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On the values at zero of partial zeta functions for ray classes of a real quadratic field III.

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Abstract. Let k be a real quadratic field. Let α be an integer of k and m be a positive rational integer. Denote by $\zeta(\mathfrak{a},(m),s)$ be a partial zeta function associated to a ray class $\mathfrak{a}$ containing the principal ideal (α) and defined with a conductor (m). We give a formula of the value of $\zeta(\mathfrak{a},(m),0)$ by means of the generalized Dedekind sum defined by H. Rademacher, *c.f.* [1]. We also give a formula of the value $L(0,\chi)$ of the L-function attatched to a ray class character χ.

1. Introduction. We studied the value at zero of a partial zeta function on a real quadratic field in [5, 6] based on the Shintani formula, *c.f.* [4]. Our explicit formula is available for a ray class group defined with a conductor of a positive integer and containing a principal ideal in a real quadratic field, which has a sum of products of values of the first Bernoulli polynomial. In the present paper, we shall convert this factor to the generalized Dedekind sum defined by H. Rademacher and show the Dedekind sum can be computed rapidly by using the explicit reciprocity law, whose algorithm is instructed in §3.3.3., [2]. We also deduce a formula of $L(0,\chi)$ for a character χ of the ray class group as an application.

2. Ray class groups. Let m be a positive integer which is greater than 1. Let S be the set of natural numbers that are prime to m. Denote by $\mathcal{O}$ the ring of integers of a real quadratic field k. Since S is a multiplicative set, we localize $\mathcal{O}$ with S and denote this localization by $\mathcal{O}_{(m)}$, which has a subring $1+m\mathcal{O}_{(m)}=\{1+mx : x\in\mathcal{O}_{(m)}\}$. Let $\mathcal{I}$ be the group of fractional ideals of k. We define a subgroup $\mathcal{I}_m$ to be

$$\mathcal{I}_m=\{I\in\mathcal{I} : S^{-1}I=\mathcal{O}_{(m)}\}.$$

An element x of k is called totally positive if it is positive in any embedding into $\boldsymbol{R}$. Let $(1+m\mathcal{O}_{(m)})_+$ be the subgroup of $1+m\mathcal{O}_{(m)}$ consisting of every totally

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positive elements. Each $z \in (1+m\mathcal{O}_{(m)})$ acts on $\mathcal{I}_m$ by $L \to Lz$, which makes $\mathcal{I}_m$ a $(1+m\mathcal{O}_{(m)})_+$-set. An orbit by this action is called a ray class. Our main concern is in the ray class which contains a principal ideal (α) for $\alpha \in \mathcal{O}_{(m)}^\times$. We denote by $\overline{\alpha}$ this ray class. Denote by $Cl_k(m)$ the ray class group and by $Cl_k^0(m)$ a subgroup of ray classes containing principal ideals, respectively. $Cl_k^0(m) = \{\overline{\alpha} : \alpha \in \mathcal{O}_{(m)}^\times\}$ is expressed as a set of double cosets:

(1) $Cl_k^0(m) \cong \mathcal{O}^\times \backslash \mathcal{O}_{(m)}^\times / (1+m\mathcal{O}_{(m)})_+$.

This expression is useful to compute ray classes. We may describe $Cl_k^0(m)$ by means of idelic notation. Set $U = \prod_{\mathfrak{p}|m\infty} U_{\mathfrak{p}}$, $U(m\infty) = \prod_{\mathfrak{p}|m}(1+m\mathcal{O}_{\mathfrak{p}}) \times \prod_{\mathfrak{p}|\infty}(k_{\mathfrak{p}}^\times)^2$ and $U_0 = \prod_{\mathfrak{p}\nmid m\infty} U_{\mathfrak{p}}$. In [6], we define $Cl_k^0(m) = UU_0/U(m\infty)U_0\mathcal{O}^\times$, and hence

$$Cl_k^0(m) \cong U/U(m\infty)\mathcal{O}^\times.$$

Since $\mathcal{O}_{(m)}^\times$ is dense in $\prod_{\mathfrak{p}|m} U_{\mathfrak{p}}$, we see $U/U(m\infty)$ is isomorpic to

$$\mathcal{O}_{(m)}^\times/(1+m\mathcal{O}_{(m)}) \times (\boldsymbol{R}^\times \otimes \boldsymbol{Z}/2\boldsymbol{Z})^2.$$

