On Fechner's Problem

金沢大学文学部論集　行動科学・哲学編
第21号　2001年　1－25

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Summary

Replacement of differences by differentials in derivation of the logarithmic law has been criticized, but, in case of the revised Fechner's problem, differentials and integration can be introduced. Dzhafarov and Colonius (1999) presented their model, in which psychophysical distances are calculated by integration of differentials derived from psychometric functions. But their model should be considered to be based essentially on Case V of Thurstonian models. On the other hand, explicit use of Thurstonian models and psychometric functions has been proposed to derive a differential equation for relation between physical and sensation intensities by Okamoto (1993, e.g.), which is presented here with a more general approach than the previous ones. After the introduction of these models, I discuss problems in criticisms for Fechner's differential formula, other uses of differential equations in psychophysics (e.g., Eisler (1963)), psychological nonsense in general mathematical distances in Dzhafarov and Colonius' model, and so on.

Introduction

Fechner's problem is concerned to scaling of sensation, especially based on discrimination data. Minimum increments of stimulus intensities to be discriminated, e.g., intensities of 1000-Hz tones, are called JNDs (just noticeable differences or difference limens (DLs)) and thought to be the same size in the sensation scale with each other. So, one might expect that the sensation scale could be obtained by cumulating JNDs (see, Gescheider, 1985, 144-146, e.g.). Furthermore, replacing differences called JNDs by differentials, we get the differential equation which leads to the logarithmic law with Weber's law. But, this process of derivation has problems (Luce and Edwards, 1958).

When intensity of the stimulus is denoted by s and JND for intensity s by Δs, Fechner's problem can be formulated as follows:

$$\text{Find a scale } u \text{ for } s \text{ such that}$$
$$u[s+\Delta s]-u(s)=\text{constant} \tag{1}$$

In discrimination experiment, the size of JND is determined by setting discrimination probability π at some value, e.g. $\pi=0.75$, so the probability that $s+\Delta s$ is judged to be more intense than s is π. Making this dependence of Δs on π explicit, we can formulate eq.(1) as follows (Luce and Galanter, 1963, p.208) :

$$u[s+\Delta(s, \pi)]-u(s)=g(\pi) \tag{2}$$

In eq.(2), $\Delta(s, \pi)$ represents JND for discrimination probability π, and $g(\pi)$ the constant value of corresponding differences in the sensory scale u.

In an experiment, π is fixed value, so $g(\pi)$ also fixed. Luce and Galanter (1963, p.223) said that JND is an algebraic rather than probabilistic notion. In this case, functional equation analysis of eq.(2) is appropriate (Luce and Edwards, 1958) and measurement theorists have contributed (e.g., Falmagne, 1985).

But, conceptually, π can be any value of probability. When value of π in eq.(2) is allowed to vary, eq.(2) is called the revised Fechner problem (Luce and Galanter, 1963, p.210). For the revised Fechner problem, we have various values of $\Delta(s, \pi)$ corresponding to various values of π. Specifically, we can have infinitely small $\Delta(s, \pi)$ as π approaches 1/2, assuming no biases. Hence, we can derive differential equations with π approaching to 1/2.

As eq.(2) can be transformed to the following psychometric function

$$\pi=g^{-1}\{u[s+\Delta(s, \pi)]-u(s)\},$$

the differential equation is linked to a psychometric function, which is the basic idea of Dzhafarov and Colonius (1999). But, strictly speaking, the central concept in Dzhafarov and Colonius' model should be considered not to be a psychometric function, but to be something like a Thurstonian model, especially Case V (Thurstone, 1927).

A psychometric function in the strict meaning has been used in Okamoto (1993, 1994, 1995a, 1995b, 1996a, 1996b, 1997, 1999) to derive psychophysical differential equations, where a Thurstonian model is linked to a psychometric function to get relation between the physical stimulus intensities and the corresponding sensation intensities. Use of a psychometric function introduces a Weber function in some form into the psychophysical differential equation, which is not clear in Dzhafarov and Colonius (1999).

In the next section, I will explain Dzhafarov and Colonius' model in a rather simplified way to elucidate the essential of their model.

As a subsidiary assumption, in their model the equally-often-noticed-difference problem (Luce and Galanter, 1963, p.211) is accepted. Luce and Galanter (1963, p.216) suggested that this problem has a solution if and only if the variances of the sensation differences are all equal to each other. Falmagne (1985, p.144-145) showed that, in Thurstone's Case III, the equally-often-noticed-difference problem has a solution not only for the constant variance assumption but also for linearly variable variance with mean. But, in the latter case, difference is calculated in the scale constructed as logarithmic transformation of the original sensory scale (Falmagne, 1985, p.136).

The assumption of constant variance is too restrictive and has not yet been empirically confirmed. But, constant variance assumption seems to be tacitly taken in criticisms for the logarithmic law and this assumption plays an important role in some type of the criticisms. Constant variance assumption will be discussed later in this paper. Equality of variances is not assumed in Okamoto's model, which will be introduced after explanation of Dzhafarov and Colonius' model.

Dzhafarov and Colonius' Model

Dzhafarov and Colonius (1999) proposed their model for multidimensional stimuli, which includes an unidimensional model as a special case, and the basic idea can be shown most easily for unidimensional one. So, in this section, I explain their model in case of unidimensional stimuli.

