Power series rings

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In this note, we mainly consider a one-dimensional complete local domain R such that R has a coefficient field which is extendable to a certain coefficient field of its derived normal ring.

Let R be a complete local ring containing a field. Then R contains a coefficient field k. Suppose that a complete local ring R' is integral over R. If k is a perfect field then R' contains a coefficient field k' which has k as a subfield (cf.[3]). For a non-perfect k, R' does not necessarily contain such a field even though R' is the derived normal ring of R. In fact, suppose that k is a field of characteristic $p>0$ and that there exists an element a in k with $a \notin k^p$. Consider the polynomial ring $k[z]$ in a variable z. Let R be the completion of $k[z]$ at the prime ideal (t) where $t=z^p-a$, as in Cohen [1]. The residue field of R is isomorphic to $k(a^{1/p})$, and R contains the formal power series ring $k[[t]]$, over which R is finite integral. Let S be the subring $k[[zt,t]]$ of R. Then R is the derived normal ring of S, but k is not contained in a coefficient field of R.

Now we consider some elementary examples of complete local domains. An example of such a local domain may be found as the completion of the local ring at a unibranched point on some algebraic curve. Suppose that $f(X,Y)=h(X,Y)+g(X,Y) \in k[X,Y]$ is an irreducible element in $k[[X,Y]]$, where $h(X,Y)$ is the non-zero homogeneous part of degree $m>0$ and $g(X,Y) \in (X,Y)^{m+1}$. By Hensel's Lemma, h is a power of a linear form, and we may assume that $h=Y^m$. Let C be the affine curve defined by the equation $f(X,Y)=0$ and let R be the local ring of the origin on C. Then R is an analytically irreducible local domain with $dimR=1$, $e(R)=m$ and $embdimR \leq 2$, where e denotes the multiplicity. Let x and y be the canonical images of X and Y in R respectively. Since $(x,y)^n=x(x,y)^{n-1}$ for $n \geq m$, x is a superficial element of the ideal (x,y) in R. Then the first neighbourhood ring of R is equal to $R[y/x]$, which is the localization of $k[x,y/x]$ with respect to the maximal ideal $(x,y/x)$.

Generally, it may be easy to check the irreducibility of $f(X,Y)$ in the power series ring $k[[X,Y]]$. As a simple example, consider the case when $f(X,Y)=Y^m+X^{mn+1}$. The origin is a heigher order cusp of multiplicity m. In fact, after n successive blowings up we get the increasing sequence of neighbourhood rings terminated in a discrete valuation ring, which is the localization of $k[yx^{-n}]$ with respect to the maximal ideal (yx^{-n}), and hence, we conclude that the polynomial Y^m+X^{mn+1} is analytically irreducible, i.e., an irreducible element in $k[[X,Y]]$.

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Of course, the situation does not change if the above $f\ (X,Y)$ is replaced by $f\ (X,Y)+q\ (X,Y)$ with $q\ (X,Y)\in(X,Y)^{mn+2}$.

Next put $f(X,Y)=Y^p+X^{2p}+X^{(n+1)p+1}$, where $char(k)=p$. Letting $Z=Y/X$, we have a transformed polynomial $(Z+X)^p+X^{pn+1}$, and hence $f\ (X,Y)$ is analytically irreducible by the same reason as above. Generally, let m and n be relatively prime integers with $m,n\geq 2$. Then the equation: $Y^m-X^n=0$ defines a irreducible rational curve, which has one higher oder cusp at the origin.

Conversely, if a singularity at the origin is resolved in the similar way as above, then the defining polynomial may be analyticaly ireducible, and it can be stated as the following:

