Hardy Spaces on Homogeneous Groups and Littlewood-Paley Functions

HARDY SPACES ON HOMOGENEOUS GROUPS AND LITTLEWOOD-PALEY FUNCTIONS

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ABSTRACT. We establish a characterization of the Hardy spaces on the homogeneous groups in terms of the Littlewood-Paley functions. The proof is based on vector-valued inequalities shown by applying the Peetre maximal function.

1. INTRODUCTION

Let $\mathbb{R}^n$ be the n dimensional Euclidean space. In this note we assume that $n \geq 2$. We also consider a structure on $\mathbb{R}^n$ which makes $\mathbb{R}^n$ a homogeneous group $\mathbb{H}$ equipped with multiplication given by a polynomial mapping. This requires that we have a dilation family $\{A_t\}_{t>0}$ on $\mathbb{R}^n$ of the form

$$A_t x = (t^{a_1}x_1, t^{a_2}x_2, \dots, t^{a_n}x_n), \quad x = (x_1, \dots, x_n),$$

with some real numbers $a_1, \dots, a_n$ satisfying $1 = a_1 \leq a_2 \leq \cdots \leq a_n$ such that each A_t is an automorphism of the group structure (see [7], [21] and [11, Section 2 of Chapter 1]). More precisely, in addition to the Euclidean structure, $\mathbb{H}$ is equipped with a homogeneous nilpotent Lie group structure and we have the following:

(1) Lebesgue measure is a bi-invariant Haar measure;
(2) the identity is the origin 0 and $x^{-1} = -x$;
(3) $(\alpha x)(\beta x) = \alpha x + \beta x$ for $x \in \mathbb{H}$, $\alpha, \beta \in \mathbb{R}$;
(4) $A_t(xy) = (A_t x)(A_t y)$ for $x, y \in \mathbb{H}$, $t > 0$;
(5) if $z = xy$, then $z_k = P_k(x, y)$, where $P_1(x, y) = x_1 + y_1$ and $P_k(x, y) = x_k + y_k + R_k(x, y)$ for $k \geq 2$ with a polynomial $R_k(x, y)$ depending only on $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}$, which can be written as

$$R_k(x, y) = \sum_{|I| \neq 0, |J| \neq 0, a(I)+a(J)=a_k} c^{(k)}_{I,J} x^I y^J.$$

(See Sections 2.1 and 2.2 below for the notation.)

Let $|x|$ be the Euclidean norm for $x \in \mathbb{R}^n$. We have a norm function $\rho(x)$ which is homogeneous of degree one with respect to the dilation A_t; by this we mean that $\rho(A_t x) = t\rho(x)$ for $t > 0$ and $x \in \mathbb{H}$. We may assume the following:

(6) ρ is continuous on $\mathbb{R}^n$ and smooth in $\mathbb{H} \setminus \{0\}$;
(7) $\rho(x+y) \leq \rho(x) + \rho(y)$ and $\rho(xy) \leq c_0(\rho(x) + \rho(y))$ for some constant $c_0 \geq 1$ and $\rho(x^{-1}) = \rho(x)$;

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(8) we have

$$c_1|x|^{\alpha_1} \le \rho(x) \le c_2|x|^{\alpha_2} \quad \text{if } \rho(x) \ge 1,$$
$$c_3|x|^{\beta_1} \le \rho(x) \le c_4|x|^{\beta_2} \quad \text{if } \rho(x) \le 1,$$

with some positive constants c_j, α_k, β_k, $1 \le j \le 4$, $1 \le k \le 2$;

(9) $\rho(x) \le 1$ if and only if $|x| \le 1$ and the unit sphere $\Sigma = \{x \in \mathbb{H} : \rho(x) = 1\}$ with respect to ρ coincides with $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

The polar coordinate expression of the Lebesgue measure $dx = t^{\gamma-1}\, dS\, dt$ is useful, where $\gamma = a_1 + \cdots + a_n$ (the homogeneous dimension). By this we mean that

$$\int_{\mathbb{H}} f(x)\, dx = \int_0^\infty \int_\Sigma f(A_t\theta) t^{\gamma-1}\, dS(\theta)\, dt$$

with $dS = \omega\, dS_0$ for appropriate functions f, where ω is a strictly positive C^∞ function on Σ and dS_0 denotes the Lebesgue surface measure on Σ.

We recall the Heisenberg group $\mathbb{H}_1$ as an example of a homogeneous group. Let us define the multiplication

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + (x_1y_2 - x_2y_1)/2),$$

$(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$. Then this group law defines the Heisenberg group $\mathbb{H}_1$ with the underlying manifold $\mathbb{R}^3$, where the dilation $A_t(x_1, x_2, x_3) = (tx_1, tx_2, t^2x_3)$ is an automorphism.

We consider the Littlewood-Paley g function on $\mathbb{H}$ defined by

$$g_\varphi(f)(x) = \left(\int_0^\infty |f * \varphi_t(x)|^2 \frac{dt}{t} \right)^{1/2}, \tag{1.1}$$

where $f \in \mathcal{S}'$, $\varphi \in \mathcal{S}$ satisfying $\int_{\mathbb{H}} \varphi\, dx = 0$ and $\varphi_t(x) = t^{-\gamma}\varphi(A_t^{-1}x)$. Here $\mathcal{S}'$ denotes the space of tempered distributions and $\mathcal{S}$ the Schwartz space, which are the same as those in the Euclidean case (see [19]); also the convolution $F * G$ for $F, G \in L^1$ is defined by

$$F * G(x) = \int_{\mathbb{H}} F(xy^{-1})G(y)\, dy = \int_{\mathbb{H}} F(y)G(y^{-1}x)\, dy.$$

We refer to [4] and [21, 13, 14] for the study of Littlewood-Paley operators and singular integrals, respectively, on L^p spaces for homogeneous groups, $1 \le p < \infty$.

In this note we prove a characterization of Hardy spaces H^p, $0 < p \le 1$, on $\mathbb{H}$ (see Section 2.3 below) in terms of the Littlewood-Paley g functions. We first recall related results in the Euclidean case. Let $\varphi^{(\ell)}$, $\ell = 1, 2, \dots, M$, be functions in $\mathcal{S}(\mathbb{R}^n)$ satisfying the non-degeneracy condition

$$\inf_{\xi \in \mathbb{R}^n \setminus \{0\}} \sup_{t>0} \sum_{\ell=1}^M |\mathcal{F}(\varphi^{(\ell)})(t\xi)| > c \tag{1.2}$$

for some positive constant c, where $\mathcal{F}(\psi)$ is the Fourier transform:

$$\mathcal{F}(\psi) = \hat{\psi}(\xi) = \int_{\mathbb{R}^n} \psi(x) e^{-2\pi i\langle x, \xi\rangle}\, dx, \quad \langle x, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n.$$

The following result in the case of the Euclidean structure is known (see [22]).

