Stability estimates and Lagrange-Galerkin schemes for Navier-Stokes type models of flow in non-homogeneous porous media

Dissertation Abstract

Stability Estimates and Lagrange-Galerkin Scheme for Navier-Stokes Type Models of Flow in non-Homogeneous Porous Media

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1 Introduction

Fluid flow and heat transfer in porous media have received considerable attention in many kinds of applications such as in geophysics, petroleum engineering, and geothermal engineering. In geothermal engineering, simulation of fluid flow and heat transfer in porous media is a useful tool not only for the pre-exploration process but also during the exploration process. For the pre-exploration process, the simulation can be used to predict how much electricity can be generate and to determine how long the reservoir can be explored by using the physical parameters such as pressure, temperature, density, porosity, size of the reservoir, and the type of reservoir obtained from seismic data as an input parameter. From this simulation, we can also determine the feasibility of a reservoir to be explored. During the exploration, simulation is used to predict the pressure and temperature changes in the reservoir because of the injection and extraction processes. The injection is needed to maintain the balance of the mass in the reservoir and to supply the water, which will be heated by the reservoir. In the extraction process, the fluid and steam are exploited from the reservoir and used to generate electricity. Because of its process, the flow in the geothermal reservoir is not a steady flow, so we can not deal with the Darcy equations to approaches such kind of phenomena. To deal with the phenomena in the geothermal reservoir, we applied the non-steady flow equations in the porous media proposed by C.T.Hsu and P. Cheng.

C.T.Hsu and P. Cheng in (1989) applied the volume averaging technique introduced by S. Whitaker [10] into the Navier-Stokes equations [4]. In the process of the derivation, they got the expression of total drag force per unit volume due to the presence of solid particles in the integral boundary form. To overcome the difficulty, they adopted the Darcy-Brinkmann-Forchheimer model of the drag force [9, 6]. In this model, it consists of two-terms, the first term is related to Darcy term and the second term is connected to the Forchheimer term. The Forchheimer term plays an essential role to establish the stability energy estimate of the model proposed by C.T.Hsu and P. Cheng in our study.

C.T. Hsu and P. Cheng have proposed the equations of non-steady flow in the porous media. However, any mathematical and numerical analysis for their model has not been studied. The aims of this thesis are to prove the L^2-stability estimate of the model, to propose an appropriate numerical method and to perform some simulation of fluid flow in simple and complex structures of the porosity. The key idea to establish the energy stability estimate is to control the nonlinear term with the non-homogeneous porosity ϕ (convection term) by using the Forchheimer term. As a numerical scheme, we proposed a characteristic finite element method (Lagrange-Galerkin scheme). We extended the idea of the characteristics method and introduced the macroscopic average velocity w to overcome the difficulty which comes from the convection term with the non-homogeneous porosity ϕ.

2 Statement of the Problem

In this section, we introduce a mathematical framework and state the problem that we will work on. The detail of the derivation of the model we can find in [4].

The notation to be used in this paper is as follows. For $d = 2, 3$, let $\Omega \subset \mathbb{R}^d$ be a bounded domain, Γ the boundary of Ω, and T a positive constant. The boundary Γ is divided into three parts, Γ_i, $i = 0, 1, 2$, which satisfy $\bar{\Gamma} = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ and $\Gamma_i \cap \Gamma_j = \varnothing$ for all $i \neq j$. We suppose that Γ is a Lipschitz boundary, and that, for each $i \in \{0, 1, 2\}$, Γ_i is piecewise smooth, where the total number of the smooth boundaries of Γ_i is finite. The Lebesgue space on Ω for $p \in [1, \infty]$ is denoted by $L^p(\Omega)$ and the Sobolev space $W^{1,2}(\Omega)$ is denoted by $H^1(\Omega)$ with the norm

$$\|u\|_{H^1(\Omega)} \equiv \Big(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\Big)^{1/2}.$$

The vector- and matrix-valued function spaces corresponding to, e.g., $L^2(\Omega)$ are denoted by $L^2(\Omega)^d$ and $L^2(\Omega)^{d\times d}$, respectively. The inner products in $L^2(\Omega)$, $L^2(\Omega)^d$, and $L^2(\Omega)^{d\times d}$ are all represented by $(\cdot,\cdot)$.

