On Some Formulas of Integral Geometry on the Sphere.

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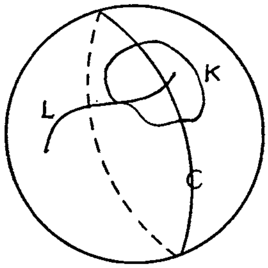
L. A. Santaló has demonstrated many fine integral formulas on the surface of consant curvature and on the general surface. In the present paper we shall demonstrate integral formulas on the sphere by an elementary method used by W. Blaschke and L. A. Santaló. This method is very simple and less complicated and applicable to integral geometry on the general surface.

§ 1. The formula of the curve which intersects with the domain.

By the term domain we will designate a bounded smaller region of the surface on the unit sphere, limited by a finite number of closed curves without double points (as to the terminogy used herein refer to the "Introduction to Integral Geometry" by L. A. Santaló). Let K be a fixed domain and L be a mobile rectifiable curve which is not a great circle on the sphere. Let S be the length of L and l be the length of L inside K.

Then we want to evaluate the integral

$$I=\int_{L\cdot K\neq O} l\,dL$$



(図 1)

We consider the mobile great circle C which intersects L inside K.
We assume that P is the point at which L intersects with C inside K.
Now we consider the integral

$$I=\int n\,dL\,dC$$

where n means the number of intersection points of L with C inside K.
First leave L fixed, then the integral gives

$$I=2\int l\,dL$$

On the other hand leave great circle C fixed, then the integral gives

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$$I = 4\int S\varphi dC$$

where φ is the length of C inside K. But we have the following formula by L. A. Santaló.

$$\int \varphi dC = \pi F$$

where F is the area of the domain K. Hence we obtain the following formula :

$$\int l dL = 2\pi\, SF.$$

This formula has the same form as that for the case of the plane. (See Blashke[1], § 3)

§ 2. Formula of small circle on sphere.

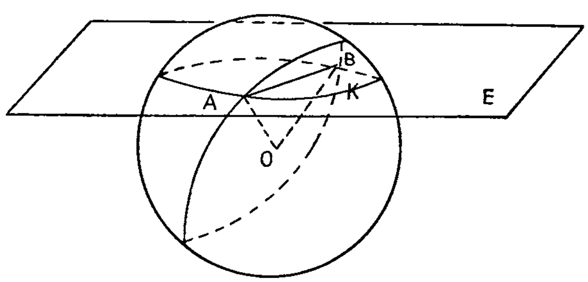
Let K be the intersection of a unit sphere O with a fixed plane E, which does not contain the center of the sphere O and let C be the great circle on the sphere O. we assume that the points A and B are intersection points of C with the circumference of K.

Put $\angle AOB = \omega$.

We want to evaluate the integral

$$I = \int_{K\cdot C \neq O} \sin\frac{\omega}{2}\, dC.$$

Let P be a point on the chord $A\,B$ and consider the integral



(図 2)

$$I = \int_{P\in AB,\ C\cdot K \neq O} dP dC$$

First leave C fixed, then the integral gives

$$I = \int 2\sin\frac{\omega}{2}\, dC = 2\int \sin\frac{\omega}{2}\, dC.$$

We may also leave the point P fixed and we get

$$I = \pi \int d\mathrm{P} = \pi F$$

where F is the area of K.

Hence we obtain the formula

$$\int \sin \frac{\omega}{2}\, dC = \frac{\pi}{2} F \quad ,,,,,,,,, \tag{2.1}$$

Now we consider the spherical crown made by plane E which cuts the sphere. We have the following formula by Santaló

$$\int \omega\, dC = \pi\, F_0 \tag{2.2}$$

where F_0 is the area of the spherical crown.
By (2.1) and (2.2) we get the formula

$$\int \left(\omega - sin \frac{\omega}{2}\right) dC = \pi \left(F_0 - \frac{F}{2}\right).$$

Bibliography

(1) W. Blaschke, Vorlesugen über Integral Geometrie.
(2) L. A. Santaló. Introduction to Integral Geometry.
(3) 〃 〃 Integral Geometry on surface.