Dzhafarov and Colonius denote a sensory scale u by ϕ, physical intensity s by x, and discrimination probability π by p, and JND $\Delta(s, \pi)$ by $w(x, p)$. Hence, eq.(2) becomes

$$\phi[x+w(x, p)]-\phi(x)=g(p) \tag{3}$$

Eq. (3) is presented in Dzhafarov and Colonius (1999, p.246).

Taking the inverse function $h=g^{-1}$ of g, we have

$$\begin{aligned} p &= g^{-1}\ \{\phi[x+w(x,\ p)]-\phi(x)\} \\ &= h\ \{\phi[x+w(x,\ p)]-\phi(x)\} \end{aligned} \tag{4}$$

If we can set with the standard normal distribution $N(z)$

$$h(t)=\int_{-\infty}^{t} N(z)\,dz,$$

we have

$$p=\int_{-\infty}^{\phi\,[x+w(x,\ p)]-\phi(x)} N(z)\,dz. \tag{5}$$

Eq. (5) corresponds to Case V of Thurstone's (1927) model. Luce and Edwards (1958, p.233) pointed out the relevance of Case V to the equally-often-noticed-difference problem.

Intuitively, from eq. (5) we have in the neighborhood of $p=1/2$,

$$dp \propto d\phi(x).$$

Hence, difference $G(b,\ a)$ of scale values of a and b can be given by

$$G(a,\ b)=\phi(b)-\phi(a)\ \propto\ \int_a^b dp(x). \tag{6}$$

That is, difference of sensory scale values, from which we can easily determine the scale, can be calculated by accumulating differentials of discrimination probability and this process of scaling is guaranteed by the model like Thurstone's Case V.

More rigorously, rewriting eq. (4) as

$$\gamma_x(y)=h\ \{\phi(y)-\phi(x)\}\ , \tag{4'}$$

where $\gamma_x(y)=p=\Pr ob$("y is more intense than x") according to Dzhafarov and Colonius' notation, and differentiating the above equation by y, we have

$$\frac{d}{dy}\gamma_x(y) = \frac{dh}{d\phi} \cdot \frac{d\phi}{dy}$$

$$= h'[\phi(y) - \phi(x)] \cdot \frac{d\phi}{dy}.$$

As notational change, setting

$$y = x$$

and

$$F(x) = \frac{d}{dy}\gamma_x(y)\Big|_{y=x} \tag{7}$$

we have

$$F(x) = h'(0) \cdot \frac{d\phi}{dx}$$

Hence, we have

$$\phi(b) - \phi(a) = \frac{1}{h'(0)} \cdot \int_a^b F(x)\,dx$$

By suitable choice of unit of the scale, i.e., replacing scale ϕ by $h'(0) \cdot \phi$, we have

$$G(a, b) = \phi(b) - \phi(a) = \int_a^b F(x)\,dx \tag{8}$$

Eq. (8) was presented by Dzhafarov and Colonius (1999, p.248) in the following form

$$G(a, b) = \int_a^b d\gamma_x(x),$$

as to which Dzhafarov and Colonius suggested that this surprisingly simple approach to $G(a, b)$ constitutes the essence of Fechnerian scaling for unidimensional continua. But, eq. (8) is trivial in view of eq. (6).

Eq. (8) is derived from eq. (4'), which is a psychometric function as a function of y. But, given the formulation eg. (5), it is more appropriate to consider eqs. (4') or (4) as a Thurstonian model. The function h in eqs.(4) or (4') is a function of difference between sensory scales. In contrast to eq.(5), a psychometric function, as a function of physical intensities, should be given as follows

$$p=\int_{-\infty}^{k(x)\frac{y-x}{x}} N(z)\,dz, \tag{9}$$

where $k(x)$ is the coefficient that is determined by the slope of the psychometric function at x.

Combination of a psychometric function such as eq.(9) with a Thurstonian model such as eq.(5) gives interesting models (Okamoto, 1993, 1994, 1995a, 1995b, 1996a, 1996 b, 1997, 1999). In Okamoto's models, variances of the sensation differences are not assumed necessarily to be constant. These variable variance models will be explained in the next section.

Variable Variance Models

Before explaining the present version of the variable variance model, I review the previous versions briefly.

The early version was a rather simple model presented in Okamoto (1993, 1994). To get a relation between physical intensities S_1 and S_2, and the corresponding sensory intensities R_1 and R_2, the following Thurstonian model and psychometric function were used:

$$\text{Prob}(\text{“}S_2 \text{ is judged to be more intense than } S_1\text{”})=\int_{-\infty}^{(R_2-R_1)/\sqrt{\sigma_1^2+\sigma_2^2}} \phi_0(x)\,dx \tag{10}$$

and

$$\text{Prob}(\text{“}S_2 \text{ is judged to be more intense than } S_1\text{”})=\int_{-\infty}^{k_{S_1}\cdot(S_2-S_1)/S_1} \phi_0(x)\,dx \tag{11}$$

where R_1 and R_2 are means of intensities of sensations produced by physical stimuli S_1 and S_2, σ_1^2 and σ_2^2 variances of sensation intensities of S_1 and S_2, k_{S_1} the coefficient determined by the slope of the psychometric function at S_1, and $\phi_0(x)$ the standard normal distribution.[1] Strictly speaking, the left sides of eqs.(10) and (11) were written in Okamoto (1994) as $P(X_2>X_1)$, where X_1 and X_2 are random variables for sensation intensities invoked by physical stimuli S_1 and S_2. Hence,

$$P(X_2 > X_1) = \text{Prob}(\text{“}S_2 \text{ is judged to be more intense than } S_1\text{”})$$