We consider the following problem of the C.T Hsu and P. Cheng model with non-homogeneous porosity [4]; find $(u,p) : \overline{\Omega}\times[0,T] \to \mathbb{R}^d \times \mathbb{R}$ such that

$$\rho\Big[\frac{\partial u}{\partial t} + (u\cdot\nabla)\frac{u}{\phi}\Big] - \nabla\cdot[2\mu D(u)] + \nabla p = f + B(u,\phi) \qquad \text{in } \Omega\times(0,T), \tag{1a}$$

$$\nabla\cdot u = 0 \qquad \text{in } \Omega\times(0,T), \tag{1b}$$

$$u = g \qquad \text{on } \Gamma_0\times(0,T), \tag{1c}$$

$$2\mu D(u)n - pn = 0 \qquad \text{on } \Gamma_1\times(0,T), \tag{1d}$$

$$[2\mu D(u)n - pn]\times n = 0 \qquad \text{on } \Gamma_2\times(0,T), \tag{1e}$$

$$u\cdot n = 0 \qquad \text{on } \Gamma_2\times(0,T), \tag{1f}$$

$$u = u^0 \qquad \text{in } \Omega,\ \text{at } t=0, \tag{1g}$$

where u is the Darcy velocity, p is the pressure, $\mu > 0$ is a dynamic viscosity, $u^0 : \Omega \to \mathbb{R}^d$ is a given initial velocity, $f : \Omega\times(0,T) \to \mathbb{R}^d$ is a given external force, $g : \Gamma_0\times(0,T) \to \mathbb{R}^d$ is a given boundary velocity, $\phi : \Omega \to (0,1]$ is a given porosity, $n : \Gamma \to \mathbb{R}^d$ is the outward unit normal vector, $D(u) : \Omega\times(0,T) \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is the strain-rate tensor define by

$$D(u) \equiv \frac{1}{2}\Big[\nabla u + (\nabla u)^T\Big],$$

and $B(u,\phi) : \Omega\times(0,T) \to \mathbb{R}^d$ is the total drag force from the micro pore structures per unit volume defined by

$$B(u,\phi) = B(u,\phi;\mu,\rho,d_p) := -\frac{\mu\phi u}{K(\phi)} - \rho\frac{F(\phi)\phi\,|u|\,u}{\sqrt{K(\phi)}}, \tag{2}$$

where $F : (0,1] \to (0,\infty)$ and $K : (0,1] \to (0,\infty]$ are functions defined by

$$F(\phi) := \frac{b}{\sqrt{a\phi^3}}, \qquad K(\phi) := \frac{d_p^2\phi^3}{a(1-\phi)^2}, \tag{3}$$

which correspond to Forchheimer constant and Kozeny–Carman absolute permeability, respectively. The constant d_p is a particle diameter, and the values of a and b are empirically given by $a = 150$ and $b = 1.75$ in [6]. The first term and the second term (non-linear term) in equation 2 is called Darcy term and Forchaimmer term, respectively.

On the boundary, we impose the Dirichlet boundary condition on Γ_0, the stress free boundary condition on Γ_1, and the slip boundary condition on Γ_2.

Throughout this thesis, the following two hypotheses are assumed to hold.

Hypothesis 3.2.1. We suppose that $\mathrm{meas}(\Gamma_0) > 0$, $f \in C([0,T];L^2(\Omega)^d)$, $g \in C([0,T];H^1(\Omega)^d)$, and $u^0 \in L^2(\Omega)^d$.

Hypothesis 3.2.1. The porosity satisfies the following.

(i) $\phi \in W^{1,\infty}(\Omega), \quad \phi_0 \equiv \operatorname*{ess.inf}_{x\in\Omega} \phi(x) > 0.$

(ii) $|\nabla\phi| \le \dfrac{2b}{d_p}(1-\phi)$ a.e. in Ω.

Let us introduce constants ϕ_1 and α defined by

$$\phi_1 \equiv \operatorname*{ess.sup}_{x\in\Omega} \phi(x) \le 1, \qquad \alpha \equiv \frac{a(1-\phi_1)^2}{d_p^2\phi_1^2} \ge 0.$$

We note that

$$\operatorname*{ess.inf}_{x\in\Omega} \frac{\phi(x)}{K(\phi(x))} \ge \alpha \ge 0. \tag{4}$$

Remark 2.1. As an example the value of $|\nabla\phi|$ in Lavrans field, Halten Terrace, Norway [2] is 4.336×10^{-5} [cm^{-1}]. In the real situation, the value of $d_p \le 0.02$ [cm] and from the empirical study, S. Ergun [3] suggested the value of $b = 1.75$. Then if we calculate the right hand side term in Hypothesis 3.2.1-(ii), it resulted 157.5 [cm^{-1}]. Obviously, the spatial derivative of the real porosity $\nabla\phi(x)$ satisfies $|\nabla\phi| \ll 157.5$ [cm^{-1}]. By this fact, Hypothesis 3.2.1-(ii) is not restrictive.

