A Note on the Theorem of K. J. Arrow and D. Levhari Concerning the Uniqueness of the Internal Rate of Return

A NOTE ON THE THEOREM OF K.J. ARROW AND D.LEVHARI CONCERNING THE UNIQUENESS OF THE INTERNAL RATE OF RETURN

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1. Introduction

Consider an investment project having a finite life of n period. Let r be an internal rate of return, then it satisfies the equation

$$0=-A_0+\sum_{k=1}^{n}\frac{A_k}{(1+r)^k}+\frac{R_n}{(1+r)^n} \tag{1}$$

where A_0 is the current replacement cost of the asset, $A_0>0$,
A_k is the expected net yield in period k,
R_n is the expected scrap value of the asset in period $n, R_n \geqq 0$.

If some of the net yields are negative, then (1) may have multiple positive solutions. To obtain the uniqueness of the internal rate of return, P.H.Karmel introduced the idea of the truncation of the period and proved by considering the positive maximum value of r at each truncated period that there exists a truncation of period at which the internal rate of return is uniquely determined [1]. K.J.Arrow and D.Levhari extended the above problem to the case of continuous variable life [2]. One of their results obtained are as follows.

Let $x(t)$ be a continuous stream of net income in period t which cuts the $t-$axis a finite number of times and vanishes for all sufficiently large t. Then the function $\psi(r)$ defined by

$$\psi(r) = \max_{0 \leq T < \infty} \left(\int_0^T e^{-rt} x(t) dt \right)$$

is continuous and monotone decreasing, therefore for any positive constant c, the zeros of $\psi(r) - c$ is uniquely determined if it exists.

However, in their situation , variable life is essentially finite since $x(t)$ is assumed to be zero for all sufficiently large t .

In this note we investigate the case that variable life is infinite and prove by following their idea that the same result holds under some assumptions.

2. The case of infinite variable life

Notations:

$$\exp(x) = e^x,$$

$$\phi(r,T) = \int_0^T \exp(-rt) x(t) dt \quad (r > 0),$$

$$\psi(r) = \max_{0 \leq T \leq \infty} \phi(r,T),$$

$$M(r) = \{T_0 \in \mathrm{R} \cup \{\infty\}; \phi(r,T_0) = \psi(r)\}.$$

Assumptions:

(1) The initial value of net income is negative, that is, $x(0) < 0$.

(2) The set S of zeros of $x(t)$ contains infinitely many elements and has no finite accumulating point, that is if $\{T_j\} \subset S$ and $T_j \to T_o$ then $T_o = \infty$.

(3) $\phi(r, \infty)$ is absolutely integrable, that is,

$$\int_0^\infty | \exp(-rt) x(t) | dt$$

converges for each $r>0$.

(4) $M(r)$ contains a finite number for any $r>0$.

Remark 1. $M(r) \subset S$ for any $r>0$.

Remark 2. The assumption 4 means that there exists a finite truncation of period which maximize the present value.

Lemma 1. $\int_0^\infty t^k exp(-rt)x(t)\mathrm{dt}$ is absolutely integrable for any $k>0$ and $r>0$.

Proof. We fix $r>0$ and $k>0$. Since

$$\lim_{t\to\infty} t^k \ \exp(-rt)=0,$$

there exists a positive number T_0 such that $t^k exp\ (-\frac{r}{2}t) \leq 1$ for all $t \geq T_0$. Moreover, by assumption (3), we may assume that the foll owing inequality holds for any $\varepsilon>0$ and $T \geq T_0$;

$$\int_T^\infty |\exp(-\frac{rt}{2})x(t)|\,dt \leq \varepsilon .$$

Therefore,

$$\int_T^\infty |\,t^k \exp(-rt)x(t)\,|\,dt \leq \int_T^\infty |\,\exp(-\frac{rt}{2})x(t)\,|\,dt \leq \varepsilon .$$

This shows that $\int_0^\infty t^k exp(-rt)x(t)dt$ is absolutely integrable. Q.E.D.