By eqs.(10) and (11), we have

$$\frac{R_2 - R_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} = k_{S_1} \cdot \frac{S_2 - S_1}{S_1}$$

Letting S_2 and R_2 approach to S_1 and R_1, respectively, and replacing S_1 and R_1 by S and R, respectively, we have

$$\frac{dR}{dS} = \frac{\sqrt{2} \cdot \sigma k_S}{S}$$

In the above equation, standard deviation (square root of variance) σ and coefficient k_S, which is closely related to a Weber function, are explicitly included. These factors, σ and a Weber function, play important roles in analysis of discrimination data. In a general case, where σ is any function of S, discrimination probabilities are not only determined by sensation differences but also affected by the sizes of variances of them. This point is important when we discuss number of JNDs between fixed differences of sensation magnitudes, and will be considered later in this paper.

R.D.Luce (personal communication, 1994 to 1995) gave me detailed comments on Okamoto (1994). Based on Luce's comments, a revised manuscript (Okamoto, 1995b) was written. In Okamoto (1995b), revision was mainly on notation. R_1, R_2, σ_1^2 and σ_2^2 were replaced by R_S, R_Z, σ_S^2 and σ_Z^2, so the Thurstonian model was presented as

$$P(X_Z > X_S) = \int_{-\infty}^{(R_Z - R_S)/\sqrt{\sigma_Z^2 + \sigma_S^2}} \phi_0(x) \cdot dx$$

I got valuable comments on Okamoto (1995b) from active specialists, including R. D.Luce (personal communication, 1995), D.Laming (personal communication, 1995) and H.Colonius (personal communication, 1995). Again, a more general model than Okamoto (1994, 1995b) was presented in Okamoto (1996b), where o-notation (see Apostol, 1974, p.192, e.g. Meaning of Apostol's notation of O (big oh) is different from

[1] Notations used here to present early versions of my models follow those in my previous papers.

that in Dzhafarov and Colonius (1999, p.267, note 3)) was introduced in response to Luce's comment. In Okamoto (1996b), it was emphasized that k_{S_1} in eq.(11) may vary as a function of S_1, so k_{S_1} was denoted as $k(S_1)$. While constancy of k_{S_1} with respect to S_1 can be derived from Weber's law, variable k_{S_1} can correspond to a case such as the near-miss to Weber's law, which case was suggested by D.R.Luce (personal communication, February 17, 1995).

In Okamoto (1996b), eqs.(10) and (11) in Okamoto (1994) were reformulated[2] as

$$P(X_2 > X_1) = \int_{-\infty}^{(\Psi(S_2)-\Psi(S_1))/\sqrt{\sigma(S_2)^2+\sigma(S_1)^2-2r\sigma(S_1)\sigma(S_2)}+o(\Psi(S_2)-\Psi(S_1))} \phi_0(x) \cdot dx$$

and

$$P(X_2 > X_1) = \int_{-\infty}^{k(S_1)\cdot\frac{S_2-S_1}{S_1}+o(S_2-S_1)} \phi_0(x) \cdot dx.$$

But, here I will present the above model with notational change as eqs.(12) and (13), because notations in Luce and Galanter (1963), Dzhafarov and Colonius (1999) and Okamoto's papers are different to each other, and I felt that some consideration on notation is needed.

The symbols used afterward are as follows:

x and y denote physical stimulus magnitudes. These symbols, x and y, play the role of independent variables in psychophysical functions and, in mathematics, independent variables are often denoted as x, y and so on. This notation accords with Dzhafarov and Colonius (1999).

X and Y denote random variables which represent sensation magnitudes invoked by physical stimulus intensities x and y, respectively.

Stevens (1975) denoted stimulus magnitude by ϕ and sensory one by Ψ. In accord to Stevens' notation, means of X and Y are denoted by $\Psi(x)$ and $\Psi(y)$, respectively. These means, $\Psi(x)$ and $\Psi(y)$, are interpreted to represent the sensory scale values of the physical stimulus intensities, x and y.

Variances of X and Y are denoted by $\sigma^2(x)$ and $\sigma^2(y)$.

[2] The term $o(S_2-S_1)$ in the psychometric function was carelessly omitted in Okamoto (1996b), but is included in this paper.

With these notations, set the following Thurstonian model:

$$\begin{aligned}&\text{Prob}(\text{“}y\text{ is judged to be more intense than }x\text{”})\\&=\text{Prob}(Y>X)\\&=\int_{-\infty}^{\{[\Psi(y)-\Psi(x)]/[\sigma^2(x)+\sigma^2(y)-2r\sigma(x)\sigma(y)]^{1/2}\}+o[\Psi(y)-\Psi(x)]} N(t)\cdot dt,\end{aligned} \tag{12}$$

where r represents the correlation coefficient between X and Y, and $N(t)$ the standard normal distribution. Signal detection theory (Green and Swets, 1966) also uses normal distributions.