Therefore, $Cl_k^0(m)$ is represented as

$$\left\{\mathcal{O}_{(m)}^\times/(1+m\mathcal{O}_{(m)}) \times (\boldsymbol{R}^\times \otimes \boldsymbol{Z}/2\boldsymbol{Z})^2\right\}/\mathcal{O}^\times.$$

The factor $(\boldsymbol{R}^\times \otimes \boldsymbol{Z}/2\boldsymbol{Z})^2$ is called the signature part. The ray class group of $\boldsymbol{Q}$ is also given by

$$Cl_{\boldsymbol{Q}}(m)^\times = \left[(\boldsymbol{Z}/m\boldsymbol{Z})^\times \times \{\pm 1\}\right]/\{\pm 1\}.$$

The quadratic field k is an extension of $\boldsymbol{Q}$ by a square root of a square-free integer $d > 1$. Put

$$w = \begin{cases} \sqrt{d} & (d \not\equiv 1 \mod 4), \\ \dfrac{1+\sqrt{d}}{2} & (d \equiv 1 \mod 4). \end{cases}$$

$\mathcal{O}$ is a free $\boldsymbol{Z}$-module generated by 1 and w. Let ε_0 be a fundamental unit of k which is greater than 1 in the first embedding into $\boldsymbol{R}$. Let ε_1 be a totally positive fundamental unit, which is $\varepsilon_1 = \varepsilon_0^2$ if $N(\varepsilon_0) = -1$ and $\varepsilon_1 = \varepsilon_0$ if dose not. Double cosets in (1) are yielded by decomposing into $(1+m\mathcal{O}_{(m)})_+$-cosets firstly, and classifying them by means of the action of $\mathcal{O}^\times$ secondly. The process of decomposition is able to be reversed. We attain the same classes in either way. Let l be the order of ε_1 in $\mathcal{O}_{(m)}^\times/(1+m\mathcal{O}_{(m)}^\times)_+$. There are positive integers a and b such that

$$\varepsilon_1^l = a + bw, \ a \equiv 1 \mod m, \ m \mid b.$$

A symbol ε is assigned to denote this unit. Since every ray class contain an integer of k, we can choose a representative of the class from $\mathcal{O}$. Let $\alpha \in \mathcal{O}$ be prime to m. We decompose it as

(2) $\alpha = c\beta z,$

where c is a positive integer such that $(c, m) = 1$, β is an element of $\mathcal{O}$ whose norm $N(\beta)$ is prime to m and which is expressed as

$$\beta = f_1 + f_2 w, \quad (f_1, f_2) = 1,$$

and where z is an element of $(1 + m\mathcal{O}_{(m)})_+\mathcal{O}^\times$. We note that c is able to be selected from positive integers less than

m. Depending on the decomposition (2), two integers are defined to be $N_0 = |N(\beta)|$ and $N = bN_0/m$. We proved $(f_2, N_0) = 1$ if $f_1 f_2 \neq 0$ in [5]. We set $f_2 = 1$ when f_1 is zero. Then, there is an integer λ so that

$$f_2\lambda \equiv f_1 + f_2 Tr(w) \mod N_0$$

holds unless $f_2 = 0$. We set $\lambda = 0$ if $f_2 = 0$. Let δ be an integer satisfying

$$\delta \equiv a + \lambda b \mod mN, \quad 0 < \delta < mN.$$

We see $\delta \equiv 1 \mod m$, and furthermore, $(\delta, mN) = 1$ by Lemma 8 in [5]. We can now introduce the formula of the value of the partial zeta function from three constants δ, mN and c.

$$(3) \quad \zeta((\alpha),(m),0) = \frac{Tr(\varepsilon)}{12mN} - B_1\left(\left\{\frac{c}{m}\right\}\right) - \sum_{s=1}^{mN} B_1\left(\frac{s}{mN}\right) B_1\left(\left\{-\frac{c}{m} + \frac{\delta s}{mN}\right\}\right),$$

where $B_1(x)$ is the first Bernoulli polynomial and $\{x\}$ is the fractional part of a real number x, *c.f.* [5].