Theorem A. *Let $0 < p \le 1$. Suppose that $\varphi^{(\ell)} \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi^{(\ell)}\, dx = 0$, $\ell = 1, 2, \dots, M$, and that the condition* (1.2) *holds. Then*

$$c_p \|f\|_{H^p} \le \sum_{\ell=1}^{M} \|g_{\varphi^{(\ell)}}(f)\|_p \le C_p \|f\|_{H^p}$$

for $f \in H^p(\mathbb{R}^n)$, where $\|\cdot\|_p$ denotes the L^p norm and $g_{\varphi^{(\ell)}}(f)$ is defined as in (1.1) *with $\varphi = \varphi^{(\ell)}$, $f * \varphi_t(x) = \int_{\mathbb{R}^n} f(x-y)\varphi_t(y)\, dy$, $\varphi_t(y) = t^{-n}\varphi(t^{-1}y)$.*

See [6] for the Hardy space $H^p(\mathbb{R}^n)$. Analogous results for L^p spaces, $1 < p < \infty$, can be found in [1], [10] and [15].

To generalize Theorem A to the case of homogeneous groups, we note that the condition (1.2) can be used to find $b \in (0,1)$, positive numbers r_1, r_2 with $r_1 < r_2$ and functions $\eta^{(1)}, \dots, \eta^{(M)} \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp} \mathcal{F}(\eta^{(\ell)}) \subset \{r_1 < |\xi| < r_2\}$, $1 \le \ell \le M$, and

$$\sum_{j=-\infty}^{\infty} \sum_{\ell=1}^{M} \mathcal{F}(\varphi^{(\ell)})(b^j\xi)\mathcal{F}(\eta^{(\ell)})(b^j\xi) = 1 \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{1.3}$$

See [16, Lemma 2.1] and also [20, Chap. V], [2]. From (1.3) it follows that

$$\sum_{j=-\infty}^{\infty} \sum_{\ell=1}^{M} \varphi^{(\ell)}_{b^j} * \eta^{(\ell)}_{b^j} = \delta \quad \text{in } \mathcal{S}', \tag{1.4}$$

where δ denotes the Dirac delta function.

Also, the condition (1.2) implies the existence of functions $\eta^{(1)}, \dots, \eta^{(M)} \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp} \mathcal{F}(\eta^{(\ell)}) \subset \{r_1 < |\xi| < r_2\}$, with positive numbers r_1, r_2 with $r_1 < r_2$, for which we have

$$\sum_{\ell=1}^{M} \int_0^\infty \varphi_t^{(\ell)} * \eta_t^{(\ell)} \frac{dt}{t} = \delta \quad \text{in } \mathcal{S}'. \tag{1.5}$$

Let Δ be the additive sub-semigroup of $\mathbb{R}$ generated by $0, a_1, \dots, a_n$ and let $\mathcal{P}_a$ be the space of polynomials on $\mathbb{H}$ of homogeneous degree less than or equal to $a \in \Delta$ (see Section 2.2 below for more details). We employ a version of (1.5) as a non-degeneracy condition for $\varphi^{(1)}, \dots, \varphi^{(M)}$ on $\mathbb{H}$ and we shall prove the following result analogous to Theorem A.

Theorem 1.1. *Let $0 < p \le 1$. We can find $d \in \Delta$ with the following property. Suppose that $\{\varphi^{(\ell)} \in \mathcal{S} : 1 \le \ell \le M\}$ is a family of functions such that*

(1)
$$\int \varphi^{(\ell)}\, dx = 0, \quad 1 \le \ell \le M;$$

(2) *there exist functions $\eta^{(\ell)} \in \mathcal{S}$, $1 \le \ell \le M$, satisfying that*

$$\sum_{\ell=1}^{M} \int_0^\infty \varphi_t^{(\ell)} * \eta_t^{(\ell)} \frac{dt}{t} = \lim_{\substack{\epsilon \to 0,\\ B \to \infty}} \sum_{\ell=1}^{M} \int_\epsilon^B \varphi_t^{(\ell)} * \eta_t^{(\ell)} \frac{dt}{t} = \delta \quad \textit{in } \mathcal{S}' \tag{1.6}$$

and that

$$\int \eta^{(\ell)} P\, dx = 0 \quad \textit{for all } P \in \mathcal{P}_d, \quad 1 \le \ell \le M.$$

Then we have

$$c_p\|f\|_{H^p} \le \sum_{\ell=1}^{M} \|g_{\varphi^{(\ell)}}(f)\|_p \le C_p\|f\|_{H^p} \quad \textit{for } f \in H^p \tag{1.7}$$

with positive constants c_p *and* C_p *independent of* f*, where* H^p *is the Hardy space on* $\mathbb{H}$*.*

Let $\mathbb{H}$ be a stratified group with a natural dilation and let h be the heat kernel on $\mathbb{H}$ (see [7]). Define $\phi^{(j)} \in \mathcal{S}$, $j = 1, 2, \dots$, by

$$\phi^{(j)}(x) = \left[\partial_t^j h(x,t)\right]_{t=1} = (-L)^j h(x,1),$$

where $\partial_t = \partial/\partial t$ and L is the sub-Laplacian of $\mathbb{H}$. As an application of Theorem 1.1 we have the following.

Corollary 1.2. *Let* $f \in H^p$*,* $0 < p \le 1$*. Then, for any* $j \ge 1$*, we have*

$$c_p\|f\|_{H^p} \le \|g_{\phi^{(j)}}(f)\|_p \le C_p\|f\|_{H^p}$$

with some positive constants c_p*,* C_p *independent of* f*.*

This almost retrieves Theorem 7.28 of [7], where the first inequality is shown under the condition that $f \in \mathcal{S}'$ vanishes weakly at infinity and $g_{\phi^{(j)}}(f) \in L^p$.

As in the case of the Euclidean structure of Theorem A, the first inequality of (1.7) of the theorem is more difficult for us to prove than the second one; the second inequality can be shown by applying a theory of vector-valued singular integrals.

Let

$$S_\varphi(f)(x) = \left(\int_0^\infty \int_{\rho(x^{-1}y)<t} |f * \varphi_t(y)|^2 t^{-\gamma-1}\, dy\, dt\right)^{1/2}$$

be the Lusin area integral on the homogeneous group $\mathbb{H}$. Then in [7], results analogous to Theorem 1.1 were proved for $S_\varphi(f)$ (see [7, Theorem 7.11 and Corollary 7.22]), while the result for the Littlewood-Paley g functions was shown only for special Littlewood-Paley functions $g_{\phi^{(j)}}$ associated with the heat kernel.