For a function $g_0 \in H^{1/2}(\Gamma_0)^d$, let us introduce function spaces $V(g_0)$, V, and Q defined by

$$V(g_0) \equiv \{v \in H^1(\Omega)^d;\ v = g_0 \text{ on } \Gamma_0,\ v\cdot n = 0 \text{ on } \Gamma_2\},\ \ V \equiv V(0),\ \ Q \equiv L^2(\Omega),$$

respectively. We define bilinear forms a_0, b, and c_0, and trilinear forms a_1 and c_1 by

$$a_0(u,v) \equiv 2\mu\big(D(u), D(v)\big), \quad b(v,q) \equiv -\,(\nabla\cdot v, q), \quad c_0(u,v) \equiv \mu\Big(\frac{\phi}{K(\phi)}u, v\Big),$$

$$a_1(u,w,v) \equiv \rho\big((u\cdot\nabla)w, v\big), \quad c_1(\theta,u,v) \equiv \rho\bigg(\frac{F(\phi)\phi\,\theta u}{\sqrt{K(\phi)}}, v\bigg).$$

The weak formulation for problem (1) is to find $\{(u,p)(t) \in V(g(t))\times Q;\ t\in(0,T)\}$ such that, for $t\in(0,T)$,

$$\begin{aligned}\rho\Big(\frac{\partial u}{\partial t}, v\Big) + a_0(u,v) + a_1\Big(u, \frac{u}{\phi}, v\Big) + b(v,p) + b(u,q) + c_0(u,v) + c_1\big(|u|, u, v\big)&\\ = (f(t), v)\,, \quad \forall (v,q)\in V\times Q,&\end{aligned} \tag{5a}$$

$$u(0) = u^0 \quad \text{in } L^2(\Omega)^d. \tag{5b}$$

3 Stability Estimates

To obtain the stability estimates of the model, we used Lemma 3.4 to compute the convection term with the non-homogeneous porosity, then control this term using the Forchaimmer term, then we have :

Theorem 3.2.1. Suppose that Hypotheses 3.2.1 and 3.2.1 hold true. Assume $g = 0$. Suppose that $(u,p) \in (C^1([0,T]; L^2(\Omega)^d) \cap L^2(0,T;V)) \times L^2(0,T;L^2(\Omega))$ satisfies (5). Then, it holds that

$$\begin{aligned}\frac{d}{dt}\Big(\frac{\rho}{2}\|u(t)\|^2_{L^2(\Omega)}\Big) + \frac{\rho}{2}\int_{\Gamma_1} \frac{|u(t)|^2}{\phi} u(t)\cdot n\,ds + \mu\beta_0^2\|u(t)\|^2_{H^1(\Omega)} + \mu\alpha\|u(t)\|^2_{L^2(\Omega)}&\\ \le \frac{1}{4\mu\beta_0^2}\|f(t)\|^2_{L^2(\Omega)},&\end{aligned} \tag{6}$$

where $\beta_0 > 0$ is a positive constant to be defined in (9) below.

Corollary 3.2.2. In addition to the same assumptions in Theorem 3.2.1, suppose that $u \cdot n \geq 0$ on $\Gamma_1 \times [0, T]$. Then, we have the following.

(i) It holds that

$$\sqrt{\rho}\|u\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{\mu}\beta_0\|u\|_{L^2(0,T;H^1(\Omega))} \leq 2\Big(\sqrt{\rho}\|u^0\|_{L^2(\Omega)} + \frac{1}{\sqrt{\mu}\beta_0}\|f\|_{L^2(0,T;L^2(\Omega))}\Big). \tag{7}$$

(ii) It holds that, for any $t \in [0, T]$,

$$\|u(t)\|_{L^2(\Omega)} \leq \exp\Big(-\frac{\mu\alpha}{\rho}\,t\Big)\|u^0\|_{L^2(\Omega)} + \frac{1}{\sqrt{2\rho\mu}\beta_0}\|f\|_{L^2(0,t;L^2(\Omega))}. \tag{8}$$

Lemma 3.2.3 (Korn's inequality, [5, 1])**.** Let Ω be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$, and let Γ_0 be a part of $\partial\Omega$ and piecewise Lipschitz-continuous. Assume $\mathrm{meas}(\Gamma_0) > 0$. Then, there exists a positive constant β_0 such that

$$\beta_0\|u\|_{H^1(\Omega)} \leq \|D(u)\|_{L^2(\Omega)}, \qquad \forall u \in \{v \in H^1(\Omega)^d;\ v = 0 \text{ on } \Gamma_0\}. \tag{9}$$