3. Dedekind sums. A generalized Dedekind sum $s(h, k : x, y)$:

$$\sum_{\mu \mod k} \left(\left(\frac{h\mu + y}{k} + x\right)\right)\left(\left(\frac{\mu + y}{k}\right)\right)$$

is defined by H. Rademacher, which has a reciprocity law, *c.f.* [1]. If we set $h = \delta$, $k = mN$, $x = -\frac{c}{m}$ and $y = 0$, then we see a similar sum which appears in the formula (3). In Lemma B, §3.3.3. in [2], the reciprocity law for a Dedekind sum

$$12\sum_{j=1}^{k-1}\left(\left(\frac{hj + c}{k}\right)\right)\left(\left(\frac{j}{k}\right)\right)$$

is given explicitly. Moreover, by using this law, it is shown that the value of the Dedekind sum is able to be computed rapidly. We apply this method to our formula (3). To this end, we define a Dedekind sum

$$s(h, k; c) = \sum_{j=1}^{k-1}\left(\left(\frac{hj + c}{k}\right)\right)\left(\left(\frac{j}{k}\right)\right),$$

which equals the Dedekind sum $s(h, k)$ when $c = 0$.

THEOREM 1. *The value of* $\zeta((\alpha),(m),0)$ *is equal to*

$$\frac{Tr(\varepsilon)}{12mN} - s(\delta, mN; (m - c)N).$$

Proof. The value of $((x))$ agrees with $B_1(\{x\})$ if $x \notin \boldsymbol{Z}$. On the contrary, we have $B_1(0) = -\frac{1}{2}$ and $B_1(1) = \frac{1}{2}$. If $\delta j - cN \equiv 0 \mod mN$, then the value of

$$\left(\left(-\frac{c}{m} + \frac{\delta j}{mN}\right)\right)\left(\left(\frac{j}{mN}\right)\right)$$

is zero. Since $(\delta, mN) = 1$ and $\delta \equiv 1 \mod m$, there is a unique integer in the interval $(0, mN)$ such that the congruence holds, that is $j = cN$. Thus, the difference

$$\sum_{s=1}^{mN} B_1\left(\left\{-\frac{c}{m} + \frac{\delta s}{mN}\right\}\right) B_1\left(\frac{s}{mN}\right) - s(\delta, mN; (m - c)N)$$

is equal to

$$\frac{1}{2}B_1\left(1-\frac{c}{m}\right)-\frac{1}{2}B_1\left(\frac{c}{m}\right).$$

Therefore, the formula follows from (3). □

REMARK 1. A formula

$$s(\delta, mN; (m-c)N) = \sum_{s=1}^{mN-1} B_1\left(\left\{-\frac{c}{m}+\frac{\delta s}{mN}\right\}\right) B_1\left(\frac{s}{mN}\right)$$

also follows from the same argument, *c.f.* Theorem 2 in [6].

We observe that the constants δ and N do not depend on the value of c. Let

$$j : Cl_{\boldsymbol{Q}}(m) \longrightarrow Cl_k(m)$$

be the natural map of ray classes. The factor c in the decomposition (2) comes from $Cl_{\boldsymbol{Q}}(m)$ through this map. Namely,

$$\overline{\alpha} = \overline{c}\overline{\beta}, \quad \overline{c} \in \operatorname{Im} j.$$

The natural map j is not always injective. A class $\overline{x} \in Cl_{\boldsymbol{Q}}(m)$ is mapped to the identity in $Cl_k(m)$ if and only if the integer x belongs to $(1+m\mathcal{O}_m)_+\mathcal{O}^\times$. More concretely, this holds for $x > 0$ if there is an integer k which satisfies $x \equiv \varepsilon_1^k \mod m$. Let $\overline{\nu} \in (\boldsymbol{Z}/m\boldsymbol{Z})^\times$ be a generator of $\operatorname{Ker} j$, where we choose it from an interval $(0, m)$. Put $H = \operatorname{Ker} j$ and $H = \{\overline{c_1}, \cdots, \overline{c_{n_1}}\}$. We decompose $Cl_k^0(m)$ into cosets relative to H:

$$Cl_k^0(m) = \bigcup_{j=1}^{n_2} H\overline{\beta_j}.$$

Since $H\overline{\alpha} = H\overline{\beta}$, we have

$$Cl_k^0(m) = \{\overline{c_i}\overline{\beta_j} : 1 \le i \le n_1, 1 \le j \le n_2\}. \tag{4}$$

EXAMPLE 1. Values of ν are as in the following table.

m	$\boldsymbol{Q}(\sqrt{2})$	$\boldsymbol{Q}(\sqrt{3})$	$\boldsymbol{Q}(\sqrt{5})$
2	1	1	1
3	2	2	2
4	1	3	1
5	4	3	1
6	5	1	5
7	1	6	6
8	1	1	5
9	8	8	8

4. The reciprocity law.

The Dedekind sum $s(h, k; c)$ has a reciprocity law, which is described explicitly in [2].