In [16] an alternative proof of the first inequality of the conclusion of Theorem A is given by applying the Peetre maximal function $F^{**}_{N,R}$ defined by

$$F^{**}_{N,R}(x) = \sup_{y \in \mathbb{R}^n} \frac{|F(x-y)|}{(1+R|y|)^N}$$

(see [12]). The proof of [16] is expected to extend to some other situations. Indeed, it has been applied to get the Littlewood-Paley function characterization of parabolic Hardy spaces of Calderón-Torchinsky [2, 3] (see [17]); see also [18] for related results on weighted Hardy spaces.

In this note we shall show that the methods of [16] can be also applied to characterize Hardy spaces on the homogeneous groups by certain Littlewood-Paley functions (Theorem 1.1). One of the ingredients of the methods is to prove a vector-valued inequality in Theorem 4.6 below in Section 4, which is stated as a weighted inequality.

In Section 2, we shall recall some results from [7] needed in this note including the definition of Hardy spaces on $\mathbb{H}$, Taylor's theorem and also we shall have the definition of weight classes. In Sections 3 and 4 we shall show key estimates Lemmas 3.1 and 4.2, respectively, which will be used to prove Theorem 4.6 in Section 4 mentioned above. The proof of Theorem 1.1 will be completed in Section 5; also the proof of Corollary 1.2 will be given there. Finally, in Section 6 we shall employ an analogue of (1.4) on $\mathbb{H}$ as a non-degeneracy condition and we shall describe results similar to Theorems 1.1 and 4.6 (Theorems 6.1 and 6.2). Also, we shall state discrete parameter versions of Theorems 1.1 and 4.6 (Theorems 6.3 and 6.4).

2. Some preliminaries

In this section we have some preliminary results. See [7] for results in Sections 2.1, 2.2 and 2.3.

2.1. **Invariant derivatives.** Let $e_j = (e_1^{(j)}, e_2^{(j)}, \dots, e_n^{(j)})$, $1 \le j \le n$, be the element of $\mathbb{H}$ such that $e_j^{(j)} = 1$ and $e_k^{(j)} = 0$ if $k \neq j$. Define the left-invariant and right-invariant differential operators, which are denoted by X_j and Y_j, respectively, by

$$X_j f(x) = \left[\frac{d}{dt} f(x(te_j))\right]_{t=0},$$
$$Y_j f(x) = \left[\frac{d}{dt} f((te_j)x)\right]_{t=0}.$$

Then we can see that $X_j(f(A_s x)) = s^{a_j}(X_j f)(A_s x)$, $Y_j(f(A_s x)) = s^{a_j}(Y_j f)(A_s x)$.

Let $\mathbb{N}_0$ denote the set of non-negative integers and let $I = (i_1, i_2, \dots, i_n) \in (\mathbb{N}_0)^n$. Define

$$|I| = i_1 + i_2 + \cdots + i_n, \quad a(I) = a_1 i_1 + a_2 i_2 + \cdots + a_n i_n.$$

Higher order differential operators X^I and Y^I are defined as

$$X^I = X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}, \quad Y^I = Y_1^{i_1} Y_2^{i_2} \dots Y_n^{i_n}.$$

Then $|I|$ is called the order of X^I and Y^I and $a(I)$ the homogeneous degree for them.

Let $I = (i_1, i_2, \dots, i_n)$ and $I' = (i_n, \dots, i_2, i_1)$. Then

$$(X^I f) * g(x) = f * (Y^{I'} g)(x),$$
$$\int_{\mathbb{H}} (X^I f)(x) g(x)\, dx = (-1)^{|I|} \int_{\mathbb{H}} f(x)(X^{I'} g)(x)\, dx,$$
$$\int_{\mathbb{H}} (Y^I f)(x) g(x)\, dx = (-1)^{|I|} \int_{\mathbb{H}} f(x)(Y^{I'} g)(x)\, dx,$$
$$X^I(f * g)(x) = (f * X^I g)(x), \quad Y^I(f * g) = (Y^I f) * g$$

for appropriate functions f, g.

2.2. **Taylor polynomials.** Let

$$P(x) = \sum c_I x^I, \quad x^I = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}, \quad I = (i_1, i_2, \dots i_n), \tag{2.1}$$

be a polynomial on $\mathbb{R}^n$. We may also consider $P(x)$ as a polynomial on $\mathbb{H}$. The degree of the polynomial P is $\max\{|I| : c_I \neq 0\}$. Also, the homogeneous degree of P is defined to be $\max\{a(I) : c_I \neq 0\}$.

If $P(x) = x^J$, then $Y^I P$ and $X^I P$ are homogeneous of degree $a(J) - a(I)$ with respect to the dilation A_t. This implies, in particular, that $Y^I P = X^I P = 0$ if $a(I) > a(J)$.

Let $\Delta = \{a(I) : I \in (\mathbb{N}_0)^n\}$. Define

$$\bar{a} = \min\{c \in \Delta : c > a\}. \tag{2.2}$$

We denote by $\mathcal{P}_a$ the space of all polynomials P in (2.1) with $a(I) \le a$ for all I.

Let $a \in \Delta$. Let f be a function which has continuous derivatives $X^I f$ in a neighborhood of $x \in \mathbb{H}$ for $a(I) \le a$. The left Taylor polynomial $P_x(y)$ of f at x of homogeneous degree a is the unique polynomial P such that $X^I P(0) = X^I f(x)$ for all I with $a(I) \le a$. The right Taylor polynomial is defined similarly with Y^I in place of X^I.

We state mean value and Taylor inequalities.

Lemma 2.1. *Suppose that f is continuously differentiable on $\mathbb{H}$. Then for $x, y \in \mathbb{H}$, we have*

$$|f(xy) - f(x)| \le C \sum_{j=1}^{n} \rho(y)^{a_j} \sup_{\rho(z) \le C_1 \rho(y)} |(X_j f)(xz)|,$$

where the constants C, C_1 are independent of x, y and f.

This can be shown by using Theorem 1.33 of [7] and the relation $Y_j \tilde{f} = -\widetilde{X_j f}$, where $\tilde{f}(x) = f(x^{-1})$.