Lemma 3.2.4. Suppose Hypothesis 3.2.1-(i) holds true. Assume $u \in H^1(\Omega)^d$ and $\nabla \cdot u = 0$ in Ω. Then, it holds that

$$\Big((u\cdot\nabla)\Big(\frac{u}{\phi}\Big), u\Big) = \frac{1}{2}\int_\Gamma \frac{|u|^2}{\phi} u\cdot n\, ds + \frac{1}{2}\Big(|u|^2, (u\cdot\nabla)\frac{1}{\phi}\Big). \tag{10}$$

Proof of Theorem 3.2.1. Substituting $(u, -p) \in V \times Q$ into (v, q) in (5), we have

$$\rho\Big(\frac{\partial u}{\partial t}, u\Big) + a_0(u,u) + a_1\Big(u, \frac{u}{\phi}, u\Big) + c_0(u,u) + c_1(|u|, u, u) = (f, u). \tag{11}$$

We evaluate each term in (11) as follows:

$$\begin{aligned}
\rho\Big(\frac{\partial u}{\partial t}, u\Big) &= \rho\int_\Omega \frac{\partial}{\partial t}\left(\frac{1}{2}u_i u_i\right)dx,\\
&= \frac{d}{dt}\left(\frac{\rho}{2}\int_\Omega |u|^2\right)dx,\\
&= \frac{d}{dt}\Big(\frac{\rho}{2}\|u\|^2_{L^2(\Omega)}\Big), && \text{(12a)}\\
a_0(u,u) &= 2\mu\int_\Omega D(u):D(u)dx,\\
&= 2\mu\|D(u)\|^2_{L^2(\Omega)} \geq 2\mu\beta_0^2\|u\|^2_{H^1(\Omega)} \qquad \text{(by Lem. 3.2.3)}, && \text{(12b)}\\
a_1\Big(u, \frac{u}{\phi}, u\Big) &= \rho((u\cdot\nabla)\frac{u}{\phi}, u),\\
&= \frac{\rho}{2}\int_{\Gamma_1}\frac{|u|^2}{\phi}u\cdot n\,ds + \frac{\rho}{2}\Big(|u|^2, (u\cdot\nabla)\frac{1}{\phi}\Big) \qquad \text{(by Lem. 3.2.4)},\\
&\geq \frac{\rho}{2}\int_{\Gamma_1}\frac{|u|^2}{\phi}u\cdot n\,ds - \Big(|u|^2, \frac{\rho|u|}{2}\Big|\nabla\frac{1}{\phi}\Big|\Big), && \text{(12c)}\\
c_0(u,u) &= \mu\Big(\frac{\phi}{K(\phi)}, |u|^2\Big) \geq \mu\alpha\|u\|^2_{L^2(\Omega)} \qquad \text{(by (4))}, && \text{(12d)}
\end{aligned}$$

$$
\begin{aligned}
c_1(|u|,u,u) &= \rho\left(\frac{F(\phi)\phi|u|u}{\sqrt{K(\phi)}},u\right),\\
&= \left(|u|^2,\rho|u|\frac{F(\phi)\phi}{\sqrt{K(\phi)}}\right), && \text{(12e)}\\
(f,u) &\le \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)},\\
&\le \mu\beta_0^2\|u\|_{L^2(\Omega)}^2+\frac{1}{4\mu\beta_0^2}\|f\|_{L^2(\Omega)}^2\\
&\le \mu\beta_0^2\|u\|_{H^1(\Omega)}^2+\frac{1}{4\mu\beta_0^2}\|f\|_{L^2(\Omega)}^2. && \text{(12f)}
\end{aligned}
$$

Here, we note the fact that Hypothesis 3.2.1 yields

$$
G_\phi := \frac{1}{2}\left|\nabla\frac{1}{\phi}\right| - \frac{F(\phi)\phi}{\sqrt{K(\phi)}} = \frac{1}{2\phi^2}\Big[|\nabla\phi| - \frac{2b}{d_p}(1-\phi)\Big] \le 0 \quad \text{a.e. in } \Omega. \tag{13}
$$