THEOREM 2. *Let h and k be positive integers such that $(h, k) = 1$ and $h < k$. Then, the following relation holds for $0 \le c < k$*

$$s(h,k;c)+s(k,h;c) = \frac{1}{12}\left(\frac{h}{k}+\frac{k}{h}+\frac{1}{hk}\right)+\frac{c^2}{2hk}-\frac{1}{2}\left\lfloor\frac{c}{h}\right\rfloor - \varepsilon(h,c),$$

where

$$\varepsilon(h,c) = \begin{cases} 0 & \text{if } h \mid c \text{ and } c \ne 0, \\ \dfrac{1}{4} & \text{if } h \nmid c \text{ or } c = 0. \end{cases}$$

The proof of this formula is instructed in Exercise 6 in §3.3.3, [2]. We follow it.

LEMMA 3. *Let h and k be integers such that $(h, k) = 1$ and $0 < h < k$. Take h' and k' so that $h'h + k'k = 1$. Then,*

$$\left\{\frac{k'j}{h}\right\}+\left\{\frac{h'j}{k}\right\}=\frac{j}{hk}+1-\delta_{\boldsymbol{Z}}\left(\frac{j}{h}\right)$$

for $1 \leq j < k$, *where* $\delta_{\boldsymbol{Z}}(x)$ *is the characteristic function of the set* $\boldsymbol{Z}$.

Proof. We choose h' and k' so that they satisfy $-k < h'$ and $0 < k' < h$.

(i) Suppose $h \mid j$ and set $j = hi$. We have

$$\left\{\frac{k'hi}{h}\right\}+\left\{\frac{h'hi}{k}\right\}=\left\{\frac{(1-k'k)i}{k}\right\}=\frac{i}{k}.$$

(ii) Suppose $j = hi+1$. We see

$$i+h' \leq \frac{k-1}{h}+h'=\frac{k-kk'}{h}.$$

This implies $i+h' < 0$. Therefore,

$$\begin{aligned}&\left\{\frac{k'(hi+1)}{h}\right\}+\left\{\frac{h'(hi+1)}{k}\right\}\\&=\left\{\frac{k'}{h}\right\}+\left\{\frac{(1-k'k)i+h'}{k}\right\}\\&=\left\{\frac{k'}{h}\right\}+\left\{\frac{i+h'}{k}\right\}\\&=\frac{k'}{h}+1+\frac{i+h'}{k}=\frac{j}{hk}+1\end{aligned}$$

(iii) Suppose $j \not\equiv 0, 1 \mod h$. We see $j \geq 2$ and $j \not\equiv 0 \mod h$. Suppose the equality is valid for $j-1$. we have

$$\left\{\frac{k'(j-1)}{h}\right\}+\left\{\frac{h'(j-1)}{k}\right\}=\frac{j-1}{hk}+1.$$

There is an integer m satisfying

$$\begin{aligned}&\left\{\frac{k'(j-1)}{h}\right\}+\frac{k'}{h}\\&\quad+\left\{\frac{h'(j-1)}{k}\right\}+\frac{h'}{k}\\&\quad=\left\{\frac{k'j}{h}\right\}+\left\{\frac{h'j}{k}\right\}+m\end{aligned}$$

holds. Applying

$$\frac{k'}{h}+\frac{h'}{k}=\frac{1}{hk},$$

to the left hand side in this equality and simplifying it, we have

$$\left\{\frac{k'j}{h}\right\}+\left\{\frac{h'j}{k}\right\}+m=\frac{j}{hk}+1.$$

Since $k'j \not\equiv 0 \mod h$ and $h'j \not\equiv 0 \mod k$, an inequality

$$m \leq \frac{j}{hk}+1-\left(\frac{1}{h}+\frac{1}{k}\right)$$

follows. Thus, $m \leq 0$. Suppose $m < 0$. We have

$$\left\{\frac{k'j}{h}\right\}+\left\{\frac{h'j}{k}\right\}>\frac{j}{hk}+2.$$

However, this inequality does not hold. Therefore, $m = 0$. The equality is also valid for j. □

Let $\Delta(h,k;c)$ be the difference between $s(h,k)$ and $s(h,k;c)$.