Lemma 2.2. *Let $a \in \Delta$, $a \ge 0$. Put $k = [a]$, where $[a]$ denotes the largest integer not exceeding a. There are constants C_a and B_a such that if f is $k+1$ times continuously differentiable on $\mathbb{H}$, $x, y \in \mathbb{H}$ and P_x is the right Taylor polynomial of f at x of homogeneous degree a, then*

$$|f(yx) - P_x(y)| \le C_a \sum_{|I| \le k+1, a(I) > a} \rho(y)^{a(I)} \sup_{\rho(z) \le B_a \rho(y)} |Y^I f(zx)|.$$

See [7, Theorems 1.33, 1.37].

2.3. **Hardy spaces.** We define

$$\|\Phi\|_{(N)} = \sup_{|I| \le N, x \in \mathbb{H}} (1 + \rho(x))^{(N+1)(\gamma+1)} |Y^I \Phi(x)|$$

(see [7, p. 35]). Put

$$B_N = \{\Phi \in \mathcal{S} : \|\Phi\|_{(N)} \le 1\}.$$

Let

$$M_{(N)}(f)(x) = \sup\{\sup_{t>0} |f * \Phi_t(x)| : \Phi \in B_N\}.$$

The Hardy space H^p on $\mathbb{H}$ for $p \in (0, 1]$ is defined as

$$H^p = \{f \in \mathcal{S}' : \|f\|_{H^p} = \|M_{(N_p)}(f)\|_p < \infty\},$$

with sufficiently large N_p. The number

$$\min\left\{N \in \mathbb{N}_0 : N \ge \min\{a \in \Delta : a > \gamma(p^{-1} - 1)\}\right\}$$

can be taken as N_p, which equals $[\gamma(p^{-1} - 1)] + 1$ when $\Delta = \mathbb{N}_0$ (see [7, Chap. 2]).

In the case of Euclidean structure, the H^p spaces can be characterized by the radial maximal function $\sup_{t>0} |f * \varphi_t|$, where $\varphi \in \mathcal{S}$ with $\int \varphi = 1$ (see [6]).

2.4. **Weight functions.** Let B be a subset of $\mathbb{H}$. Then B is a ball in $\mathbb{H}$ with center $x \in \mathbb{H}$ and radius $t > 0$ if

$$B = \{y \in \mathbb{H} : \rho(y^{-1}x) < t\}.$$

We write $B = B(x,t)$. Let $\fint_B f(y)\,dy = |B|^{-1}\int_B f(y)\,dy$, where $|B|$ denotes the Lebesgue measure of B. Let w be a weight function on $\mathbb{H}$ and $1 < p < \infty$. We say that w belongs to the class A_p of Muckenhoupt if

$$\sup_B \left(\fint_B w(x)\,dx\right)\left(\fint_B w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{H}$.

The Hardy-Littlewood maximal operator is defined by

$$M(f)(x) = \sup_{x\in B}\fint_B |f(y)|\,dy,$$

where the supremum is taken over all balls B in $\mathbb{H}$ containing x. (See [8, 9].)

We denote by $\|f\|_{L^p_w}$ the weighted L^p norm

$$\left(\int_{\mathbb{H}} |f(x)|^p w(x)\,dx\right)^{1/p}.$$

We shall apply the following weighted vector-valued inequalities.

Lemma 2.3. *Let $1 < \mu, \nu < \infty$. Suppose that $w \in A_\nu$. Then for appropriate functions $G(x,t)$ on $\mathbb{H}\times(0,\infty)$ we have*

$$\left(\int_{\mathbb{H}}\left(\int_0^\infty M(G^t)(x)^\mu \frac{dt}{t}\right)^{\nu/\mu} w(x)\,dx\right)^{1/\nu} \le C\left(\int_{\mathbb{H}}\left(\int_0^\infty |G(x,t)|^\mu \frac{dt}{t}\right)^{\nu/\mu} w(x)\,dx\right)^{1/\nu},$$

where $G^t(x) = G(x,t)$.

This is a version of a result in [5] (see [9, pp. 265–267]).

3. Some basic estimates

For $\eta, \psi \in \mathcal{S}$ and $t, L > 0$, let

(3.1)

$$C(\eta,\psi,t,L,x) = (1+\rho(x))^L(\eta * \psi_t(x)), \quad C(\eta,\psi,t,L) = \int_{\mathbb{H}} |C(\eta,\psi,t,L,x)|\,dx.$$

Define the Peetre maximal function on $\mathbb{H}$ by

$$F^{**}_{N,R}(x) = \sup_{y\in\mathbb{H}} \frac{|F(xy^{-1})|}{(1+R\rho(y))^N} = \sup_{y\in\mathbb{H}} \frac{|F(y)|}{(1+R\rho(y^{-1}x))^N}. \tag{3.2}$$

Let $f \in \mathcal{S}'$. We say that f vanishes weakly at infinity if $f * \phi_t \to 0$ in $\mathcal{S}'$ as $t \to \infty$ for all $\phi \in \mathcal{S}$ (see [7, p. 50]).

Lemma 3.1. *Suppose that $\varphi^{(\ell)}, \eta^{(\ell)} \in \mathcal{S}$, $1 \le \ell \le M$, satisfy $\int \varphi^{(\ell)} = 0$, $1 \le \ell \le M$, and (1.6). Suppose that $f \in \mathcal{S}'$ vanishes weakly at infinity and that $\psi \in \mathcal{S}$. Let $b \in (0,1)$. Then*

(3.3)

$$(f*\psi_t)^{**}_{L,t^{-1}}(x) \le \sum_{\ell=1}^M \sum_{j=-\infty}^\infty C_L b^{-Lj_+}\int_b^1 C(\eta^{(\ell)},\psi,u^{-1}b^{-j},L)(f*\varphi^{(\ell)}_{ub^jt})^{**}_{L,b^{-j}t^{-1}}(x)\,\frac{du}{u},$$

where $j_+ = \max(0, j)$.