Psychometric function[3] is assumed to be written as

$$\text{Prob}(\text{“}y\text{ is judged to be more intense than }x\text{”})=\int_{-\infty}^{\frac{1}{k(x)}\cdot\frac{y-x}{x}+o(y-x)} N(t)\cdot dt \tag{13}$$

o-notations in eqs. (12) and (13) are introduced to show that at least asymptotic representations by the standard normal distribution $N(t)$ around x is sufficient in this model. This approximate type of modeling, which is for Fechner's problem, was introduced as M.F.Norman's by Krantz (1971, p.596). More generally, use of $N(t)$ is not necessary and a model without use of $N(t)$ will be introduced later. Representation of a psychometric function by a cumulative normal of the physical stimulus is called the phi-gamma hypothesis and supported with respect to some sensory continuum (Gescheider, 1985, Pp.65-67). The phi-gamma hypothesis depends on the physical scale used. Dependency of a model or a theory on a particular system of representation is also observed in physics. Newtonian mechanics stands on inertial systems (e.g., Arya, 1979).

By eqs. (12) and (13), we have

$$\frac{\Psi(y)-\Psi(x)}{[\sigma^2(x)+\sigma^2(y)-2r\sigma(x)\sigma(y)]^{1/2}}+o[\Psi(y)-\Psi(x)]=\frac{1}{k(x)}\cdot\frac{y-x}{x}+o(y-x)$$

With some smoothness conditions, letting y approach to x, we have

$$\frac{d\Psi}{dx}=\frac{\sigma(x)\cdot\sqrt{2(1-r)}}{k(x)\cdot x} \tag{14}$$

[3] $k(S_1)$ is replaced by $\frac{1}{k(x)}$, to make the meaning of the term $k(x)$ more straightforward. Explanation of $k(x)$ will be given later.

k (x) and Weber function

That $k(x)$ in eq.(14) is closely related to Weber function can be shown as follows.

Let $z_{0.75}$ and $x_{0.75}$ be the values corresponding to discrimination probability 0.75 such that

$$0.75 = \int_{-\infty}^{z_{0.75}} N(t) \cdot dt$$

$$= \int_{-\infty}^{\frac{1}{k(x)} \cdot \frac{x_{0.75}-x}{x} + o(x_{0.75}-x)} N(t) \cdot dt$$

Then we have

$$z_{0.75} \approx \frac{1}{k(x)} \cdot \frac{x_{0.75}-x}{x}$$

$$k(x) \approx \frac{1}{z_{0.75}} \cdot \frac{x_{0.75}-x}{x}$$

Because $\frac{x_{0.75}-x}{x}$ is Weber fraction for discrimination probability 0.75 and $\frac{1}{z_{0.75}}$ constant coefficient, we can see that $k(x)$ is essentially Weber function.

More general approach[4]

In eqs.(12) and (13), the standard normal distribution $N(t)$ is used. But $N(t)$ is not necessarily needed. More generally, we can proceed as follows:

Set

$$\text{Prob}(Y>X) = T\left\{\frac{\Psi(y)-\Psi(x)}{[\sigma^2(x)+\sigma^2(y)-2r\sigma(x)\sigma(y)]^{1/2}}\right\} \tag{15}$$

and

$$\text{Prob}(Y>X) = W\left(\frac{1}{k(x)} \cdot \frac{y-x}{x}\right), \tag{16}$$

where $T(z)$ and $W(z)$ are cumulative distribution functions, i.e.

$$\lim_{z\to-\infty} T(z) = \lim_{z\to-\infty} W(z) = 0,\ \lim_{z\to+\infty} T(z) = \lim_{z\to+\infty} W(z) = 1,$$

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[4] This general approach is tried in response to stimulation given by D.Laming (personal communication, 1995).

and it is assumed that $T(z)$ and $W(z)$ are not decreasing, and strictly increasing in the neighborhood of 0.

By eqs. (15) and (16), we have

$$\frac{\Psi(y)-\Psi(x)}{[\sigma^2(x)+\sigma^2(y)-2r\sigma(x)\sigma(y)]^{1/2}}=T^{-1}\left\{W\left[\frac{1}{k(x)}\cdot\frac{y-x}{x}\right]\right\}$$

at least in the neighborhood of x, where T^{-1} is defined.

When $h(z)=T^{-1}[W(z)]$ is linear[5] in the neighborhood of 0, i.e.,

$$\begin{aligned} h(z)&=T^{-1}[W(z)]\\ &=\alpha\cdot z+o(z), \end{aligned}$$

we have

$$\frac{\Psi(y)-\Psi(x)}{[\sigma^2(x)+\sigma^2(y)-2r\sigma(x)\sigma(y)]^{1/2}}=\alpha\cdot\frac{1}{k(x)}\cdot\frac{y-x}{x}+o(y-x)$$

Let y approach x, we have

$$\frac{d\Psi}{dx}=\frac{\alpha\cdot\sigma(x)\cdot\sqrt{2(1-r)}}{k(x)\cdot x}$$

This equation is essentially the same as eq. (14).