Lemma 2. Let $\{r_j\}$ be a positive sequence which converges to $r_0>0$ and let $\{T_j\}$ be a non-negative sequence which converges to T_0, here T_0 may be infinity ∞ . Then it holds that

$$\lim_{j\to\infty} \phi\,(r_j, T_j) = \phi\,(r_0, T_0).$$

Proof. We show the case when T_0 is infinity. It is easily seen that

$$|h|\,t\exp(\frac{r_0 t}{2})\geq |\,1-\exp(-ht)\,| \qquad \text{when } |h|<\frac{r_0}{2} \text{ and } t\geq 0.$$

From this inequality,

$$|\int_0^{T_j}\exp(-r_j t)x(t)dt-\int_0^{\infty}\exp(-r_0 t)x(t)dt\,|\leq \int_0^{T_j}|\,\{\exp(-r_j t)-\exp(-r_0 t)\}x(t)\,|dt$$

$$+\int_{T_j}^{\infty}\exp(-r_0 t)x(t)\,|dt\leq |r_j-r_0|\int_0^{\infty}|t\exp(-\frac{r_0 t}{2})x(t)\,|dt+\int_{T_j}^{\infty}|\,\exp(-r_0 t)x(t)\,|dt.$$

Therefore by assumption (3) and Lemma 1,

$$\lim_{j\to\infty}|\int_0^{T_j}\exp(-r_j t)x(t)dt-\int_0^{\infty}\exp(-r_0 t)x(t)dt\,|=0. \qquad \text{Q.E.D.}$$

In the same manner as Lemma 2, we can show the following remark.

Remark 3. Put

$$y^{(\varepsilon)}=\{x\in R;\ |x-y|<\varepsilon\},\ \infty^{(\varepsilon)}=\{x\in R;\ |x|>\frac{1}{\varepsilon}\},\text{and } (M(r))^{(\varepsilon)}=\bigcup_{T\in M(r)}T^{(\varepsilon)}.$$

Then for any $\varepsilon>0$, there exists a $\delta>0$ such that for r with $|r-r_0|<\delta$, the inclusion relation $M(r)\subset(M(r_0))^{(\varepsilon)}$ holds.

Lemma 3. For any $r_0>0$, $\min_{T\in M(r_0)}\phi_r(r_0,T)$ exists, where $\phi_r(r_0,T)=\frac{\partial\phi(r_0,T)}{\partial r}$.

Proof. Since $\phi_r(r_0,T)$ is bounded with respect to T,

$$\inf_{T\in M(r_0)}\phi_r(r_0,T)=\lambda$$

is a finite number. Take a sequence $\{T_j\}\subset M(r_0)$ such that $\{\phi_r(r_0,T_j)\}$ converges to λ. If $\{T_j\}$ is a finite set, then this lemma is trivial. If this set is infinite, then infinity ∞ is an accumulating point and since the value $\phi(r_0, T_j)$ does not depend on j, it holds that $\infty \in M(r_0)$ and

$$\lambda = \lim_{j\to\infty} \phi_r(r_0, T_j) = \phi(r_0, \infty). \qquad \text{Q.E.D.}$$

Lemma 4. For any $r_0>0$ and for any finite element $T\in M(r_0)$, there exists a positive constant c which depends only on $r_0>0$ such that

$$\phi_r(r_0, T) < -c.$$

This Lemma has been proved in [2]. But for the sake of self-containment we will give a simple proof.

Let T be a finite element of $M(r_0)$. Since the number of zeros of $x(t)$ not exceeding T is even, we can put them $t_1 < t_2 \dots < t_{2m-1} < t_{2m} = T$ (Fig.1).

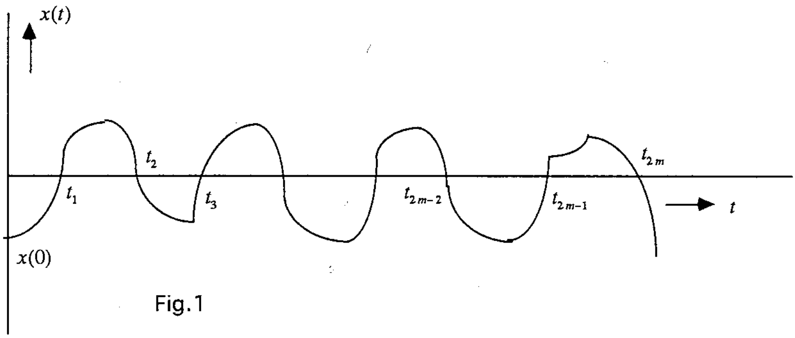

Fig.1

Since $T\in M(r_0)$, it holds that

$$\int_0^T \exp(-r_0 t)x(t)dt - \int_0^{t_{2m-2}} \exp(-r_0 t)x(t)dt = \int_{t_{2m-2}}^T \exp(-r_0 t)x(t)dt \geq 0.$$