Combining (12) with (11) and using (13), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{\rho}{2}\|u(t)\|_{L^2(\Omega)}^2\Big)+\frac{\rho}{2}\int_{\Gamma_1}\frac{|u(t)|^2}{\phi}u(t)\cdot n\,ds+\mu\beta_0^2\|u(t)\|_{H^1(\Omega)}^2+\mu\alpha\|u(t)\|_{L^2(\Omega)}^2\\
&\qquad\le \frac{1}{4\mu\beta_0^2}\|f(t)\|_{L^2(\Omega)}^2+\big(|u(t)|^2,\rho|u(t)|G_\phi\big)\le\frac{1}{4\mu\beta_0^2}\|f(t)\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Thus, we obtain (6). □

4 Lagrange-Galerkin Scheme

We derive the Lagrange-Galerkin scheme by extending the method of characteristic and introduce the macroscopic average velocity $w : \overline{\Omega}\times[0,T]\to\mathbb{R}^d$ to overcome the difficulty comes from the non-homogeneous porosity. Then for the Darcy velocity u and the porosity ϕ in problem (1); the material derivative D/Dt with respect to w defined by

$$
w \equiv \frac{u}{\phi}, \qquad\qquad \frac{D}{Dt}\equiv\frac{\partial}{\partial t}+w\cdot\nabla.
$$

Then, we can rewrite $\partial u/\partial t+(u\cdot\nabla)(u/\phi)$ by

$$
\frac{\partial u}{\partial t}+(u\cdot\nabla)\frac{u}{\phi}=\phi\Big[\frac{\partial w}{\partial t}+(w\cdot\nabla)w\Big]=\phi\frac{Dw}{Dt}. \tag{14}
$$

Let τ be a time increment, $N_T\equiv\lfloor T/\tau\rfloor$ the total number of time steps, and $t^k\equiv k\tau$ for $k\in\{0,1,\dots,N_T\}$. For a function ψ defined in $\overline{\Omega}\times[0,T]$ or $\Gamma_0\times[0,T]$, we denote $\psi(\cdot,t^k)$ simply by ψ^k. Let $X:[0,T]\to\mathbb{R}^d$ be a solution of the following ordinary differential equation,

$$
X'(t)=w(X(t),t),\quad t\in[0,T], \tag{15}
$$

subjected to an initial condition $X(t^k)=x$. Physically, $X(t)$ represents the position of a fluid particle with respect to the macroscopic average velocity w at time t. For a given velocity $v:\Omega\to\mathbb{R}^d$, let $X_1(v,\tau):\Omega\to\mathbb{R}^d$ be the mapping defined by

$$
X_1(v,\tau)(x)\equiv x-v(x)\tau, \tag{16}
$$

which is an upwind point of x with respect to the velocity v and a time increment τ.

Now, we derive the second-order approximation of $\partial u/\partial t+(u\cdot\nabla)(u/\phi)$ at (x,t^k) by the Adams–Bashforth method as follows:

$$\begin{aligned}
&\Big[\frac{\partial u}{\partial t}+(u\cdot\nabla)\frac{u}{\phi}\Big](x,t^k)=\phi(x)\frac{Dw}{Dt}(x,t^k)=\phi(x)\frac{d}{dt}\left(w(X(t),t)\right)_{|t=t^k}\\
&=\frac{\phi(x)}{2\tau}\Big[3w^k-4w^{k-1}\circ X_1\big(w^k,\tau\big)+w^{k-2}\circ X_1\big(w^k,2\tau\big)\Big](x)+O(\tau^2)\\
&=\frac{\phi(x)}{2\tau}\Big[3w^k-4w^{k-1}\circ X_1\big(w^{(k-1)*},\tau\big)+w^{k-2}\circ X_1\big(w^{(k-1)*},2\tau\big)\Big](x)+O(\tau^2)\\
&=\frac{1}{2\tau}\Big[3u^k-\phi\big[4w^{k-1}\circ X_1(w^{(k-1)*},\tau)-w^{k-2}\circ X_1(w^{(k-1)*},2\tau)\big]\Big](x)+O(\tau^2),
\end{aligned}\tag{17}$$

where the symbol "$\circ$" denotes the composition of functions,

$$[v\circ X_1(v,\tau)](x)=v(X_1(v,\tau)(x)),$$

and $w^{(k-1)*}$ is a second-order approximation of w^k defined by

$$w^{(k-1)*}\equiv 2w^{k-1}-w^{k-2}.$$

The idea of (17) has been proposed and employed in [8].