LEMMA 4. *Let* h *and* k *be positive integers such that* $h < k$ *and* $(h,k) = 1$. *Let* c *be an integer such that* $0 < c < k$. *We have*

$$\Delta(h,k;c) = -\frac{1}{2}\left(\left(\frac{h'c}{k}\right)\right) + \sum_{j=1}^{c}\left(\left(\frac{h'j}{k}\right)\right),$$
$$\Delta(k,h;c) = -\frac{1}{2}\left(\left(\frac{k'c}{h}\right)\right) + \sum_{j=1}^{c}\left(\left(\frac{k'j}{h}\right)\right).$$

Proof. Since $(h,k) = 1$, we change the variable with $j \equiv hi \mod k$. Then, the Dedekind sum is expressed as

$$s(h,k;c) = \sum_{i=1}^{k-1}\left(\left(\frac{i+c}{k}\right)\right)\left(\left(\frac{h'i}{k}\right)\right).$$

Therefore, $\Delta(h,k;c)$ equals

$$\sum_{j=1}^{k-1}\left[\left(\left(\frac{j+c}{k}\right)\right) - \left(\left(\frac{j}{k}\right)\right)\right]\left(\left(\frac{h'j}{k}\right)\right).$$

Note the value of $\left(\left(\frac{j+c}{k}\right)\right) - \left(\left(\frac{j}{k}\right)\right)$ equals

$$\begin{cases} \dfrac{c}{k} & (j < k-c), \\ \dfrac{c}{k} - \dfrac{1}{2} & (j = k-c), \\ \dfrac{c}{k} - 1 & (j > k-c). \end{cases}$$

Suppose $c \geq 2$. We divide the sum into three parts corresponding to cases, and extract terms $\frac{c}{k}\left(\left(\frac{h'j}{k}\right)\right)$. We have

$$\frac{c}{k}\sum_{j=1}^{k-1}\left(\left(\frac{h'j}{k}\right)\right) - \frac{1}{2}\left(\left(\frac{h'(k-c)}{k}\right)\right) - \sum_{j=k-c+1}^{k-1}\left(\left(\frac{h'j}{k}\right)\right).$$

By changing j to $c-j$, we obtain

$$\sum_{j=k-c+1}^{k-1}\left(\left(\frac{h'j}{k}\right)\right) = -\left(\left(\frac{h'c}{k}\right)\right) + \sum_{j=1}^{c}\left(\left(\frac{h'j}{k}\right)\right).$$

Hence, the above expression of $\Delta(h,k;c)$ is simplified:

$$-\frac{1}{2}\left(\left(\frac{h'c}{k}\right)\right) + \sum_{j=1}^{c}\left(\left(\frac{h'j}{k}\right)\right).$$

When $c = 1$, it is easy to show

$$\Delta(h,k;c) = -\frac{1}{2}\left(\left(\frac{h'(k-1)}{k}\right)\right) = \frac{1}{2}\left(\left(\frac{h'}{k}\right)\right).$$

Let k_1 and c_1 be integers in the interval $[0,h)$ satisfying $k \equiv k_1 \mod h$ and $c \equiv c_1 \mod h$. Since $k'k_1 \equiv 1 \mod h$, we have

$$\Delta(k_1,h;c_1) = -\frac{1}{2}\left(\left(\frac{k'c_1}{h}\right)\right) + \sum_{j=1}^{c}\left(\left(\frac{k'j}{h}\right)\right).$$

Let r be integer such that $c = c_1 + rh$. The second formula follows from

$$\sum_{j=c_1+1}^{c_1+rh}\left(\left(\frac{k'j}{h}\right)\right) = 0.$$

□

We note Lemma 3 implies

LEMMA 5. *Let h and k be positive integers such that $(h,k) = 1$ and $h < k$. We have*

$$\left(\left(\frac{h'j}{k}\right)\right) + \left(\left(\frac{k'j}{h}\right)\right) = \frac{j}{hk} - \frac{1}{2}\delta_Z\left(\frac{j}{h}\right)$$

for $1 \leq j < k$.