Proof. Define $\zeta \in \mathcal{S}$ by

$$\zeta = \sum_{\ell=1}^{M} \int_1^\infty \varphi_t^{(\ell)} * \eta_t^{(\ell)} \,\frac{dt}{t}.$$

The fact that $\zeta \in \mathcal{S}$ and $\int \zeta = 1$ can be seen from [7, p. 51]. We have

$$\begin{aligned} f * \psi_t &= \lim_{\substack{k\to\infty,\\ m\to\infty}} (f * \zeta_{tb^{m+1}} * \psi_t - f * \zeta_{tb^{-k}} * \psi_t) \\ &= \lim_{\substack{k\to\infty,\\ m\to\infty}} \sum_{j=-k}^{m} \sum_{\ell=1}^{M} \int_b^1 f * \varphi_{utb^j}^{(\ell)} * (\eta^{(\ell)} * \psi_{u^{-1}b^{-j}})_{utb^j} \,\frac{du}{u} \\ &= \sum_{j=-\infty}^{\infty} \sum_{\ell=1}^{M} \int_b^1 f * \varphi_{utb^j}^{(\ell)} * (\eta^{(\ell)} * \psi_{u^{-1}b^{-j}})_{utb^j} \,\frac{du}{u}, \end{aligned}$$

if $f \in \mathcal{S}'$ vanishes weakly at infinity (see Proposition 1.49 and the proof of Theorem 1.64 in [7]). Noting that

$$\eta^{(\ell)} * \psi_{u^{-1}b^{-j}}(x) = (1+\rho(x))^{-L} C(\eta^{(\ell)}, \psi, u^{-1}b^{-j}, L, x),$$

we see that

$$\begin{aligned} (3.4)\quad |f * \psi_t(z)| \le \sum_{\ell=1}^{M} \sum_{j=-\infty}^{\infty} \int_b^1 \int |f * \varphi_{utb^j}^{(\ell)}(y)| (1 + t^{-1}b^{-j}\rho(y^{-1}z))^{-L} \\ \times |C(\eta^{(\ell)}, \psi, u^{-1}b^{-j}, L, A_{utb^j}^{-1}(y^{-1}z)| (utb^j)^{-\gamma} \,dy \,\frac{du}{u}, \end{aligned}$$

since $b \le u \le 1$ in the integral. We observe that

(3.5)

$$(1 + t^{-1}b^{-j}\rho(y^{-1}z))^{-L}(1 + t^{-1}\rho(z^{-1}x))^{-L} \le 2^L c_0^L b^{-Lj_+}(1 + t^{-1}b^{-j}\rho(y^{-1}x))^{-L},$$

where c_0 is as in (7) of Section 1. To see this, we first note that

$$\begin{aligned} &(1 + t^{-1}b^{-j}\rho(y^{-1}z))(1 + t^{-1}\rho(z^{-1}x)) \\ &\qquad = b^{-j}t^{-2}\left(b^j t^2 + t\rho(y^{-1}z) + b^j t\rho(z^{-1}x) + \rho(y^{-1}z)\rho(z^{-1}x)\right) \end{aligned}$$

and

$$\begin{aligned} I &:= (1 + t^{-1}b^{-j}\rho(y^{-1}z))(1 + t^{-1}\rho(z^{-1}x))(1 + t^{-1}b^{-j}\rho(y^{-1}x))^{-1} \\ &= \frac{b^j t^2 + t\rho(y^{-1}z) + b^j t\rho(z^{-1}x) + \rho(y^{-1}z)\rho(z^{-1}x)}{t(b^j t + \rho(y^{-1}x))} \\ &\ge \frac{b^j t^2 + t\rho(y^{-1}z) + b^j t\rho(z^{-1}x)}{t(b^j t + \rho(y^{-1}x))}. \end{aligned}$$

If $j \ge 0$, since $\rho(y^{-1}x) \le c_0(\rho(y^{-1}z) + \rho(z^{-1}x))$, $c_0 \ge 1$ and $b^j \le 1$,

$$I \ge b^j c_0^{-1} \frac{t^2 + t\rho(y^{-1}x)}{t(b^j t + \rho(y^{-1}x))} \ge b^j c_0^{-1}.$$

Next let $j \le 0$. If $b^j t \ge \rho(y^{-1}x)$, then

$$I \ge \frac{b^j t^2 + t\rho(y^{-1}z) + b^j t\rho(z^{-1}x)}{2t^2 b^j} \ge \frac{1}{2}.$$

If $b^j t < \rho(y^{-1}x)$, since $b^j \geq 1$,

$$I \geq \frac{b^j t^2 + t\rho(y^{-1}z) + b^j t\rho(z^{-1}x)}{2t\rho(y^{-1}x)}$$
$$\geq \frac{b^j t + \rho(y^{-1}z) + \rho(z^{-1}x)}{2\rho(y^{-1}x)} \geq \frac{c_0^{-1}\rho(y^{-1}x)}{2\rho(y^{-1}x)} \geq \frac{1}{2}c_0^{-1}.$$

Combining results, we can easily get (3.5).

Multiplying both sides of (3.4) by $(1+t^{-1}\rho(z^{-1}x))^{-L}$ and using (3.5), we have

$$|f*\psi_t(z)|(1+t^{-1}\rho(z^{-1}x))^{-L}$$
$$\leq C\sum_{\ell=1}^{M}\sum_{j=-\infty}^{\infty} b^{-Lj_+}\int_b^1\int \frac{|f*\varphi^{(\ell)}_{utb^j}(y)|}{(1+t^{-1}b^{-j}\rho(y^{-1}x))^L}$$
$$\times |C(\eta^{(\ell)},\psi,u^{-1}b^{-j},L,A^{-1}_{utb^j}(y^{-1}z)|(utb^j)^{-\gamma}\,dy\,\frac{du}{u}$$
$$\leq C\sum_{\ell=1}^{M}\sum_{j=-\infty}^{\infty} b^{-Lj_+}\int_b^1 C(\eta^{(\ell)},\psi,u^{-1}b^{-j},L)(f*\varphi^{(\ell)}_{ub^jt})^{**}_{L,b^{-j}t^{-1}}(x)\,\frac{du}{u}.$$

The inequality (3.3) follows from this by taking supremum in z. □

To estimate $C(\eta,\psi,t,L)$ in (3.1) we apply the following result.

Lemma 3.2. *Let $\eta,\psi\in\mathcal{S}$.*

(1) *Let $t\geq 1$. Suppose that $a\in\Delta$ and $\int \eta P\,dx = 0$ for all $P\in\mathcal{P}_a$. Then, for any $M\geq 0$, we have*
$$|\eta*\psi_t(x)| \leq B_1(\eta,\psi,a,M)t^{-\bar a-\gamma}(1+t^{-1}\rho(x))^{-M}$$
for all $x\in\mathbb{H}$ with some constant $B_1(\eta,\psi,a,M)$ (see (2.2) for $\bar a$).

(2) *Let $0<t\leq 1$. If $a\in\Delta$ and $\int\psi P dx = 0$ for all $P\in\mathcal{P}_a$, then, for any $M\geq 0$,*
$$|\eta*\psi_t(x)| \leq B_2(\eta,\psi,a,M)t^{\bar a}(1+\rho(x))^{-M}$$
for all $x\in\mathbb{H}$ with some constant $B_2(\eta,\psi,a,M)$.