Asymmetry of psychometric functions

Psychometric function eq. (11) is symmetric around S_1 as a function of S_2. But, if Weber's law holds strictly, the psychometric function is not symmetric (Drösler, 2000). This asymmetry can be seen as follows:

Let ΔS be JND of stimulus S for discriminability probability π and k Weber fraction $k=\frac{\Delta S}{S}$ and set

$$T=S+\Delta S,$$

then

[5] Luce and Galanter (1963, p.200) said that the psychometric function is very nearly a straight line in a region approximately one JND, or a little more, above and below the PSE.

$$T = S + \Delta S$$
$$= (1+k)S$$

Hence, we have

$$\frac{T-S}{T} = \frac{T - \frac{1}{1+k} \cdot T}{T}$$

$$= \frac{k}{1+k}$$

With $\Delta\_T = \frac{k}{1+k} \cdot T$,
We have

$$T - \Delta\_T = T - (T-S)$$
$$= S$$

That is,

$$\mathrm{Pr}ob(\text{"}T - \Delta T \text{ is less intense than } T\text{"}) = \mathrm{Pr}ob(\text{"}S \text{ is less intense than T"})$$
$$= \pi$$

in case of a forced two-choice experiment.

This equation shows that Weber fraction $\frac{\Delta\_T}{T}$ in the negative direction is $\frac{k}{1+k}$ and not equal to that value k in the positive direction. This asymmetry necessitates the introduction of o-notation into eq. (13).

Discussion

The logarithmic and power laws

Based on the idea of cumulating JNDs, the logarithmic law is derived. But, this logarithmic function for sensory intensity has been criticized theoretically that, in derivation, differences are replaced by differentials, and Weber's law is assumed. There are also empirical criticisms against the logarithmic law. For example, Stevens (1936, p.412) pointed out that summating the same number of JNDs for two tones of different frequency does not yield equal loudnesses.

Against the logarithmic law, the power law is proposed (Stevens, 1975). Baird (1997, p.50) says that the power law must be squarely faced by any theory of psychophysics. The power law has been supported empirically by experiments such as magnitude estimation, cross-modality matching, partition and so on. In some theoretical approaches, power functions are derived (e.g., Luce, 1990; Krantz, 1972).

Just like as the logarithmic law can be derived from the Fechner's assumption (Krueger, 1989, p.253)

$$\Delta\psi = \text{constant}, \tag{17}$$

the power law can be derived from Ekman's law (Stevens, 1975, p.235)

$$\frac{\Delta\psi}{\psi} = \text{constant}, \tag{18}$$

where $\Delta\psi$ is the increment of sensory magnitude corresponding to JND. But, to confirm eq.(18) empirically, Ekman (1959) assumed the power law. So, it is tautological to justify the power law by Ekman's law.

As another hypothesis than Fechner's or Ekman's law, Krueger (1989, p.260) presents the following

$$\Delta\psi \propto m^{-1}, \tag{19}$$

where m is the total number of JNDs in a continuum.

Eq.(19) is also considered to be a model without the constant variance assumption, interpreting

$$\sigma(x) \propto m^{-1}.$$

But, in the model eq.(19), variance varies only over frequencies of tones, say. Over intensities of a fixed frequency, the variance is constant.

Krueger (1989, p.265), in the last paragraph, refers to the problem of constancy of $\Delta\psi$. In view of eq.(14), this problem is fundamental and should be given more attention by researchers.

In general, laws of empirical sciences like psychology should be supported ultimately by empirical facts. In view of validity of experimental methods, Laming (1997) criticized the power law. From their results of the scale discrimination experiment, Namba and Kuwano (1991, p.238) concluded that it is possible that whether a ratio or interval scale is found depends on the instruction given to the subjects. In their experiment (experiment 2, pp.236-238), the subjects were instructed to judge whether the stimulus spacing produced equal perceived ratios, equal perceived intervals, or neither. For the stimuli spaced 10-dB steps, 30 out of 44 subjects perceived equal ratios, but for 3-dB steps stimuli, 32 out of 44 subjects perceived equal intervals. Here, it should be noted that, mathematically, structures of ratios and intervals (i.e. differences) are isomorphic to each other, hence theoretical analysis to distinguish between them is difficult.

When, to criticize the logarithmic law, Stevens (1936, p.412) said "it has been found that summating the same number of DL's for two tones of different frequency does not yield equal loudness", it sounds like that constancy of variance of sensory intensity was assumed, which is clearly inferred in view of Thurstonian model like eq. (10). Hence, if the constancy of variance is abandoned, Stevens' criticism as to number of JNDs loses force.

In a sense, Ekman's law eq.(18) denies the constancy of variance, because it can be interpreted in the framework of eq.(14) as setting

$$\sigma(x) \propto \Psi(x) \tag{20}$$

In this case, by eq. (14) and Weber's law

$$k(x) = \text{constant}, \tag{21}$$

we can derive the power law as follows:

By eq.(14) and eq.(20), we have

$$\frac{d\Psi}{dx} = \frac{\Psi \cdot \sqrt{2(1-r)}}{k(x) \cdot x}$$

By eq.(21), we have

$$\frac{d\Psi}{dx} = \alpha \cdot \frac{\Psi}{x},$$

where $\alpha=\frac{\sqrt{2(1-r)}}{k(x)}=$constant.

Hence, we have

$$\log\Psi=\alpha\cdot\log x+\beta,\quad \beta=\text{constant}$$
$$\Psi=e^{\beta}\cdot x^{\alpha}$$
$$=\gamma\cdot x^{\alpha},\qquad \gamma=e^{\beta}:\text{ constant}$$

On the other hand, Fechner's assumption eq,(17) can be formulated as follows

$$\sigma(x)=c,$$

where c is the size of the constant variance.

