{(\mu_1,\mu_3)} \Delta b_{m_2+1}$$
$$=O(\sqrt{\log \mu_1}\, \sqrt{\log \mu_3} \sum_{\mu_2=0}^{m_2} \mu_2 \Delta^2 b_{\mu_2})+O(\sqrt{\log \mu_1}\, \sqrt{\log \mu_3}\, m_2\, \Delta b_{m_2})$$
$$=O(\sqrt{\log \mu_1}\, \sqrt{\log \mu_3}) \quad \text{a.e.}$$

Hence we get

the second term in (1)

$$=O(b_{m_3} \sqrt{\log m_3} \sum_{\mu_1=2^{M_1^2}+1}^{m_1} \sqrt{\log \mu_1}\, \Delta b_{\mu_1})+O(b_{m_1}\, b_{m_3} \sqrt{\log m_1}\, \sqrt{\log m_3})$$
$$=O(b_{m_1} b_{m_3} \sqrt{\log m_1}\, \sqrt{\log m_3})=O(\frac{1}{\sqrt{p_{m_1}}\, \sqrt{p_{m_3}}})=o(1) \qquad \text{a.e.}$$

Similary,

$$\text{the third term in (1)}=O(\frac{1}{\sqrt{p_{m_1}}\, \sqrt{p_{m_2}}})=o(1) \qquad \text{a.e.}$$

For the first term in (1), we can write

$$\sum_{\mu_2=0}^{m_2} \sum_{\mu_3=0}^{m_3} B_{\mu_2,\mu_3}\, \Delta b_{\mu_2}\, \Delta b_{\mu_3}$$

(2) $$=\sum_{\mu_1=2^{M_1^2}+1}^{m_1} \sum_{\mu_2=0}^{m_2} \sum_{\mu_3=0}^{m_3} T_{\mu_1,\mu_2,\mu_3} \Delta b_{\mu_1} \Delta b_{\mu_2} \Delta b_{\mu_3}$$

$$+\sum_{\mu_2=0}^{m_2}\sum_{\mu_3=0}^{m_3} T_{m_1,\mu_2,\mu_3}\Delta b_{\mu_2}\Delta b_{\mu_3}\cdot b_{m_1+1}$$
$$-\sum_{\mu_2=0}^{m_2}\sum_{\mu_3=0}^{m_3} T_{2M_1^2,\mu_2,\mu_3}\Delta b_{\mu_2}\Delta b_{\mu_3}\cdot b_{2M_1^2+1}\ .$$

To estimate the first term in (2), we let

$$U_{\mu_1,\mu_2,\mu_3}=\sum_{\nu_1=0}^{\mu_1}\sum_{\nu_2=0}^{\mu_2}\sum_{\nu_3=0}^{\mu_3} T_{\nu_1,\nu_2,\nu_3}=\sum_{\nu_1=0}^{\mu_1}\sum_{\nu_2=0}^{\mu_2}\sum_{\nu_3=0}^{\mu_3} S_{\nu_1,\nu_2,\nu_3}(g,x)$$
$$U_{\mu_1,\mu_2}=\sum_{\nu_1=0}^{\mu_1}\sum_{\nu_2=0}^{\mu_2}\sum_{\mu_3=0}^{m_3} T_{\nu_1,\nu_2,\mu_3}\Delta b_{\mu_3}$$
$$U_{\mu_1}=\sum_{\nu_1=0}^{\mu_1}\sum_{\mu_2=0}^{m_2}\sum_{\mu_3=0}^{m_3} T_{\nu_1,\mu_2,\mu_3}\Delta b_{\mu_2}\Delta b_{\mu_3}\ .$$

Then, by the Abel transformation, we have

$$\text{the first term in (2)}=\sum_{\mu_1=2^{M_1^2}+1}^{m_1} U_{\mu_1}\Delta^2 b_{\mu_1}+U_{m_1}\Delta b_{m_1+1}-U_{2^{M_1^2}}\Delta b_{2^{M_1^2}+1}$$