Proof. Let $t\geq 1$ to prove part (1). Let $P_x(y)$ be the right Taylor polynomial of ψ at x of homogeneous degree $a\in\Delta$. Then, if $R_x(y)=\psi(yx)-P_x(y)$,

$$|R_x(y)| \leq C(\psi,a,M)\rho(y)^{\bar a}(1+\rho(x))^{-M} \tag{3.6}$$

for any $a\in\Delta$, $M>0$, provided that $\rho(x)\geq D_a\rho(y)$ with sufficiently large D_a. This can be shown by applying Lemma 2.2. Indeed, if $D_a\geq 2c_0B_a$, $\rho(z)\leq B_a\rho(y)$ and $\rho(x)\geq D_a\rho(y)$, where B_a is as in Lemma 2.2, then it can be easily shown that $c_0\rho(zx)\geq\rho(x)/2$.

If $\int\eta P\,dx = 0$ for $P\in\mathcal{P}_a$,

$$\int \eta(y)t^{-\gamma}\psi(A_t^{-1}(y^{-1}x))\,dy = \int\eta(y)t^{-\gamma}R_{A_t^{-1}x}(A_t^{-1}y^{-1})\,dy =: J.$$

By (3.6) we have

$$|R_{A_t^{-1}x}(A_t^{-1}y^{-1})| \leq Ct^{-\bar a}\rho(y)^{\bar a}(1+t^{-1}\rho(x))^{-M} \tag{3.7}$$

if $\rho(x) \geq D_a\rho(y)$. Let $J = J_1 + J_2$, where

$$J_1 = \int\limits_{D_a\rho(y)\leq\rho(x)} \eta(y)t^{-\gamma}R_{A_t^{-1}x}(A_t^{-1}y^{-1})\,dy, \quad J_2 = \int\limits_{D_a\rho(y)>\rho(x)} \eta(y)t^{-\gamma}R_{A_t^{-1}x}(A_t^{-1}y^{-1})\,dy.$$

Then, (3.7) implies that

$$|J_1| \leq Ct^{-\bar{a}-\gamma}(1+t^{-1}\rho(x))^{-M}\int \rho(y)^{\bar{a}}|\eta(y)|\,dy \leq Ct^{-\bar{a}-\gamma}(1+t^{-1}\rho(x))^{-M}. \tag{3.8}$$

Next we estimate J_2. By Lemma 2.2

$$|R_x(y)| \leq C(\psi, a)\sum_{|I|\leq[a]+1, a(I)>a}\rho(y)^{a(I)},$$

which implies that

$$|R_{A_t^{-1}x}(A_t^{-1}y^{-1})| \leq C\sum_{|I|\leq[a]+1, a(I)>a} t^{-a(I)}\rho(y)^{a(I)} \leq Ct^{-\bar{a}}\sum_{|I|\leq[a]+1, a(I)>a}\rho(y)^{a(I)}.$$

Thus

$$\begin{aligned}|J_2| &\leq Ct^{-\bar{a}-\gamma}\int\limits_{D_a\rho(y)>\rho(x)}|\eta(y)|\left(\sum_{|I|\leq[a]+1, a(I)>a}\rho(y)^{a(I)}\right)dy \\ &\leq C_{M,a}t^{-\bar{a}-\gamma}(1+\rho(x))^{-M} \leq C_{M,a}t^{-\bar{a}-\gamma}(1+t^{-1}\rho(x))^{-M}.\end{aligned} \tag{3.9}$$

By (3.8) and (3.9) we have, for any $M \geq 0$,

$$|J| \leq Ct^{-\bar{a}-\gamma}(1+t^{-1}\rho(x))^{-M} \tag{3.10}$$

for $t \geq 1$. This completes the proof of part (1).

To prove part (2), let $0 < t \leq 1$. We note that

$$(\eta * \psi_t)\tilde{}\,(x) = s^{\gamma}\tilde{\psi} * \tilde{\eta}_s(A_s x), \quad s = t^{-1} \geq 1.$$

Thus by (3.10), if $M \geq 0$ and $\int \psi P\,dx = 0$ for $P \in \mathcal{P}_a$, we have, for $x \in \mathbb{H}$,

$$|\eta * \psi_t(x)| \leq Cs^{\gamma}s^{-\bar{a}-\gamma}(1+\rho(x))^{-M} = Ct^{\bar{a}}(1+\rho(x))^{-M}.$$

This concludes the proof. □

Remark 3.3. The constants $B_j(\eta, \psi, a, M)$, $j = 1, 2$, in Lemma 3.2 can be taken independent of η and ψ if $\|\eta\|_{(L)} \leq 1$ and $\|\psi\|_{(L)} \leq 1$ and if L is sufficiently large depending on a, M.

4. Maximal function of Peetre and vector-valued inequalities

For the maximal function $(f*\varphi_t)^{**}_{N,t^{-1}}$ we have the estimate in Lemma 4.2 below. We first prove the following.

Lemma 4.1. *Let F be continuously differentiable on $\mathbb{H}$. Let $r > 0$, $N = \gamma/r$ and let $0 < u \leq 1$. Then for $x \in \mathbb{H}$, we have*

$$F^{**}_{N,1}(x) \leq C_r u^{-N}M(|F|^r)^{1/r}(x) + C_r u\sum_{j=1}^{n}(X_jF)^{**}_{N,1}(x).$$

Proof. For $u, r > 0$ and $x, z \in \mathbb{H}$ we have

(4.1)
$$|F(xz^{-1})| = \left(\fint_{B(xz^{-1},u)} |F(y) + (F(xz^{-1}) - F(y))|^r \, dy \right)^{1/r}$$
$$\le c_r \left(\fint_{B(xz^{-1},u)} |F(y)|^r \, dy \right)^{1/r} + c_r \left(\fint_{B(xz^{-1},u)} |F(xz^{-1}) - F(y)|^r \, dy \right)^{1/r},$$

where $c_r = 1$ if $r \ge 1$ and $c_r = 2^{-1+1/r}$ if $0 < r < 1$.