(1) *Suppose that* $f(X,Y)=h\ (X,Y)+g\ (X,Y)$ *satisfies the above situation, that is,* $h(X,Y)=V^m$ *with* $V=aX+bY$ *a linear form and* $g\ (X,Y)\in(X,Y)^{m+1}$, *and letting* $\tilde{f}(U,V)=V^m+\tilde{g}(U,V)$ *be a transformed polynomial after a suitable non-singular linear change,* $\tilde{g}(U,V)$ *is regular in U. Let* $f_1(X_1,\ Y_1)$ *be the strict quadratic transformation of* $\tilde{f}(U,V)$ *with* $X_1=U$ *and* $Y_1=V/U$. *If the curve* $f_1(X_1,\ Y_1)=0$ *has a singular point at the origin* O_1, *we assume that the situation for the polynomial* $f_1(X_1,Y_1)$ *is the same as that of* $f(X,Y)$. *Then the above process may be applied to the polynomial* $f_1(X_1,Y_1)$. *Continuing the similar process, if we arrive at a curve* C': $f_n(X_n,Y_n)=V_n+g_n(X_n,Y_n)=0$ *with* V_n *a linear form*$\neq 0$, *then we may conclude that the defining polynomial* $f(X,Y)$ *is irreducible in* $k[[X,Y]]$. *Thus, a polynomial* $f(X,Y)$ *is an irreducible element in* $k[[X,Y]]$ *if and only if the above procedure can be taken for the polynomial* $f(X,Y)$.

For example, let C_m be the projective plane curve defined by the equation $X_2^mX_0^2-X_1^{m+1}X_0+X_2^{m+2}=0$. On $X_0\neq 0$, the affine form of C_m is given by $f(X,Y)=Y^m-X^{m+1}+Y^{m+2}=0$. Then $f(X,Y)$ satisfies the above situation, and hence it is irreducible in $k[[X,Y]]$. Clearly, C_m is an irreducible curve with an m-fold point at $P(1,0,0)$. Since C_m has one simple point in its first neighbourhood, the genus $g(C_m)=(m+1)m/2-m(m-1)/2=m$. Let v be the valuation centered at P on C_m, and let I be the ideal generated by the canonical images of f_X and f_Y in the affine coodinate ring of C_m. Calculating the analytic branch at P, we have $v\ (I)=(m+1)(m-1)$, and hence the class of $C_m=(m+2)(m+1)-v(I)=3m+3$ $(char(k)=0,\ m>1)$.

(2) *The curve* C_m *is a hyperelliptic curve of genus m and class 3m+3. It has one sigular point P, an m-fold simple cusp, and hence just one blowing up P already resolves the singularity of* C_m. *The linear system cut out by the lines through P defines a degree 2 morphism:* $\tilde{C}_m\to P^1$.

In general, the value $v(I)$ should be replaced by $v(J)$, where J is the conductor at P. Fortunately, the singularity at P is resolved by only one blowing up, and hence $J=M_p^{m-1}$, where M_p is the maximal ideal of the local ring at P, i.e., $v(J)=(m-1)v(M_P)=(m-1)(m+1)$.

Now we consider the following example. The surface F defined by $ZX^m-Y^m+X^{m+1}=0$ contains the line L: $X=Y=0$ as a singular subvariety with multiplicity m. Let R be the local ring of this subvatiety L on F. Then R is localization of $k(z)[x,y]$ at the maximal ideal (x,y), where x,y and z are the canonical images of X,Y and Z respectively in the coordinate ring of the surface F, and the first neibourhood ring of R is $R_1=R[y/x]$. Since $z=v^m-x$ with $y=vx$, R_1 is a regular local ring with maximal ideal xR_1 and residue field $k(z^{1/m})$. Define a flat morphism $F\to L$ by $(X,Y,Z)\to Z$. The fibre over $a\in k$ is an irreducible curve defined by $aX^m-Y^m+X^{m+1}=0$. Let R_a be its local ring at the origin. Then we have the following:

(3) *R is a 1-dimensional local domain with multiplicity $e(R)=m$, and its associated graded ring is an integral domain. Associated with R, there is an infinite family of 1-dimensional local domains R_a with $e(R_a)=m$ for all $a\in k$. If char $(k)=0$, then R_a is unibranched only at $a=0$. If char$(k)=m$, then R_a is unibranched at all a.*

In what follows, R will denote a 1-dimensional Macaulay local ring with maximal ideal M. Then we have the following:

(4) *The associated graded ring of R is an integral domain if and only if its first neighbourhood ring $\tilde{R}$ is a discrete valuation ring with maximal ideal $\tilde{M}=M\tilde{R}$ and $M^n R\cap R=M^n$ for all n. In this case, R is analytically irreducible, and the multiplicity $e(R)$ is equal to $|\tilde{R}/\tilde{M}: R/M|$.*