In this case, we get the logarithmic law. By eq.(14) and the above equation, we have

$$\frac{d\Psi}{dx}=\frac{c\cdot\sqrt{2(1-r)}}{k(x)\cdot x}$$

With Weber's law eq.(21), we have

$$\frac{d\Psi}{dx}=\alpha\cdot\frac{1}{x},$$

where $\alpha=\frac{c\cdot\sqrt{2(1-r)}}{k(x)}=$constant.

Hence, we have

$$\Psi=\alpha\cdot\log x+\beta$$

with β a constant.

Some unification attempts of the logarithmic and the power law have been tried. Baird (1997, p.84) said that researchers in psychophysics implicitly understand the sameness of local and global psychophysics. Reviewing researches of scales of magnitude, partition or category, summated JND and neurelectric, Krueger (1989) seeked a common unified sensory scale. For example, he noted that a logarithmic function can be approximated by a power function with a low exponent and presented a figure (Fig.1 in his paper) to show this fact concretely (p.254). The power law is only an approximation in case of the Sensory Aggregate Model (Baird, 1997, p.46; also see p.66). Norwich (1993) derived his law of sensation theoretically and showed that his law includes both the logarithmic and power laws as extreme cases. But, in the present state where experimental validity of the power law is doubted (Laming, 1997), before trying to unify both laws, we should scrutinize experimental results that are considered to support the power law.

Problems in Criticisms for Fechner's Differential Formula

Counter examples have been proposed concerning the process of deriving differential formula from difference ones. But, these examples are inappropriate in some sense. I will show this point using examples in Dzhafarov and Colonius (1999).

To show self-contradiction as to the derivation, they set the following equations (p.245):

$$\Delta x = w(x) \tag{22}$$

and

$$\Delta \phi = c, \tag{23}$$

where ϕ represents sensation scale for physical stimulus x, and $w(x)$ increment of stimulus intensity corresponding to the increment c of sensation.

By eqs.(22) and (23), they gave

$$\frac{\Delta x}{w(x)} = \frac{\Delta \phi}{c} \tag{24}$$

and, replacing differences by differentials, finally presented

$$\frac{dx}{w(x)} = \frac{d\phi}{c} \tag{25}$$

As criticism for this replacement of differences by differentials, Dzhafarov and Colonius (1999, p.245) presented superficially self-contradictory examples. But, scrutinizing these examples, we can see that problems are not in replacement of differences by differentials, but rather in carelessness of proposers of so-called self-contradictory examples.

Δx and $\Delta\phi$ are interpreted as JND and its corresponding difference of sensations. Letting Δx and $\Delta\phi$ approach to 0, we have $w(x)$ and c also approaching to 0. So, in this case, eq.(25) becomes the following nonsense equation

$$\frac{dx}{0}=\frac{d\phi}{0}$$

To avoid this nonsense, Dzhafarov and Colonius (1999) should have taken an approach such as the following:

Let return to eqs.(22) and (23). Actually, $w(x)$ and c are functions of discrimination probability π, so these should be denoted as $w(x, \pi)$ and $c(\pi)$. With these notational changes, eq.(24) becomes

$$\frac{\Delta x}{w(x, \pi)}=\frac{\Delta\phi}{c(\pi)}$$

So, we have

$$\frac{\Delta\phi}{\Delta x}=\frac{c(\pi)}{w(x, \pi)} \tag{26}$$

As π approaches to 1/2, $\dfrac{\Delta\phi}{\Delta x}=\dfrac{c(\pi)}{w(x, \pi)}$ converges to $\dfrac{d\phi}{dx}$.

At this point, we should note that when, e.g.,

$$\frac{d^2\phi}{dx^2}<0,$$

we have

$$\frac{c(\pi_1)}{w(x, \pi_1)}>\frac{c(\pi_2)}{w(x, \pi_2)} \quad \text{for } \pi_1<\pi_2$$

The above equation is the cause of so-called self-contradiction using finite difference. That is, in general, coefficient $\dfrac{c}{w(x)}$ in differential equation is different from the value of $\dfrac{c}{w(x)}$ for finite differences Δx or $\Delta\phi$.

Consider the following example in Dzhafarov and Colonius (1999, p.245)

$$G(x, x+kx)=\frac{c}{k}\log(1+k)\neq c,$$

where $G(x, x+kx)$ denotes difference of sensory scale values, the coefficient $\frac{c}{k}$ is the limit of $\frac{x \cdot c(\pi)}{w(x, \pi)}$ as π approaches 1/2. The constants c and k are nonsense when considered individually, because these values should be 0 as limit values.

For discrimination probability π, set

$$\frac{\Delta x}{x}=\frac{w(x, \pi)}{x}=k(\pi) \tag{27}$$

and

$$\Delta\phi=c(\pi) \tag{28}$$

In the above equations, Weber fraction $\frac{\Delta x}{x}$ and corresponding sensory difference $\Delta\phi$ are explicitly denoted as functions of π.