Let $w = xz^{-1}$, $y \in B(xz^{-1}, u)$. By Lemma 2.1

$$|F(w) - F(y)| = |F(y(y^{-1}w)) - F(y)| \le C \sum_{j=1}^{n} \rho(y^{-1}w)^{a_j} \sup_{\rho(v) \le C_1 \rho(y^{-1}w)} |(X_j F)(yv)|.$$

Since $0 < u \le 1$,

$$|F(w) - F(y)| \le Cu \sum_{j=1}^{n} \sup_{\rho(v) \le C_1 \rho(y^{-1}w)} |(X_j F)(yv)|.$$

We note that

$$\rho(y^{-1}xz^{-1}) = \rho(y^{-1}w) < u, \quad \rho(y^{-1}x) = \rho(x^{-1}y) \le c_0(u + \rho(z)).$$

Therefore

$$\sup_{\rho(v) \le C_1 \rho(y^{-1}w)} |(X_j F)(yv)| \le C \sup_{\rho(v) \le C_1 \rho(y^{-1}w)} \frac{|(X_j F)(xx^{-1}yv)|}{(1 + \rho(x^{-1}yv))^N} (1 + u + \rho(z))^N$$
$$\le C (X_j F)^{**}_{N,1}(x) (1 + \rho(z))^N.$$

It follows that

$$(4.2) \qquad \left(\fint_{B(xz^{-1},u)} |F(xz^{-1}) - F(y)|^r \, dy \right)^{1/r} \le Cu \sum_{j=1}^{n} (X_j F)^{**}_{N,1}(x) (1 + \rho(z))^N.$$

We observe that $B(xz^{-1}, u) \subset B(x, c_0(u + \rho(z)))$, since we have $\rho(y^{-1}x) \le c_0(u + \rho(z))$ if $\rho(y^{-1}(xz^{-1})) \le u$. Thus

(4.3)
$$\left(\fint_{B(xz^{-1},u)} |F(y)|^r \, dy \right)^{1/r} \le C \left(u^{-\gamma} (u + \rho(z))^{\gamma} \fint_{B(x, c_0(u+\rho(z)))} |F(y)|^r \, dy \right)^{1/r}$$
$$\le C u^{-\gamma/r} (1 + \rho(z))^{\gamma/r} M(|F|^r)(x)^{1/r}.$$

If $N = \gamma/r$, combining (4.1), (4.2) and (4.3), we have

$$\frac{|F(xz^{-1})}{(1 + \rho(z))^{\gamma/r}} \le C u^{-\gamma/r} M(|F|^r)(x)^{1/r} + Cu \sum_{j=1}^{n} (X_j F)^{**}_{N,1}(x).$$

Taking supremum in z, we get the conclusion. □

Lemma 4.2. *Let $N = \gamma/r$, $r > 0$, $0 < \delta \le 1$. Let $f, \varphi \in \mathcal{S}$. Then we have*

$$(f * \varphi_t)^{**}_{N,t^{-1}}(x) \le C_r \delta^{-N} M(|f * \varphi_t|^r)^{1/r}(x) + C_r \delta \sum_{j=1}^{n} (f * (X_j \varphi)_t)^{**}_{N,t^{-1}}(x)$$

for all $t > 0$.

To prove this we apply the following.

Lemma 4.3. *Define the operator* T_t *by* $(T_t f)(x) = f(A_t x)$. *Then, for appropriate functions* F, f, g *on* $\mathbb{H}$ *we have*

(1) $(T_t F^{**}_{N,R})(x) = (T_t F)^{**}_{N,tR}(x)$ *for all* $t, N, R > 0$;
(2) $T_t(f * g)(x) = t^\gamma (T_t f) * (T_t g)(x)$ *for every* $t > 0$;
(3) $T_t(M(f))(x) = M(T_t f)(x)$ *for every* $t > 0$.

This can be shown by direct computation.

Proof of Lemma 4.2. By (1), (2) of Lemma 4.3

$$T_t(f * \varphi_t)^{**}_{N,t^{-1}}(x) = (T_t f * \varphi)^{**}_{N,1}(x).$$

Using Lemmas 4.1, we have

$$(T_t f * \varphi)^{**}_{N,1}(x) \le C\delta^{-N} M(|T_t f * \varphi|^r)^{1/r}(x) + C\delta \sum_{j=1}^{n} (T_t f * X_j \varphi)^{**}_{N,1}(x).$$

Applying $T_{t^{-1}}$ to both sides of this inequality, we can get the conclusion, since by Lemma 4.3 we have

$$\begin{aligned} T_{t^{-1}}(T_t f * \varphi)^{**}_{N,1}(x) &= (f * \varphi_t)^{**}_{N,t^{-1}}(x), \\ T_{t^{-1}} M(|T_t f * \varphi|^r)^{1/r}(x) &= M(|f * \varphi_t|^r)^{1/r}(x), \\ T_{t^{-1}}(T_t f * X_j \varphi)^{**}_{N,1}(x) &= (f * (X_j \varphi)_t)^{**}_{N,t^{-1}}(x). \end{aligned}$$

□

Let $a, b, L \ge 0$ and

$$\begin{aligned} \mathcal{C}^{(1)}_{a,L} &= \{(\eta, \psi) \in \mathcal{S} \times \mathcal{S} : \sup_{t \ge 1} t^a C(\eta, \psi, t, L) < \infty\}, \\ \mathcal{C}^{(2)}_{b,L} &= \{(\eta, \psi) \in \mathcal{S} \times \mathcal{S} : \sup_{0 < t \le 1} t^{-b} C(\eta, \psi, t, L) < \infty\}, \\ \mathcal{C}_{a,b,L} &= \mathcal{C}^{(1)}_{a,L} \cap \mathcal{C}^{(2)}_{b,L}, \end{aligned}$$

where $C(\eta, \psi, t, L)$ is as in (3.1).

By Lemma 3.2 we have the following results.

Remark 4.4. Let a, b, c, d, L, N be non-negative numbers and $\eta, \psi \in \mathcal{S}$.

(1) If $\alpha \in \Delta$, $\bar{\alpha} \ge a + L$ and $\int \eta P \, dx = 0$ for all $P \in \mathcal{P}_\alpha$, then $(\eta, \psi) \in \mathcal{C}^{(1)}_{a,L}$.
(2) If $\beta \in \Delta$, $\bar{\beta} \ge b$ and $\int \psi P \, dx = 0$ for all $P \in \mathcal{P}_\beta$, then $(\eta, \psi) \in \mathcal{C}^{(2)}_{b,N}$. In particular, $(\eta, \psi) \in \mathcal{C}^{(2)}_{\epsilon,N}$ for some $\epsilon > 0$ and for all N if $\int \psi \, dx = 0$.
(3) We have $\mathcal{C}^{(j)}_{a,L} \subset \mathcal{C}^{(j)}_{b,L}$ if $a \ge b$ and $\mathcal{C}^{(j)}_{a,L} \subset \mathcal{C}^{(j)}_{a,N}$ if $L \ge N$ for $j = 1, 2$. The set $\mathcal{C}_{a,b,L}$ is decreasing in each of the parameters a, b, L when the other two are fixed.

