A Supplement to “On Normality of a Family of Holomorphic Functions”

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In our previous paper[1], we proved the following :

Let D be a domain in C^n and let $\{f_j\}$ *be a sequence of holomorphic functions in D such that*

(i) ; *$\{f_j\}$ is bounded at each point of D, i. e., $\{f_j(p)\}$ is a bounded subset of the complex plane C for each point $p \in D$,*

(ii) ; *the sequence $\{G_j\}$ of graphs of f_j converges analytically to an analytic set A in $D \times C$.*

Then the sequence $\{f_j\}$ converges uniformly to a holomorphic function in D on every compact subset of D.

In this note we show that the condition (ii) cited above is weakened, that is we show the following

THEOREM. *Let D be a domain in C^n and let $\{f_j\}$ be a sequence of holomorphic functions in D such that*

(i) ; *$\{f_j\}$ is bounded at each point of D,*

(ii) ; *the sequence $\{G_j\}$ of graphs of f_j converges geometrically to a proper analytic set A in $D \times C$.*

Then $\{f_j\}$ converges uniformly to a holomorphic function in D on every compact subset of D.

Proof. Let E be a set of non fine point of A, that is $p \notin E$ *if and only if the set* $A(p) = A \cap \{(p, w) \in C^{n+1}\}$ has no finite accumulating point. Then E is a proper analytic set in D[2]. Let $P_0 \in D - E$, then since $\{f_j(p_0)\}$ is bounded there exists at least one limit point q_0. By the definition of the geometric convergence, it holds that $(p_0, q_0) \in A$. Since A is proper and since $p_0 \notin E$ we can take an open polydisc $\Delta \Subset D - E$ with center at p_0 and an open disc $U \Subset C$ with center at q_0 such that $A \cap (\Delta \times \partial U) = \phi$. Then there

1) On normality of a family of holomorphic functions, Publications RIMS, Vol 9, No. 3, 1974.

2) See ibid., section 2.

exists a positive integer j_0 such that $G_j \cap (\Delta \times \partial U) = \phi$ and $G_j \cap (\Delta \times U) \neq \phi$ for all $j \geqq j_0$. Let $\pi : \Delta \times U \to \Delta$ be the natural projection and put

$$\pi_j = \pi \mid G_j \cap (\Delta \times U) : G_j \cap (\Delta \times U) \longrightarrow \Delta.$$

Then it is easily seen that π_j is a proper mapping, so that $(G_j \cap (\Delta \times U), \pi_j, \Delta)$ is an analytic cover[3]. Thus π_j is onto. This means that $\{f_j\}$ is uniformly bounded on Δ, that is $\{f_j\}$ is locally uniformly bounded on $D-E$. Let $p_0 \in E$. Since E is a proper analytic set in D, by a linear change of coordinate if necessary, we may assume that there exists a polydisc Δ with center at p_0 such that E does not meet with the distinguished boundary of Δ. Then by the maximum principle of the holomorphic functions, $\{f_j\}$ is bounded on Δ. That is $\{f_j\}$ is locally uniformly bounded on D, and then $\{f_j\}$ is a normal family. If a subsequence $\{f_{\nu_j}\}$ of $\{f_j\}$ converges to a holomorphic function f on every compact set in D, then $\{G_{\nu_j}\}$ converges geometrically to the graph of f and A =graph of f. From this, $\{f_j\}$ itself converges uniformly to f on every compact set in D.

REMARK 1. Under the condition of the above theorem, the set E is in fact empty since A is a graph of a holomorphic function.

REMARK 2. The conditions (i), (ii) are said in other words as follows.

(i) ; $\{f_j\}$ is bounded at each point of D and equicontinuous at some point $p_0 \in D$,

(ii) ; the sequence $\{G_j\}$ converges geometrically to an analytic set A in $D \times C$.

In fact, we have only to show that A is proper. There exists an open polydisc $\Delta \subset D$ and a positive constant M such that $|f_j(p)| \leqq M$ if $p \in \Delta$. Thus if $(p_0, q) \in A$ then $|q| \leqq M$, that is A is a proper analytic set.

3) See, for example, Gunning R.C., and Rossi H., Analytic functions of several complex variables, Chapter III, B.