By eqs. (27) and (28), we have

$$\frac{c(\pi)}{k(\pi)}=x \cdot \frac{\Delta\phi}{\Delta x}=\frac{x \cdot c(\pi)}{w(x, \pi)}$$

Letting π approach to 1/2, we have

$$\lim_{\pi\to 1/2}\frac{c(\pi)}{k(\pi)}=x \cdot \frac{d\phi}{dx}=\lim_{\pi\to 1/2}\frac{x \cdot c(\pi)}{w(x, \pi)}$$

Hence, setting the limit as a ratio of some non-zero values c_0 and k_0, i.e.,

$$\frac{c_0}{k_0}=\lim_{\pi\to 1/2}\frac{c(\pi)}{k(\pi)},$$

we have

$$\frac{d\phi}{dx}=\frac{c_0}{k_0} \cdot \frac{1}{x}$$

and

$$\phi(x)=\frac{c_0}{k_0} \cdot \log x+\text{constant}$$

Because $\frac{c_0}{k_0}$ is given as the limit value with Δx approaching to 0, it is trivial that

$\Delta\phi$ is not equal to c_0 for arbitrary finite Δx. Hence, it is not appropriate to consider that this fact of $\Delta\phi$ being not equal to c_0 is self-contradictory.

But, constancy of $\Delta\phi$ for constant Weber fraction $\frac{\Delta x}{x}$ can be shown as follows;

Set

$$\frac{\Delta x}{x}=\delta=\text{constant},$$

then we have

$$\begin{aligned}\Delta\phi=\phi(x+\delta\cdot x)-\phi(x)&=\frac{c_0}{k_0}\cdot\log\frac{x+\delta\cdot x}{x}\\&=\frac{c_0}{k_0}\cdot\log(1+\delta)\\&=\text{constant}\end{aligned}$$

As a more general case, Dzhafarov and Colonius (1999, p.245) presented the following equation

$$G(a,\ a+\Delta a)=c\int_a^{a+w(a)}\left(\frac{dx}{w(x)}\right) \tag{29}$$

They said that the right side of eq. (29) should be constant, although it is not in general. I find two problems in their discussion concerning this point.

First, coefficient c in eq. (29) is nonsense, because it is 0 as a limit value.

Second, the right side, if correctly interpreted, should be constant against Dzhafarov and Colonius' explanation. The reason is as follows:

By eq. (26), ratio $\frac{c(\pi)}{w(x,\ \pi)}$ has limit value $\frac{d\phi}{dx}$ with Δx approaching to 0. That is, we have

$$\frac{d\phi}{dx}=\lim_{\substack{\Delta x\to 0\\ or\\ \pi\to 1/2}}\frac{c(\pi)}{w(x,\ \pi)}=V(x),\ \text{say.}$$

Set

$$w(x)=\frac{c_0}{V(x)}\ \text{for any positive constant } c_0,$$

then

$$\frac{d\phi}{dx}=\frac{c_0}{w(x)} \tag{30}$$

$w(x)$ is determined as a ratio of any positive constant c_0 to $V(x)$.

Then, we have

$$\int_a^{a+\Delta a}\left(\frac{c_0}{w(x)}\right)\cdot dx=\int_a^{a+\Delta a}\left(\frac{d\phi}{dx}\right)\cdot dx=\phi(a+\Delta a)-\phi(a)$$

Hence, when eq.(30) holds and $\phi(a+\Delta a)-\phi(a)$ is constant for $\frac{\Delta a}{a}$ constant, then the following integration

$$\int_a^{a+\Delta a}\left(\frac{c_0}{w(x)}\right)\cdot dx \tag{31}$$

is also constant.

But, it should be noted that, in general, the constant value is not equal to c_0, because the value of $\phi(a+\Delta a)-\phi(a)$ is determined by ratio $\frac{\Delta a}{a}$, and the value of c_0 depends on $w(x)$. When a pair of c_0 and $w(x)$ are used in eq.(31), then for any non-zero constant α, the pair of αc_0 and $\alpha w(x)$ can be used as c_0 and $w(x)$ in eq.(31).

Differential Equation in Psychophysics

Eisler (1963) used a differential equation to represent intrasubjective relations in psychophysics, e.g. relation between magnitude and category scales. By assuming proportionality between the two scales, x and y (according to Eisler's notation, x and y represent subjective scales of stimulus Φ), he derived the following equation

$$\frac{dy}{dx}=\frac{\sigma_y(y)}{\sigma_x(x)}, \tag{32}$$

where $\sigma_x(x)$ and $\sigma_y(y)$ are Weber functions of independent variables x and y.

Okamoto (1994) applied eq.(32) to cross modality matching. He set the following model:

$$\frac{dR^{(i)}}{\sigma_i}=\alpha\frac{dR^{(j)}}{\sigma_j}, \tag{33}$$

where $R^{(i)}$ and $R^{(j)}$ are sensory values (mean values) of physical simuli $S^{(i)}$ and $S^{(j)}$, which are of two modalities, i and j, respectively. σ_i and σ_j are variances of sensations of stimuli $S^{(i)}$ and $S^{(j)}$, and α is a constant.

Eq. (33) represents some correspondence between discriminabilities of sensations, when $\frac{dR^{(i)}}{\sigma_i}$ and $\frac{dR^{(j)}}{\sigma_j}$ are considered as indices of discriminability. Moreover, eq. (33) may be considered to reflect dynamic range (Teghtsoonian and Teghtsoonian, 1997). By eq. (33), Okamoto (1994) derived a power function for magnitude estimation and a logarithmic function for category scale.