Therefore,

$$\int_0^T t\exp(-r_0 t)x(t)dt=\int_0^{t_{2m-2}} t\exp(-r_0 t)x(t)dt+\int_{t_{2m-2}}^{t_{2m-1}} t\exp(-r_0 t)x(t)dt$$

$$+\int_{t_{2m-1}}^{T} t\exp(-r_0 t)x(t)dt\geq\int_0^{t_{2m-2}} t\exp(-r_0 t)x(t)dt+t_{2m-1}\int_{t_{2m-2}}^{t_{2m-1}}\exp(-r_0 t)x(t)dt$$

$$+t_{2m-1}\int_{t_{2m-1}}^{T}\exp(-r_0 t)x(t)dt=\int_0^{t_{2m-2}} t\exp(-r_0 t)x(t)dt+t_{2n-1}\int_{t_{2m-2}}^{T}\exp(-r_0 t)x(t)dt$$

$$\geq\int_0^{t_{2m-2}} t\exp(-r_0 t)x(t)dt+t_{2m-3}\int_{t_{2m-2}}^{T}\exp(-r_0 t)x(t)dt.$$

In the same manner as above,

$$\int_0^{t_{2m-2}} t\exp(-r_0 t)x(t)dt\geq\int_0^{t_{2m-4}} t\exp(-r_0 t)x(t)dt+t_{2m-3}\int_{t_{2m-4}}^{t_{2m-3}}\exp(-r_0 t)x(t)dt$$

$$+t_{2m-3}\int_{t_{2m-3}}^{t_{2m-2}}\exp(-r_0 t)x(t)dt=\int_0^{t_{2m-4}} t\exp(-r_0 t)x(t)dt+t_{2m-3}\int_{t_{2m-4}}^{t_{2m-2}}\exp(-r_0 t)x(t)dt.$$

From these inequalities, we can obtain

$$\int_0^T t\exp(-r_0 t)x(t)dt\geq\int_0^{t_{2m-4}} t\exp(-r_0 t)x(t)dt+t_{2m-3}\int_{t_{2m-4}}^{T}\exp(-r_0 t)x(t)dt.$$

By continuing this process, we obtain

$$\int_0^T t\exp(-r_0 t)x(t)dt\geq\int_0^{t_{2m-4}} t\exp(-r_0 t)x(t)dt+t_1\int_{t_{2m-4}}^{T}\exp(-r_0 t)x(t)dt.$$

$$\geq c+t_l\int_0^T\exp(-r_0 t)x(t)dt,$$

where,

$$c=\int_0^{t_1}(t-t_1)\exp(-r_0 t)x(t)dt>0,$$

that is, $\phi_r(r_0,T)=\int_0^T t\exp(-r_0 t)x(t)dt<-c.$ Q.E.D.

Remark 4. If $M(r_0)$ contains infinitely many elements, then Lemma 4 holds for $T=\infty$. But if $\{\infty\}\in M(r_0)$ and if $M(r_0)-\{\infty\}$ is a finite set, then it is open that whether Lemma 4 holds or not.

Theorem 1. The function $\psi(r)$ is continuous and strictly monotone decreasing on $(0,\infty)$.

Proof. (i). Continuity of $\psi(r)$. Suppose that $\psi(r)$ is not continuou s at $r_0>0$. Then there exists an $\varepsilon_0>0$ which satisfies the following condition:

for any $\delta>0$, there is an r with $|r-r_0|<\delta$ such that $|\psi(r)-\psi(r_0)| >\varepsilon_0$.

Then we can choose a positive sequence $\{r_j\}$ converging to r_0 such that $|\psi(r_j)-\psi(r_0)|>\varepsilon_0$. Let $T_j\in M(r_j)$ $(T_j<\infty)$, and let T_0 be an accumulating point of $\{T_j\}$ (T_0 may possibly infinity). For simplicity, we may assume that $\{T_j\}$ converges to T_0. Then $\psi(r_j)=\phi(r_j,T_j)$ conve rges to $\phi(r_0,T_0)$ by Lemma 2.
Now, for any $T\geq 0$, $\phi(r_j,T_j)$, hence

$$\phi(r_0,T_0)=\lim_{j\to\infty}\phi(r_j,T_j)\geq\lim_{j\to\infty}\phi(r_j,T)=\phi(r_0,T).$$

This means that $T_0\in M(r_0)$ and $\phi(r_0,T_0)=\psi(r_0)$. But this is a contr-adiction since $|\psi(r_j)-\psi(r_0)|>\varepsilon_0$.