Let $\mathcal{T}_h\equiv\{e\}$ be a triangulation of $\overline{\Omega}\,(=\cup_{e\in\mathcal{T}_h})$, h_e the diameter of $e\in\mathcal{T}_h$, and $h\equiv\max_{e\in\mathcal{T}_h}h_e$ the maximum element size. We define the function spaces X_h,M_h,V_h and Q_h by

$$\begin{aligned}
X_h&\equiv\big\{v_h\in C(\overline{\Omega})^d;\ v_{h|e}\in P_2(e)^d,\ \forall e\in\mathcal{T}_h\big\},\\
M_h&\equiv\big\{q_h\in C(\overline{\Omega});\ q_{h|e}\in P_1(e),\ \forall e\in\mathcal{T}_h\big\},
\end{aligned}$$

$V_h\equiv X_h\cap V$, and $Q_h:=M_h\cap Q=M_h$, respectively, where $P_k(e)$ is the (scalar-valued) polynomial space of degree $k\in\mathbb{N}$ on e.

Let $u_h^0\in X_h$ and $\{g_h^k\}_{k=1}^{N_T}\subset X_h$, approximations of u^0 and g, be given. Our new Lagrange–Galerkin scheme of second-order in time for solving problem (1) is to find $\big\{(u_h^k,p_h^k)\big\}_{k=1}^{N_T}\subset V_h(g_h^k)\times Q_h$ such that, for all $(v_h,q_h)\in V_h\times Q_h$,

(initial step)

$$\begin{aligned}
\Big(\frac{u_h^1-\phi[w_h^0\circ X_1(w_h^0,\tau)]}{\tau},v_h\Big)&+a_0(u_h^1,v_h)+b(v_h,p_h^1)+b(u_h^1,q_h)\\
&+c_0(u_h^1,v_h)+c_1(|u_h^0|,u_h^1,v_h)=(f^1,v_h),
\end{aligned}\tag{18a}$$

(general step)

$$\begin{aligned}
&\Big(\frac{1}{2\tau}\Big[3u_h^k-\phi\big[4w_h^{k-1}\circ X_1(w_h^{(k-1)*},\tau)-w_h^{k-2}\circ X_1(w_h^{(k-1)*},2\tau)\big]\Big],v_h\Big)\\
&\quad+a_0(u_h^k,v_h)+b(v_h,p_h^k)+b(u_h^k,q_h)+c_0(u_h^k,v_h)+c_1(|u_h^{(k-1)*}|,u_h^k,v_h,)\\
&\quad=(f^k,v_h),\quad k=2,\dots,N_T,
\end{aligned}\tag{18b}$$

where w_h^k and $w_h^{(k-1)*}$ are defined by

$$w_h^k\equiv\frac{u_h^k}{\phi},\qquad w_h^{(k-1)*}\equiv 2w_h^{k-1}-w_h^{k-2}.$$

We compute (u_h^1,p_h^1) by (18a) and $\{(u_h^k,p_h^k)\}_{k=2}^{N_T}$ by (18b). This idea on the initial step treatment has been proposed for the Navier–Stokes equations, cf. [9], where the second-order convergence in time in $L^2(\Omega)$-norm has been proved. Here, we apply it to problem (1).

5 Numerical Results

5.1 Experimental Order of Convergence

To test our numerical scheme, for the solution (u_h, p_h) of scheme (18) we define errors $Er1$ and $Er2$ by

$$Er1 \equiv \max_{n=0,\ldots,N_T} \| u_h^n - u^n \|_{H^1(\Omega)}, \quad Er2 \equiv \max_{n=0,\ldots,N_T} \| p_h^n - p^n \|_{L^2(\Omega)} .$$

Figure 1 shows the graphs of $Er1$ and $Er2$ versus h $(= \tau)$ in logarithmic scale. The values of $Er1$, $Er2$ and slopes are represented in Table 1. We can see that both $Er1$ and $Er2$ are almost of second order in h $(= \tau)$.