Mathematical distance and Psychological Meaning

Although the problem is not psychological but merely mathematical, Dzhafarov and Colonius (1999, p.248) pointed out the possibility that the derivative in eq. (7) equals 0 or infinity. In this case, they proposed the following derivative (ibid. p.249),

$$F(x)=\lim_{\Delta x\to 0+}\frac{\sqrt[\mu]{\gamma_x(x+\Delta x)-\gamma_x(x)}}{\Delta x},$$

where μ is some positive constant such that $F(x)$ has a finite positive value.

In case of $\mu=1$, by eq. (8) their distance

$$G(a,\ b)=\int_a^b F(x)\cdot dx$$

gives discrimination probability p by eq. (4'). That is, calculated distances are connected with the probabilities, from which they are calculated. But, in case of $\mu\neq 1$, it is not clear how to calculate the discrimination probability from the distance

$$G(a,\ b)=\int_a^b \lim_{\Delta x\to 0+}\frac{\sqrt[\mu]{\gamma_x(x+\Delta x)-\gamma_x(x)}}{\Delta x}\cdot dx$$

Without the way to calculate the probabilities from which the distances are determined, these distances for $\mu\neq 1$ are mere mathematical objects without psychological meaning.

Conclusion

For the fixed discrimination probability, just noticeable differences are of finite sizes. As a solution for Fechner's problem, we can accumulate the JNDs to get a sensory scale. But, when this cumulative method goes beyond the simple summation, we see troubles, which can be revealed by strict mathematical consideration (Luce and Edwards, 1958).

From theoretical point of view, discrimination probability can be any value between 0 and 1. For the revised Fechner's problem given by Luce and Galanter (1963), discrimination probability π takes any value, so the corresponding JND varies as π varies. Hence, for the revised Fechner's problem, derivation of differential equation can be justified. Considering in the framework of the revised Fechner's problem, Dzhafarov and Colonius (1999) introduced differentials and gave a sensory scale by integration of differentials. But their approach has problems: (1) Although their approach seems to be based on a psychometric function, its core should be considered to belong to Thurstonian model, especially to Case V. So, its relation to characteristics of discrimination in terms of physical continuum such as Weber's function is not clear. (2) Possibility of variability of variances of sensory intensities are not considered. That is, constancy of variances are implicitly assumed.

When we explicitly combine a Thurstonian model with a psychometric function, we get the differential equation (14), where physical and sensory scales are related to Weber function and variance of sensory intensity. In this model eq.(14), we need not to assume the Weber function and the variances to be constant. Hence, eq.(14) can treat of a wide range of possibilities as to sensory scales.

Possibility of variability of variances of sensory intensities is important in case of Stevens' (1936) criticism for number of JNDs beween the same differences of sensation. When variances vary, numbers of JNDs between the same sensory differences are not the same as each other.

Constancy of Weber function, i.e. Weber's law, which is assumed in derivation of the logarithmic law, does not hold strictly. Eq.(14) can be used in case where Weber function is not constant, e.g. in case of the near-miss to Weber's law (Baird, 1997, pp. 56-58; Luce, 1993, p.124).

Fechner's problem treats of construction of a sensory scale. As to sensory scales, two types of scales, the logarithmic and the power laws, are the famous opponents in psychophysics. The logarithmic law is derived from constancy of variance and Weber's law. The power functions are proposed against the logarithmic ones and derived from data of direct methods, or from Ekman's law as the logarithmic law from constant variance of sensation.

Some theorists attempted to unify the two laws. Krueger (1989) reviews works on the four major types of scales, magnitude, partition or category, summated jnd, and neurelectic, and tried to find a common basic underlying scale (ibid. p.252). Norwich (1993) derived theoretically his equation, which includes both the logarithmic and power laws as extreme cases. How to unify the two types of scales, scales constructed from the data of discrimination experiments and scales from those of direct methods, depends on validity of the scales.

The power law is supported by direct methods. But, those methods are criticized not only in the current years (Laming, 1997), but also were rejected, according to Scheerer (1987, pp.199-200), in about a century ago. Baird (1997, chap.8) presented the results of the simulations where the power law was obtained by Number Preference Model. According to Number Preference Model, response numbers are not determined based on sensation ratios, but selected based on the subject's preference for numbers.

When we restrict the problem within discrimination experiments, we must determine how the variances of sensations vary. As to cumulating JNDs, Luce and Edwards (1958, p.237) said in the last paragraph "(most psychophysicists) have stubbornly summated jnd's in the obvious and correct way" (words in parentheses are added by the author). But, this summation of JNDs depends on the assumption that variances of sensations are constant. When we read Stevens' (1936) criticism as to number of JNDs, we find the necessity of empirical supports of this constant variance assumption.

In derivation of the logarithmic law, Weber function is assumed to be constant. However, it is well known that Weber function is not constant (Baird, 1997, pp.54-59). Cases, where variances may vary or Weber functions are not constant, can be treated in the unified framework eq.(14).

Eq.(14) treats exclusively of discrimination data, but includes cases other than of constant variances of sensation or Weber's law. Relation between eq.(14) and direct methods is not discussed in this paper.

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