In fact, let $G=R[Mt]/MR[Mt]$ be the associated graded ring, and assume that G is an integaral domain. Then R is an integral domain, and since (0) is the unique relevant prime divisor of the zero ideal in G, M^s-M^{s+1} is equal to the set of surperficial elements of degree s (cf.[2]). Applying the theory of degree 0 localization to the blowup algebra $R[Mt]$, $M\tilde{R}$ is the unique maximal (principal) ideal of $\tilde{R}$, and hence $\tilde{R}$ is a discrete valuation ring. Finally, we have $M^n R\cap R=M^n$ for all $n=1,2,\ldots$, since $M\tilde{R}$ has no irrelevant prime divisor. Conversely, assume that these conditions are satisfied for R. Since $M^n\tilde{R}\cap R=M^n$ for all $n=1,2,\ldots$, the ideal (0) is unmixed of height 0, and since $M\tilde{R}$ is prime, so is $MR[Mt]$ also. Thus G is an integral domain. Since the associated graded ring of R is isomorphic to that of the completion R^* of R, R^* is an integral domain, hence an analytically irreducible local domain. The last assertion is obvious since $e(R)=length_R\tilde{R}/M\tilde{R}=length_R\tilde{R}/\tilde{M}$.

Now assume that R contains a field and satifies the conditions stated above, and also assume that the residue field is a perfect field. Then the completion R^* satisfies the similar conditions. It is integral and contains a coefficient field k which is perfect. The integral closure V of R^* is a complete discrete valuation ring with the unique coefficient field K which contains k. The completion of $\tilde{R}$ is identified with V, and V itself is the first neighbourfood ring of R^*. In particular, we have the following:

(5) *In the situation as above, assume that R is complete. Then R contains a transcendental element t and a finite k-module $L\subseteq K$ with $k\subseteq L$ such that*

(a) *$R=k[[f_1,f_2,\ldots,f_r]]\subseteq K[[t]]$, where $f_i=a_i t+\cdots\in tK[[t]]$ and a_i, $1\le i\le r$ are linearly independent over k.*

(b) *Letting $L=ka_1+\cdots+ka_r$, we have $k(L)=K$, and hence $L^c=K$, $L^{c-1}\neq K$ for some $c\geq 0$.*

(c) *The associated graded ring is isomorphic to $k[Lt]$, and the multiplicity $e=e(R)$ is equal to $|K: k|$.*

In fact, since $\tilde{R}$ contains a unique coefficient field K with $k\subseteq K$ and is complete with $\tilde{M}=$

$M\tilde{R}$, we have $\tilde{R}=K[[t]]$ for some $t \in R$ and $k[[t]] \subseteq R \subseteq K[[t]]$. Let L be the set of elements $a \in K$ with $f=at+\cdots \in R$. *Since* K is finite algebraic over k, L is a finite-dimensional k-module with $k \subseteq L$. Since $M/M^2 \cong M/t^2K[[t]] \cap R \cong M+t^2K[[t]]/t^2K[[t]] \cong L$ by (4) and G is integral, R is equal to a power series ring stated as in (a), and G is isomorphic to a polynomial ring $k[Lt]=k+Lt+L^2t^2+\cdots$. Thus the proof is complete.

As a corollary to (5), we have the following:

Let K be a finite extension field of a field k, and let L be a k-module contained in K with $k(L)=K$ and $k \subseteq L$. Let $f_i=a_i t+\cdots \in tK[[t]]$, $1 \leq i \leq r$ be such that the coefficients a_i generate L over k. Then the following are equivalent for the power series ring $R=k[[f_1, f_2, \cdots, f_r]]$:

(a) *The associated graded ring of R is an integral domain.*

(b) *For any $f=at^n+bt^{n+1}+\cdots \in R$ with $n \geq 0$, $a \in L^n$.*

For example, let $f=t, g=(\sqrt{2}+\sqrt{3})t$ and $h=t+\sqrt{3}t^2$. Then $Q[[f,g,h]]$ does not satisfy the above condition.