(ii). We will show that the following inequality (2) holds for any $r_0>0$:

$$\psi_r^-(r_0)=\lim\frac{\psi(r_0+h)-\psi(r_0)}{h}<0 \tag{2}$$

In fact, let $T_0\in M(r_0), T_0<\infty$. Then

$$\begin{aligned}\psi(r_0-h)&\geq\phi(r_0+h,T_0)=\phi(r_0,T_0)+\phi(r_0,T_0)h+o(h)\\&=\psi(r_0)+\phi_r(r_0,T_0)h+o(h),\end{aligned}$$

here $o(h)$ is a Landau's symbol. Therefore if $h<0$,

$$\frac{\psi(r_0+h)-\psi(r_0)}{h} \leq \phi_r(r_0,T_0)+\frac{o(h)}{h}$$

The above inequality implies that

$$\limsup_{h\to -0}\frac{\psi(r_0+h)-\psi(r_0)}{h} \leq \phi_r(r_0,T_0) \qquad (3)$$

Next, for $T(h)\in M(r_0+h)$, it holds that

$$\psi(r_0+h)=\phi(r_0+h,T(h))=\phi(r_0,T(h))+\phi_r(r_0,T(h))h+o(h)$$
$$\leq \psi(r_0)+\phi_r(r_0,T(h))h+o(h)$$

Therefore if $h<0$,

$$\frac{\psi(r_0+h)-\psi(r_0)}{h} \geq \phi_r(r_0,T(h))+\frac{o(h)}{h}.$$

Put

$$\alpha=\liminf_{h\to -0}\frac{\psi(r_0+h)-\psi(r_0)}{h}$$

then there exists a negative sequence $\{h_j\}$ which converges to 0 such that

$$\lim_{j\to\infty}\frac{\psi(r_0+h_j)-\psi(r_0)}{h_j}=\alpha.$$

Let T^* be an accumulating point of $\{T(h_j)\}$. For simplicity, we may assume that $T(h_j)\to T^*$. Then $T^*\in M(r_0)$ and $\phi(r_0,T^*)\leq\alpha$, so that

$$\phi_r(r_0,T^*)\leq\liminf_{h\to -0}\frac{\psi(r_0+h)-\psi(r_0)}{h}$$

$$\leq\limsup_{h\to -0}\frac{\psi(r_0+h)-\psi(r_0)}{h} \leq\phi_r(r_0,T_0).$$

Now we will show that inequality (3) holds for $T_0=\infty$.

For $t \geq$ and h with $-\frac{r_0}{2} < h < 0$, put

$$f(t) = \frac{h^2 t^2}{2}\exp(\frac{r_0 t}{2}) - ht - \exp(-ht) + 1.$$

Then $f(0)=0$ and

$$f'(t) = h^2(\frac{r_0 t^2}{4} + t)\exp(\frac{r_0 t}{2}) - h + h\exp(-ht),$$

$$f''(t) = r_0 h^2 t(1 + \frac{r_0 t}{8})\exp(\frac{r_0 t}{2}) + h^2(\exp(\frac{r_0 t}{2}) - \exp(-ht)) > 0$$

Consequently, it holds that

$$\frac{h^2 t^2}{2}\exp(\frac{r_0 t}{2}) \geq \exp(-ht) + ht - 1.$$

By the same way,

$$-\frac{h^2 t^2}{2}\exp(\frac{r_0 t}{2}) \leq \exp(-ht) + ht - 1.$$

Therefore we can get

$$|\exp(-(r_0+h)t) + ht\exp(-r_0 t) - \exp(-r_0 t)| \leq \frac{h^2 t^2}{2}\exp(-\frac{r_0 t}{2})$$

Then

$$\left| \int_0^\infty \{\exp(-(r_0+h)t) + ht\exp(-r_0 t) - \exp(-r_0 t)\} x(t)\,dt \right|$$

$$\leq \frac{h^2}{2}\int_0^\infty t^2 \exp(-\frac{r_0 t}{2}) \, |x(t)| \, dt = Kh^2,$$

here, K is a positive constant. Hence

$$\psi(r_0+h) \geq \phi(r_0+h,\infty) \geq \phi(r_0,\infty) + h\phi_r(r_0,\infty) - Kh^2 = \psi(r_0) + h\phi_r(r_0,\infty) - Kh^2.$$

$$= \psi(r_0) + h\phi_r(r_0,\infty) - Kh^2.$$

Thus, if $h<0$, we can obtain

$$\limsup_{h\to -0}\frac{\psi(r_0+h)-\psi(r_0)}{h} \leq \phi_r(r_0,\infty),$$

that is, (3) holds for $T_0=\infty$.

Now, if we take T_0 such as

$$\phi_r(r_0,T_0)=\min_{T\in M(r_0)}\phi_r(r_0,T),$$

then

$$\phi_r(r_0,T_0) \leq \phi_r(r_0,T^*) \leq \liminf \frac{\psi(r_0+h)-\psi(r_0)}{h}$$

$$\leq \limsup \frac{\psi(r_0+h)-\psi(r_0)}{h} \leq \phi_r(r_0,T_0)$$

Therefore (2) holds by Lemma 4 for any $r_0>0$.