Now, we can prove the reciprocity law. It is verified directly when $c = 0$. Suppose $c > 0$. A sum $\Delta(h, k; c) + \Delta(k, h; c)$ is written as

$$-\frac{1}{2}\left[\left(\left(\frac{h'c}{k}\right)\right)+\left(\left(\frac{k'c}{h}\right)\right)\right] +\sum_{j=1}^{c}\left[\left(\left(\frac{h'j}{k}\right)\right)+\left(\left(\frac{k'j}{h}\right)\right)\right]$$

from Lemma 4. By virtue of Lemma 5, we have $\Delta(h, k; c) + \Delta(k, h; c)$ is equal to

$$\frac{c^2}{2hk}-\frac{1}{2}\left\lfloor\frac{c}{h}\right\rfloor+\frac{1}{4}\delta_Z\left(\frac{c}{h}\right).$$

It is easy to check

$$\frac{1}{4}\delta_Z\left(\frac{c}{h}\right)=-\varepsilon(h,c)+\frac{1}{4}$$

holds. This complete the proof.

Let δ' be a positive integer such that $\delta'\delta \equiv 1 \bmod m$ holds. We make $\Delta(\delta, mN; (m-c)N)$ short by starting the formula

$$-\frac{1}{2}\left(\left(\frac{\delta'(m-c)N}{mN}\right)\right)+\sum_{j=1}^{(m-c)N}\left(\left(\frac{\delta'j}{mN}\right)\right).$$

Since

$$\begin{aligned}\sum_{j=1}^{(m-c)N}\left(\left(\frac{\delta'j}{mN}\right)\right) &= -\sum_{j=(m-c)N+1}^{mN}\left(\left(\frac{\delta'j}{mN}\right)\right)\\ &= \sum_{t=0}^{cN-1}\left(\left(\frac{\delta't}{mN}\right)\right)\\ &= \sum_{t=0}^{cN-1}\left(\left(\frac{\delta't}{mN}\right)\right)-\left(\left(\frac{\delta'c}{m}\right)\right),\end{aligned}$$

$\Delta(h, k; c) + \Delta(k, h; c)$ is equal to

$$-\frac{1}{2}\left(\left(\frac{\delta'(m-c)}{m}\right)\right)+\sum_{t=1}^{cN}\left(\left(\frac{\delta't}{mN}\right)\right)-\left(\left(\frac{\delta'c}{m}\right)\right).$$

Furthermore, this equals

$$-\frac{1}{2}\left(\left(\frac{c}{m}\right)\right)+\sum_{t=1}^{cN}\left(\left(\frac{\delta't}{mN}\right)\right),$$

because of $\delta' \equiv 1 \bmod m$. We define a function $\varphi(x,\beta)$ of $x \in \boldsymbol{N}$ to be

$$\varphi(x,\beta)=\sum_{t=1}^{xN}\left(\left(\frac{\delta't}{mN}\right)\right),$$

for the constant N depends only upon β. Recall that a positive integer ν generates $\operatorname{Ker} j$.

LEMMA 6. *The function $\varphi(x,\beta)$ has the following properties:*

(i) $\varphi(x+m,\beta)=\varphi(x,\beta)$.

(ii) *The difference $\varphi(x\nu,\beta) - \varphi(x,\beta)$ equals*

$$\frac{1}{2}\left(\left(\frac{x\nu}{m}\right)\right)-\frac{1}{2}\left(\left(\frac{c}{m}\right)\right).$$

(iii) *We have a formula*

$$\Delta(\delta, mN, (m-c)N)=-\frac{1}{2}\left(\left(\frac{c}{m}\right)\right)+\varphi(c,\beta).$$

Proof. By virtue of an equation

$$\frac{\delta'(t+(k-1)N)}{mN}=\frac{k-1}{m}+\frac{\delta't}{mN},$$

we convert $\varphi(x,\beta)$ to a double sum

$$\sum_{k=1}^{x}\sum_{t=1}^{N}\left(\left(\frac{k-1}{m}+\frac{\delta't}{mN}\right)\right).$$