Let R be a complete local domain as in (5). If $dim_k L=2$, then R is Gorenstein. Generally we have the following:

(6) *Let d_i denotes $dim_k L^i$, $i=0,1,\cdots,c-1$ $(d_0=1)$. With the notation as in* (5), *R is Gorenstein if and only if the following equality holds: $ec=2(d_0+d_1+\cdots+d_{c-1})$. $(d_{-1}=0)$*

In fact, the conductor between R and $\tilde{R}=K[[t]]$ is equal to the ideal$D=t^cK[[t]]$, and hence $length_R R/D=d_0+d_1+\cdots+d_{c-1}$, since $M^i/M^{i+1} \cong L^i$ for $i=0,1,\cdots,c$-1 and $M_c=D$. On the other hand, $length_R\ K[[t]]/R=length_R K[[t]]/D-length_R R/D=ec-length_R R/D$, and hence the assertion is proved.

In particular, we see that for a complete local domain R with $K[[t]]$ as its derived normal ring, the value group of R may be closely related to the structure of R only in the case where the coefficient field of R is equal to K. Finally, we add an ementary example, different from the above type. Let R be a power series ring $K[[t^{e_1},\cdots,t^{e_r}]]$ contained in $V=K[[t]]$. We assume that $1<e_1<e_2<\cdots<e_r$ and these r integers are relatively prime. Then, by $H=<e_1, \cdots, e_r>$ we denote the semigroup which is generated by these r elements $(0 \in H)$. Assume that they form a minimal set of generators of H. It contains all large integers, and hence V is the derived normal ring of R. By $T(H)$ we denote the semigroup which is generated by H and all differences e_i-e_1. Let $d(H)$ denote e_2-e_1 if $r \geq 2$, 0 if $r=1$ and R_n the n-th neighbourhood ring of R.

Then we have the following:

(7) (a) *The least integer n with $R_n=V$ is equal to the integer m with $d(T^{m-1}(H))=1$. $(T^0(H)=H)$*

(b) *Suppose that $R_1=V$. Then the associated graded ring G of R is Macaulay, i.e., it has a regular homogeneous element of positive degree if and only if $\{e \in H \mid e \geq ne_1\} \subseteq H_n+H$, $n=1,2,\cdots$, where H_n is the set of elements z such that z is a sum of n elements in $\{e_1,\cdots, e_r\}$. In this case, $e_r<2e_1$.*

Let M denotes the maximal ideal $(t^{e_1},\cdots,t^{e_r})$, and let v denotes the associated valuation of V. The assertion (a) is easy. In fact, since V is finite over R, the multiplicity $e=e(M)=e_V(MV)=e_1=e_V(t^{e_1}V)=e(t^{e_1}R)=length_R R/t^{e_1}R$. Then $length_R M^{n+1}/M_n t^{e_1}=length_R R/M^n t^{e_1}-length_R R/M^{n+1}=length_R R/M^n+length_R M^n/M^n t^{e_1}-length_R R/M^{n+1}=en-r+e-(e\ (n+1)-r)=0$ for all large n.

Thus, we see that $M^{n+1}=M^n t^{e_1}$ for all large n. Hence t^{e_1} is a superficial element of M.

This implies that the first neighbourhood ring of R is equal to $\tilde{R}=R[Mt^{-e_1}]$, and hence that $T(H)$ is the semigroup attached to $\tilde{R}$. Now, suppose that $\tilde{R}=V$. Then, $R[t^{e_2-e_1},\cdots,t^{e_r-e_1}]=K[[t]]$. This implies $e_2-e_1=1$. Conversely, if this equality holds, then $t=t^{e_2}/t^{e_1}\in\tilde{R}$, and hence $\tilde{R}=V$. According to these results, we get the assertion (a).

As for (b), we note that G is Macauly if and only if $M^n V\cap R=M^n$ *for* $n=1,2,\cdots$. We can easily prove this by the theory of degree zero localization and primary decomposition, and we omit the proof. With the notation as in (b), M^n is the set of formal sums of elements at^e with $a\in K$ and $e\in H_n+H$, and $M^n V\cap R$ is equal to the set of formal sums of elements at^e in R such that $a\in K$, $e\in H$ and $e\geq ne_1$. Then the assertion (b) easily follows from these facts. It may be interesting to find some concrete types of the semigroup H which satisfies the condition (b).

References

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