(iii). In the next place we will show that $\psi(r)$ is strictly monotone decreasing. Let $0<r_1<r_2$ and suppose that $\psi(r_1)\leq\psi(r_2)$. Since $\psi(r)$ is continuous,

$$\psi(r_0)=\max_{r_1\leq r\leq r_2}\psi(r)$$

exists. If $r_1<r_0\leq r_2$, then for r with $r_1<r<r_0$, it holds that

$$\frac{\psi(r)-\psi(r_0)}{r-r_0}\geq 0$$

But from (2), this is a contradiction so that $r_1=r_0$ and $\psi(r_1)=\psi(r_2)$.

Next, put

$$\psi(r^{\hat{}})=\min_{r_1\leq r\leq r_2}\psi(r).$$

If $\psi(r^{\hat{}})<\psi(r_2)$ then, by restricting $\psi(r)$ on $[r^{\hat{}},r_2]$, we get the same contradiction, therefore $\psi(r)$ must be constant on $[r_1,r_2]$. But this is impossible by (2). Therefore $\psi(r_1)>\psi(r_2)$, that is, strictly monotone decreasing. Q.E.D.

Now we add one more assumption:

(5) $\int_0^{\infty} x(t)dt=\infty$.

If the internal rate of return is 0, then the sum of total net income during the infinite period will be infinity. Therefore (5) seems to be a natural assumption.

Now, by summing up the contents obtained so far, we can get the following main theorem under assumptions (1)−(5).

Theorem 2. For any positive constant c, there exists a unique internal rate of return r which satisfies the equality $\psi(r)=c$.

Proof. It is enough to show that

$$\lim_{r\to\infty}\psi(r)=0, \qquad \lim_{r\to 0}\psi(r)=\infty.$$

(i). Take an increasing sequence $\{r_j\}$ such that $r_j\to\infty$. Let $T_j\in M(r_j)$, $T_j<\infty$. Fix a positive number h and let $\overline{r_j}=r_j-h$. Evidently we may assume that $\overline{r}_j>0$. Put $f(t)=\exp(-ht)x(t)$ and rewrite $\overline{r_j}$ by r_j. Then

$$\int_0^{t_1}\exp(-r_j t)f(t)dt<\exp(-r_j t_1)\int_0^{t_1}f(t)dt \qquad ,$$

$$\int_{t_1}^{t_2}\exp(-r_j t)f(t)dt<\exp(-r_j t_1)\int_{t_1}^{t_2}f(t)dt.$$

Therefore, it holds that

$$\int_0^{t_2}\exp(-r_j t)f(t)dt<\exp(-r_j t_1)\int_0^{t_2}f(t)dt.$$

By proceeding this process, we can obtain

$$\int_{t_{2k-2}}^{t_{2k}} \exp(-r_j t) f(t) dt < \exp(-r_j t_{2k-1}) \int_{t_{2k-2}}^{t_{2k}} f(t) dt, \qquad (1 \leq k \leq m),$$

here, $t_0=0$ and $t_{2m}=T_j$.

From these inequalities,

$$\psi(r_j) = \int_0^{T_j} \exp(-r_j t) f(t) dt \leq \sum_{k=1}^{m} \exp(-r_j t_{2k-1}) \int_{t_{2k-2}}^{t_{2k}} f(t) dt \leq \sum_{k=1}^{m} \exp(-r_j t_1) \int_{t_{2k-2}}^{t_{2k}} |f(t)| dt$$

$$= \exp(r_j t_1) \int_0^{T_j} |f(t)| dt \leq \exp(r_j t_1) \int_0^{\infty} |f(t)| dt.$$

Therefore $\psi(r_j) \to 0 (r_j \to \infty)$. Since $\psi(r)$ is monotone decreasing, this shows that $\lim_{r\to\infty} \psi(r) = 0$.

(ii). For any $T>0$,it holds that

$$\lim_{r\to 0} \psi(r) \geq \lim_{r\to 0} \int_0^T \exp(-rt) x(t) dt = \int_0^T x(t) dt.$$

Then by assumption (5), $\lim_{r\to 0} \psi(r) = \infty$. Q.E.D.

Reference

[1]. Karmel,P.H., " The Marginal Efficiency of Capital," Economic Record,Vol. XXXV, No. 72 (1959), 423-434.

[2]. Arrow,K.J., Levhari, D., " Uniqueness of the Internal Rate of Return with Variable Life of Investment," The Economic Journal, Sept.,1969, 560-566.