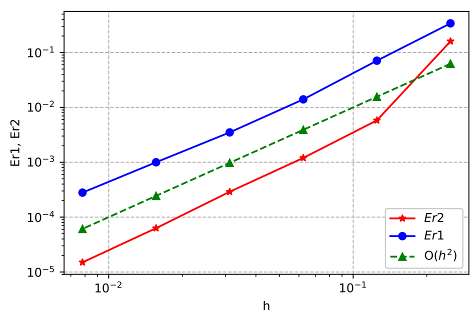


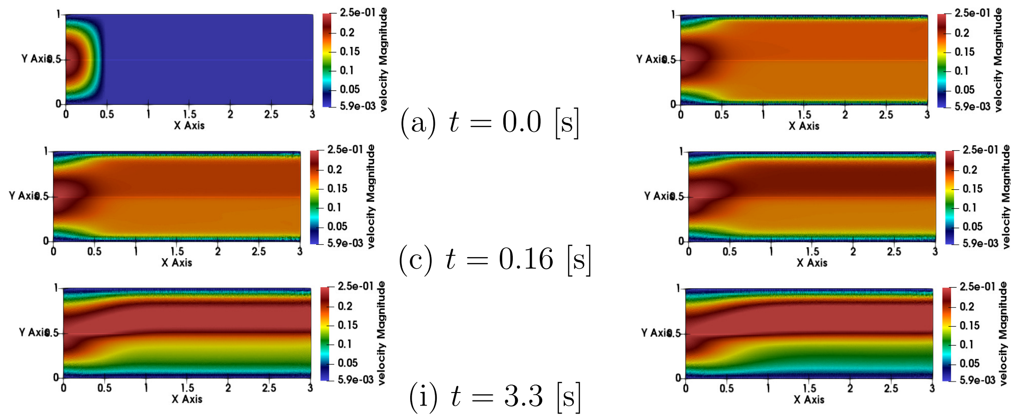
Figure 1: The order of convergence for scheme (18).

Table 1: Values of $Er1$ and $Er2$ and their slopes for the problem 1 by scheme (18).

N	$Er1$	$Er2$	Slope of $Er1$	Slope of $Er2$	CPU times (s)
4	3.4×10^{-1}	1.6×10^{-1}	–	–	0.96
8	7.1×10^{-2}	5.8×10^{-3}	2.2	4.8	5
16	1.4×10^{-2}	1.2×10^{-3}	2.3	2.3	15.1
32	3.5×10^{-3}	2.9×10^{-4}	2.0	2.0	69.7
64	1.0×10^{-3}	6.3×10^{-5}	1.80	2.2	550
128	2.8×10^{-4}	1.5×10^{-5}	1.84	2.1	24.570

5.2 Simulation of Flow in Two Layers of Porosity

This simulation motivated by the real condition of the geothermal reservoir which has porosity function of the depth. In the top of the reservoir, the value of porosity is large, while in the bottom, the value of porosity is small due to the existence of pressure which comes from the mass of the soils and rocks. Figure 2-(a) is the initial condition of the simulation. From this figure, we can see the profile distribution of velocity is symmetric. As long as the time increasing, the profile distribution becomes asymmetric; this happens because of the difference of values of the porosity. It can be understood that high porosity implies high permeability. High permeability means the resistance of fluids to flow is small so that the fluid can flow faster rather than the area with small porosity.



(a) $t = 0.0$ [s] (b) $t = 0.083$ [s]

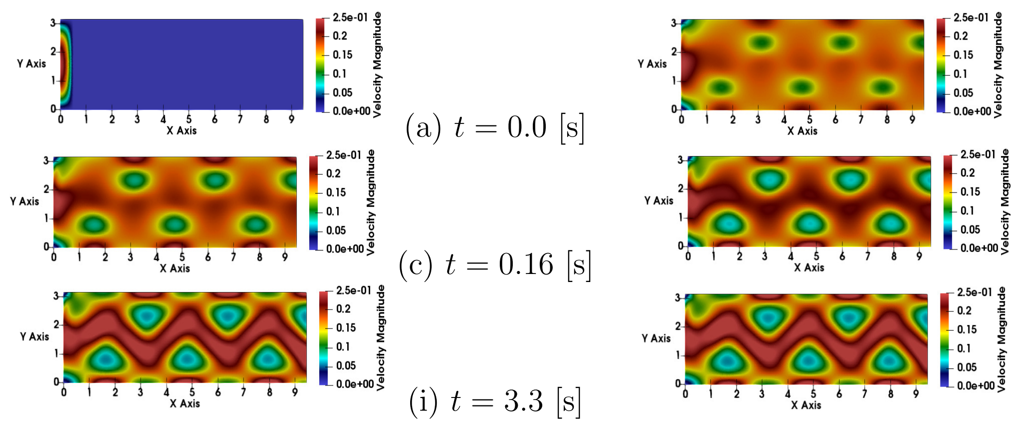
(c) $t = 0.16$ [s] (d) $t = 0.33$ [s]

(i) $t = 3.3$ [s] (j) $t = 5.0$ [s]

Figure 2: Time evolution of velocity magnitude.