Since $(\delta', mN) = 1$, we see $\varphi(x+m, \beta)$ is equal to

$$\varphi(x,\beta) + \sum_{k=x+1}^{x+m} \sum_{t=1}^{N} \left(\left(\frac{k-1}{m} + \frac{\delta' t}{mN}\right)\right).$$

This proves $\varphi(x+m, \beta) = \varphi(x, \beta)$, because of

$$\sum_{k=1}^{m} \sum_{t=1}^{N} \left(\left(\frac{k-1}{m} + \frac{\delta' t}{mN}\right)\right) = 0.$$

Let c' be an integer such that $1 \leq c' < m$ and $c\nu \equiv c' \bmod m$. Since $\zeta((c\beta), (m), 0) = \zeta((c'\beta), (m), 0)$, we have $\Delta(\delta, mN, (m-c)N) = \Delta(\delta, mN, (m-c')N)$. Thus,

$$-\frac{1}{2}\left(\left(\frac{c}{m}\right)\right) + \varphi(c, \beta) = -\frac{1}{2}\left(\left(\frac{c'}{m}\right)\right) + \varphi(c', \beta).$$

Since $\varphi(c', \beta) = \varphi(c\nu, \beta)$ holds by virtue of the decomposition (2), the difference $\varphi(c\nu, \beta) - \varphi(c, \beta)$ equals

$$\frac{1}{2}\left(\left(\frac{c\nu}{m}\right)\right) - \frac{1}{2}\left(\left(\frac{c}{m}\right)\right).$$

We can enlarge this equality to $x = c + rm$ for an arbitrary positive integer r. □

5. Applications. It is showed in §3.3.4, [2] that the value of $s(h, k; c)$ is able to be calculated rapidly by applying the reciprocity law. The formula of $s(1, k)$ is well-known:

$$s(1,k) = -\frac{1}{4} + \frac{1}{6k} + \frac{k}{12}.$$

Since $s(1, k; c) = s(1, k) + \Delta(1, k; c)$, a formula

$$s(1,k;c) = -\frac{c}{2} + \frac{k}{12} + \frac{1}{6k} + \frac{c^2}{2k} \tag{7}$$

follows. We apply the Euclidean algorithm. Let h_j, k_j and c_j for $j = 0, 1, 2, \cdots, n+1$ be series generated by the following procedure:

(i) Put $h_0 = h$, $k_0 = k$ and $c_0 = c$.

(ii) Compute h_j, k_j, c_j for $j \geq 1$, recursively. Put $k_j = h_{j-1}$. Determine h_j and c_j to be integers $0 \leq h_j, c_j < k_j$ so that

$$h_j \equiv k_{j-1} \bmod k_j,\ c_j \equiv c_{j-1} \bmod k_j$$

holds.

(iii) The algorithm stops at $j = n+1$ when the decreasing series h_j attains $h_{n+1} = 1$.

We define $\tau(h, k; c)$ to be

$$\tau(h,k;c) = s(h,k;c) + s(k,h;c).$$

The value of the Dedekind sum is equal to the following sum

$$\sum_{j=0}^{n} (-1)^j \tau(h_j, k_j; c_j) + (-1)^{n+1} s(1, k_{n+1}; c_{n+1}).$$

EXAMPLE 2. We compute in $\boldsymbol{Q}(\sqrt{3})$ and make correction of tables in [6].

$m = 7$, $l = 8$

α	order	N_0	$\zeta(\mathfrak{a}, \mathfrak{m}, 0)$
1	1	1	17/21
2	3	1	5/21
3	3	1	−1/21

(continued)

α	order	N_0	$\zeta(\mathfrak{a},\mathfrak{m},0)$
$1+w$	6	-2	$13/21$
$3+w$	2	6	$11/21$
$2+4w$	6	-11	$13/21$
$8+2w$	6	13	$-19/21$
$8+w$	6	61	$-13/21$
$3+7w$	6	-133	$1/21$
$2+7w$	6	-143	$-5/21$
$1+7w$	2	-146	$-17/21$
$3+8w$	2	-183	$-11/21$

$m=9,\ l=18$

α	order	N_0	$\zeta(\mathfrak{a},\mathfrak{m},0)$
1	1	1	$1/18$
2	3	1	$13/18$
4	3	1	$-11/18$
$4+9w$	6	-227	$11/18$
$2+9w$	6	-239	$-3/18$
$1+9w$	2	-242	$-1/18$