Since now we know that from Lemma 1 (ii)

$$U_{\mu_1,\mu_2,\mu_3}=O(\mu_1\mu_2\mu_3)\qquad \text{a.e.,}$$

so we get

$$\begin{aligned}U_{\mu_1,\mu_2} &=\sum_{\mu_3=0}^{m_3} U_{\mu_1,\mu_2,\mu_3}\Delta^2 b_{\mu_3}+U_{\mu_1,\mu_2,m_3}\Delta b_{m_3+1}\\ &=O(\mu_1\ \mu_2\sum_{\mu_3=0}^{m_3}\mu_3\Delta^2 b_{\mu_3})+O(\mu_1\mu_2 m_3\Delta b_{m_3})=O(\mu_1\mu_2)\quad \text{a.e.}\\ U_{\mu_1} &=\sum_{\mu_2=0}^{m_2} U_{\mu_1,\mu_2}\Delta^2 b_{\mu_2}+U_{\mu_1,m_2}\Delta b_{m_2+1}\\ &=O(\mu_1\sum_{\mu_2=0}^{m_2}\mu_2\Delta^2 b_{\mu_2})+O(\mu_1 m_2\Delta b_{m_2})=O(\mu_1)\quad \text{a.e.}\end{aligned}$$

and

the first term in (2)

$$=O(\sum_{\mu_1=2^{M_1^2}+1}^{m_1}\mu_1\Delta^2 b_{\mu_1})+O(m_1\Delta b_{m_1})+O(2^{M_1^2}\Delta b_{2^{M_1^2}})=o(1)\quad \text{a.e.}$$

For the second term in (2), we let similarly

$$V_{\mu_2,\mu_3}=\sum_{\nu_2=0}^{\mu_2}\sum_{\nu_3=0}^{\mu_3} T_{m_1,\nu_2,\nu_3}=\sum_{\nu_2=0}^{\mu_2}\sum_{\nu_3=0}^{\mu_3} S_{m_1,\nu_2,\nu_3}(g,x)$$
$$V_{\mu_2}=\sum_{\nu_2=0}^{\mu_2}\sum_{\mu_3=0}^{m_3} T_{m_1,\nu_2,\mu_3}\Delta b_{\mu_3}\ .$$

Then by the Abel transformation, we have

$$\text{the second term in (2)}=b_{m_1+1}\sum_{\mu_2=0}^{m_2} V_{\mu_2}\Delta^2 b_{\mu_2}+b_{m_1+1}V_{m_2}\Delta b_{m_2+1}\ .$$

Since by Lemma 1 (ii)

$$V_{\mu_2,\mu_3}=O(\mu_2\mu_3\sqrt{\log m_1})\qquad \text{a.e.,}$$

so we get

$$V_{\mu_2} = \sum_{\mu_3=0}^{m_3} V_{\mu_2,\mu_3}\Delta^2 b_{\mu_3} + V_{m_3}\Delta b_{m_3+1}$$
$$= O(\mu_2 \sqrt{\log m_1} \sum_{\mu_3=0}^{m_3} \mu_3 \Delta^2 b_{\mu_3}) + O(\mu_2 \sqrt{\log m_1} \cdot m_3 \Delta b_{m_3})$$
$$= O(\mu_2 \sqrt{\log m_1}) \qquad \text{a.e.}$$

and

the second term in (2)

$$= O(b_{m_1} \sqrt{\log m_1} \sum_{\mu_2=0}^{m_2} \mu_2 \Delta^2 b_{\mu_2}) + O(b_{m_1} \sqrt{\log m_1} \cdot m_2 \Delta b_{m_2})$$
$$= O(b_{m_1} \sqrt{\log m_1}) = O(\frac{1}{\sqrt{p_{m_1}}}) = o(1) \quad \text{a.e.}$$

Simiarly

the third term in (2) $= O(\frac{1}{\sqrt{p_{2^{M_1^2}}}}) = o(1)$ a.e.

From these estimates we conclude that the first term in (1) is $o(1)$ a.e. and the theorem is proved.

In the above proof the reason why we use the indices "2^{M^2}" is due to Lemma 2, and if we appeal to Theorem 2, "2^{M^2}" may be replaced by "2^M".

REMARK. I don't know whether the factor $\log(|j_1|+2)\cdots\log(|j_n|+2)$ can be improved and an analogy of the theorem of Sjölin or Nikishin is valid or not for $n \geqq 3$.

Next we shall give a sufficient condition on the modulus of continuity of a function ensuring the almost everywhere convergence. For $s=(s_1,\cdots,s_n) \in [-1,1]^n$ we put

$$\Delta_{s_k}(f, x) = f(x_1,\cdots,x_k+s_k,\cdots,x_n) - f(x_1,\cdots,x_k,\cdots,x_n) \quad (k=1,\cdots,n)$$

and

$$\Delta_s(f, x) = \Delta_{s_1}\cdots\Delta_{s_n}(f, x).$$

So

$$\Delta_s(f, x) = f(x+s) - f(x) \qquad \text{when } n=1,$$

and

$$\Delta_s(f, x) = f(x_1+s_1, x_2+s_2) - f(x_1+s_1, x_2) - f(x_1, x_2+s_2) + f(x_1, x_2) \text{ when } n=2.$$

Then for $f \in L^2(T^n)$, it can be shown easily by Parseval's equality that

$$\sum_{j_1=-\infty}^{\infty}\cdots\sum_{j_n=-\infty}^{\infty} |f(j_1,\dots,j_n)|^2 \log(|j_1|+2)\cdots\log(|j_n|+2) < \infty$$

if and only if

$$\int_0^1 \cdots \int_0^1 \| \Delta_s(f) \|_2^2 \frac{1}{s_1 \cdots s_n} ds_1 \cdots ds_n < \infty .$$

Hence we get the following corollary from Theorem 3.