5.3 Simulation of Flow in Complex Structure of Porosity

This simulation is motivated by the real condition of the porosity distribution in the rock structure, such as in carbonate rock, where the value of porosity is irregular. Figure 3-(a) is the initial velocity



(a) $t = 0.0$ [s] (b) $t = 0.083$ [s]

(c) $t = 0.16$ [s] (d) $t = 0.33$ [s]

(i) $t = 3.3$ [s] (j) $t = 5.0$ [s]

Figure 3: Time evolution of magnitude velocity.

magnitude of the simulation. From Figure 3, we can see that the fluid is flowing faster in the area which has a large porosity; for the area which has small porosity, the fluid is flowing slowly. In the area which has small porosity, we can see the gradation motion of the fluid clearly; this fact emphasizes us that scheme (18) can deal with the irregular pattern of porosity. Figure 3 has a good agreement with the natural flow in the irregular design of porous media qualitatively.

6 Conclusions

To approaches the phenomena in the geothermal reservoir, we deal with the equations of non-steady flow in the non-homogeneous porous media proposed by C.T. Hsu and P. Cheng. In this work, we succeeded to prove the L^2-stability estimates of the model by introducing Lemma 3.2.4 to extract the influence of the non-homogeneity of the porosity. To established the energy stability estimates, we control this term with the Forchheimer term comes from the Darcy-Brinkmann-Forchheimer model. As a numerical scheme, we proposed a characteristic finite element method

(Lagrange-Galerkin scheme). We extended the idea of the characteristics method and introduced the macroscopic average velocity w to overcome the difficulty which comes from the convection term with the non-homogeneous porosity ϕ. To check the convergence order of the scheme, we compared a simple problem with the analytical solution and showed that our scheme has second-order accuracy both in space and in time. From the numerical simulation presented in Subsection 5.2 and 5.3, our results have a good agreement with the natural flow in the simple and complex structures of porosity qualitatively.

In this work, we succeeded to propose the Lagrange-Galerkin scheme for solving the model. However the theoretical convergence of this scheme has not been proved yet. Another challenge to improve the stability estimates of our results is to extend the Hypothesis 3.1.2.(i) to allow the condition if ϕ has a jump.

For the next work, we plan to couple our system with the thermal energy to simulate the fluid flow and heat transfer in the geothermal reservoir in 3D to predict the electrical generating capacity and the life time of the reservoir.

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令和　元年　8月　6日

学位論文審査報告書（甲）

1．学位論文題目（外国語の場合は和訳を付けること。）

Stability estimates and Lagrange-Galerkin schemes for Navier-Stokes type models of flow in non-homogeneous porous media（不均質な多孔質媒体における流れのナビエ・ストークス型モデルに対する安定性評価とラグランジュ・ガラーキン解法）

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(2) 氏名（ふりがな）　Imam Wijaya（いまむ うぃじゃや）

3．審査結果の要旨（600～650字）

Imam Wijaya さんは、2016年10月に自然科学研究科数物科学専攻に入学した(数物科学グローバル人材育成コース(MEXT-GHR)給付生)。それ以来、不均質な多孔質媒体における流れのナビエ・ストークス型モデル（NS型不均質多孔質媒体流モデル）の研究、特に、数値解法の開発を行ってきた。Imam さんの仕事は、次の２つに集約される。（i）NS 型不均質多孔質媒体流モデルの安定性評価を得た。（ii）同モデルのためのラグランジュ・ガラーキン数値解法を開発した。

通常、多孔質媒体中の流れは、ダルシー則を用いて計算されることが多いが、多孔質媒体の浸透率が不均質かつ比較的高い場合には、ダルシー則では不十分であることが知られており、これに対処するために Hsu－Cheng によって 1990 年に NS 型不均質多孔質媒体流モデルが提案された。実際、Imam さんの母国インドネシアでは、地熱発電によるエネルギーが重要なエネルギー資源であり、NS 型不均質多孔質媒体流モデルは重要であると考えられる。しかしながら、1990 年のモデル提案以降、数学的解析結果は得られていなかった。Imam さんは、NS 型不均質多孔質媒体流モデルに対して、安定性評価を得た。ここに、非線形項の評価手法は極めて秀逸である。また、有限要素（ガラーキン）法と流体粒子の軌跡に沿って離散化を行うラグランジュ法を組み合わせ、さらに、時間について２次精度化した、ラグランジュ・ガラーキン解法を提案し、数値シミュレーションによって、有効性を示した。ここに、同数値解法の導出では、新たにダルシー流速（流速を浸透率で割ったもの）を考えることで、物理的視点から自然なラグランジュ法の適用に成功している。数値シミュレーション結果は、妥当でありながら興味深いものである。これらの結果は原著論文１報にまとめられた。

以上により本論文は、博士（理学）を授与するに値すると判断した。

4．審査結果　(1) 判　定（いずれかに○印）　合　格　・　不合格

(2) 授与学位　博　士（　理学　）