The second application is that for a formula of the value of the L-function $L(0,\chi)$ attached to a character χ of $Cl_k(m)$ in a real quadratic field k of class number one. Recall $H = \operatorname{Im} j$ consists of n_1 classes $\overline{c_i}$, which are chosen so that $1 \le c_i < m$ holds. Let ν be a positive integer generating $\operatorname{Ker} j$. Put $n_3 = |\operatorname{Ker} j|$. The ray class group is decomposed into n_2 cosets $\overline{\beta_i}H$. Since the constants N_0, N and δ depend only upon β_j, we write them as $N_{0,j}$, N_j and δ'_j. The L-function is defined to be

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\chi(c_i\beta_j)\zeta((c_i\beta_j),(m),s).$$

The value of $L(0,\chi)$ is given by

$$\sum_{i,j}\chi(c_i\beta_j)\left(\frac{Tr(\varepsilon)}{mN_j} - s(\delta_j, mN_j, (m-c_i)N_j)\right),$$

where

$$s(\delta_j, mN_j; (m-c_i)N_j) = s(\delta_j, mN_j) - \frac{1}{2}\left(\!\left(\frac{c_i}{m}\right)\!\right) + \varphi(c_i,\beta_j).$$

Put

$$A_1 = \sum_{i=1}^{n_1}\chi(c_i), \qquad B_1 = \sum_{i=1}^{n_1}\chi(c_i)\left(\!\left(\frac{c_i}{m}\right)\!\right),$$

$$L_j = \sum_{i=1}^{n_1}\chi(c_i)\varphi(c_i,\beta_j).$$

Corresponding each term, we introduce U, V and W to represent $L(0,\chi)$ as $U-V-W$, that is

$$U = \frac{A_1 Tr(\varepsilon)}{m}\sum_{j=1}^{n_2}\frac{\chi(\beta_j)}{N_j},$$

$$V = A_1\sum_{j=1}^{n_2}\chi(\beta_j)s(\delta_j, mN_j),$$

$$W = \sum_{j=1}^{n_2}\chi(\beta_j)\left(-\frac{B_1}{2} + \sum_{i=1}^{n_1}\chi(c_i)\varphi(c_1,\beta_j)\right).$$

We define $\chi(x) = 0$ if $(x,m) \ne 1$ and extend χ onto $\boldsymbol{N}$.

$$B = \sum_{x=1}^{m}\chi(x)\left(\!\left(\frac{x}{m}\right)\!\right).$$

Since

$$L_j = \sum_{i,k} \chi(c_i)\varphi(c_i\nu^k, \beta),$$

we see

$$L_j = \sum_i \chi(c_i) \sum_k \left(\varphi(c_i, \beta_j) + \frac{1}{2}\left(\left(\frac{c_i\nu^k}{m}\right)\right) - \frac{1}{2}\left(\left(\frac{c_i}{m}\right)\right) \right)$$

$$= n_3 \sum_{i=1}^{n_1} \chi(c_i)\varphi(c_1, \beta_j) + \frac{B}{2} - \frac{n_3 B_1}{2}.$$

By solving this equation, we have

$$\sum_{i=1}^{n_1} \chi(c_i)\varphi(c_1, \beta_j) = \frac{L_j}{n_3} - \frac{B}{2n_3} + \frac{B_1}{2}.$$

Thus,

$$W = \frac{1}{n_3}\sum_{j=1}^{n_2} \chi(\beta_j)\left(L_j - \frac{B}{2}\right).$$

THEOREM 7. *The following formula holds:*

$$L(0,\chi) = \frac{A_1 Tr(\varepsilon)}{m}\sum_{j=1}^{n_2}\frac{\chi(\beta_j)}{N_j} - A_1\sum_{j=1}^{n_2}\chi(\beta_j)s(\delta_j, mN_j) - \frac{1}{n_3}\sum_{j=1}^{n_2}\chi(\beta_j)\left(L_j - \frac{B}{2}\right).$$

REMARK 2. $L(0,\chi)$ is zero if $\chi(\sigma) = 1$ holds for a complex conjugation σ of a complex place of the ray class field $k(m)$. Thus, if $k(m)$ is not a CM-field, the L-function vanishes at 0.

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