COROLLARY. *If $f \in L^2(T^n)$ satisfy the condition*

$$(*) \qquad \int_0^1 \cdots \int_0^1 \| \Delta_s(f) \|_2^2 \frac{1}{s_1 \cdots s_n} ds_1 \cdots ds_n < \infty$$

then $S_{m_1,\cdots,m_n}(f, x)$ converges to $f(x)$ almost everywhere in T^n as $m_1,\cdots,m_n \to \infty$.

Now we consider the following conditions.

$$(1^\circ) \qquad \sup_{|s_k| \leqq \delta} \| \Delta_{s_k}(f) \|_2 = O\left(\frac{1}{(\log \frac{1}{\delta})^{\frac{n}{2}+\varepsilon}}\right) \quad (k=1,\cdots,n)$$

$$(2^\circ) \qquad \sup_{\substack{|s_k| \leqq \delta_k \\ k=1,\dots,n}} \| \Delta_s(f) \|_2 = O\left(\frac{1}{(\log \frac{1}{\delta_1})^{\frac{1}{2}+\varepsilon} \cdots (\log \frac{1}{\delta_n})^{\frac{1}{2}+\varepsilon}}\right)$$

$$(3^\circ) \qquad \sup_{|s| \leqq \delta} \| \Delta_s(f) \|_2 = O\left(\frac{1}{(\log \frac{1}{\delta})^{\frac{n}{2}+\varepsilon}}\right)$$

Then it is obvious that (1°) implies (2°), (2°) implies (3°) and (2°) implies (*). So under the condition (1°) or (2°), $S_{m_1,\cdots,m_n}(f, x)$ converges almost everywhere in T^n as $m_1,\cdots,m_n \to \infty$. These results are given without proof in the paper of L.V. Zhizhiashvili [11] also.

REMARK. I don't know whether under the condition (3°) $S_{m_1,\cdots,m_n}(f, x)$ converges almost everywhere or not. When $n=2$, this problem has a positive answer by P. Sjölin [8] and M. Bakhbukh [1]. Also I don't know whether under these conditions with $\varepsilon=0$, $S_{m_1,\cdots,m_n}(f, x)$ converges almost everywhere or not. When $n=2$ this problem has a negative answer by M. Bakhbukh and E.M. Nikishin [2].

REFERENCES

[1] M. BAKHBUKH, On sufficient conditions for the convergence of double Fourier series over rectangles, Math. Notes, **15** (1974), 501–503.

[2] M. BAKHBUKH AND E.M. NIKISHIN, The convergence of double Fourier series of continuous functions, Siberian Math. Journ., **14** (1973), 832–839.

[3] C. FEFFERMAN, On the divergence of multiple Fourier series, Bull. Amer. Math. Soc., **77** (1971), 191–195.

[4] C. FEFFERMAN, On the convergence of multiple Fourier series, Bull. Amer. Math. Soc., **77** (1971), 744–745.

[5] S. KACZMARZ, Zur Theorie der Fourierschen Doppelreihen, Studia Math., **2** (1930), 91–96.

[6] E.M. NIKISHIN, Weyl multipliers for multiple Fourier series, Math. USSR Sbornik, **18** (1972), 351–360.

[7] P. SJOLIN, Convergence almost everywhere of certain singular integrals and multiple Fourier series, Arkiv Matem., **9** (1971), 65–90.

[8] P. SJOLIN, On the convergence almost everywhere of double Fourier series, Arkiv Matem., **14** (1976), 1–8.

[9] N.R. TEVZADZE, On the convergence of double Fourier series of square summable functions, Soobšč. Akad. Nauk Gruzin. SSR., **58** (1970), 277–279.

[10] L. V. ZHIZHIASHVILI, Some problems in the theory of simple and multiple trigonometric and orthogonal series, Russian Math. Surveys, **28** (1973), 65–127.

[11] L. V. ZHIZHIASHVILI, On the convergence of multiple trigonometric Fourier series, Soobšč. Akad. Nauk Gruzin. SSR., **80** (1975), 17–19.

[12] A. ZYGMUND, *Trigonometric series,* Cambridge, 1959.