Here we give a proof of part (1). Part (2) can be shown similarly. Let $t \ge 1$. By part (1) of Lemma 3.2, if $M > L + \gamma$ and $\bar{\alpha} \ge a + L$, we have

$$\begin{aligned} C(\eta, \psi, t, L) &\le C t^{-\bar{\alpha} - \gamma + M} \int_{\rho(x) \ge t} \rho(x)^{L-M} \, dx + C t^{-\bar{\alpha} - \gamma} \int_{\rho(x) \le t} (1 + \rho(x))^L \, dx \\ &\le C t^{-\bar{\alpha} + L} \le C t^{-a}. \end{aligned}$$

This completes the proof.

Using Lemmas 3.1 and 4.2, we can prove the following result.

Theorem 4.5. *Let $q \geq 1, r > 0$ and $N = \gamma/r$. Let $\varphi^{(\ell)} \in \mathcal{S}$, $\int \varphi^{(\ell)} = 0$, $1 \leq \ell \leq M$. Suppose that there exist $\eta^{(\ell)} \in \mathcal{S}$, $1 \leq \ell \leq M$, for which we have* (1.6)*. Let $f \in \mathcal{S}$. If $(\eta^{(m)}, X_k\varphi^{(\ell)}) \in \mathcal{C}^{(1)}_{N+\epsilon,N}$ with some $\epsilon > 0$ for all $k = 1, \dots, n$ and $\ell, m = 1, \dots, M$, then*

$$\sum_{\ell=1}^{M} \int_0^\infty (f * \varphi_t^{(\ell)})^{**}_{N,t^{-1}}(x)^q \,\frac{dt}{t} \leq C \sum_{\ell=1}^{M} \int_0^\infty M(|f * \varphi_t^{(\ell)}|^r)(x)^{q/r} \,\frac{dt}{t}. \tag{4.4}$$

Proof. By the assumption of the theorem and (2), (3) of Remark 4.4 we have $(\eta^{(m)}, X_k\varphi^{(\ell)}) \in \mathcal{C}_{N+\epsilon,\epsilon,N}$ for some $\epsilon > 0$. Thus by (3.3) of Lemma 3.1 we have

$$\begin{aligned}
&(f * (X_k\varphi^{(\ell)})_t)^{**}_{N,t^{-1}}(x) \\
&\leq \sum_{m=1}^{M} \sum_{j=-\infty}^{\infty} C_N b^{-Nj_+} \int_b^1 C(\eta^{(m)}, X_k\varphi^{(\ell)}, u^{-1}b^{-j}, N)(f * \varphi^{(m)}_{ub^jt})^{**}_{N,b^{-j}t^{-1}}(x) \,\frac{du}{u} \\
&\leq \sum_{m=1}^{M} \sum_{j=-\infty}^{\infty} C_{N,b} b^{\epsilon|j|} \int_b^1 (f * \varphi^{(m)}_{ub^jt})^{**}_{N,b^{-j}t^{-1}}(x) \,\frac{du}{u}.
\end{aligned}$$

Using this and Lemma 4.2, we see that

$$\begin{aligned}
(f * \varphi_t^{(\ell)})^{**}_{N,t^{-1}}(x) &\leq C\delta^{-N} M(|f * \varphi_t^{(\ell)}|^r)^{1/r}(x) \\
&\quad + C\delta \sum_{m=1}^{M} \sum_{j=-\infty}^{\infty} C_{N,b} b^{\epsilon|j|} \int_b^1 (f * \varphi^{(m)}_{ub^jt})^{**}_{N,b^{-j}t^{-1}}(x) \,\frac{du}{u}.
\end{aligned}$$

Thus, applying Hölder's inequality when $q > 1$, we have

$$\begin{aligned}
(f * \varphi_t^{(\ell)})^{**}_{N,t^{-1}}(x)^q &\leq C\delta^{-Nq} M(|f * \varphi_t^{(\ell)}|^r)(x)^{q/r} \\
&\quad + C_{N,b,q,M}\delta^q \sum_{m=1}^{M} \sum_{j=-\infty}^{\infty} b^{q\epsilon|j|/2} \left(\int_b^1 (f * \varphi^{(m)}_{ub^jt})^{**}_{N,b^{-j}t^{-1}}(x) \,\frac{du}{u} \right)^q.
\end{aligned}$$

Since $q \geq 1$, Hölder's inequality implies that

$$\left(\int_b^1 (f * \varphi^{(m)}_{ub^jt})^{**}_{N,b^{-j}t^{-1}}(x) \,\frac{du}{u} \right)^q \leq (\log(1/b))^{q/q'} \int_b^1 (f * \varphi^{(m)}_{ub^jt})^{**}_{N,b^{-j}t^{-1}}(x)^q \,\frac{du}{u}. \tag{4.5}$$

So we see that

$$\begin{aligned}
(f * \varphi_t^{(\ell)})^{**}_{N,t^{-1}}(x)^q &\leq C\delta^{-Nq} M(|f * \varphi_t^{(\ell)}|^r)(x)^{q/r} \\
&\quad + C_{N,b,q,M}\delta^q \sum_{m=1}^{M} \sum_{j=-\infty}^{\infty} b^{q\epsilon|j|/2} \int_b^1 (f * \varphi^{(m)}_{ub^jt})^{**}_{N,b^{-j}t^{-1}}(x)^q \,\frac{du}{u}.
\end{aligned} \tag{4.6}$$

By integration of both sides of the inequality (4.6) over $(0,\infty)$ with respect to the measure dt/t, it follows that

$$\begin{aligned}
&\sum_{\ell=1}^{M}\int_0^\infty (f*\varphi_t^{(\ell)})^{**}_{N,t^{-1}}(x)^q\,\frac{dt}{t} \le C\delta^{-Nq}\sum_{\ell=1}^{M}\int_0^\infty M(|f*\varphi_t^{(\ell)}|^r)(x)^{q/r}\,\frac{dt}{t}\\
&+C\delta^q\left[\sum_{j=-\infty}^{\infty} b^{q\epsilon|j|/2}\right]\sum_{\ell=1}^{M}\int_b^1\int_0^\infty (f*\varphi_t^{(\ell)})^{**}_{N,ut^{-1}}(x)^q\,\frac{dt}{t}\,\frac{du}{u}\\
&\le C\delta^{-Nq}\sum_{\ell=1}^{M}\int_0^\infty M(|f*\varphi_t^{(\ell)}|^r)(x)^{q/r}\,\frac{dt}{t}\\
&+C\delta^q\left(\int_b^1 u^{-Nq}\,\frac{du}{u}\right)\left[\sum_{j=-\infty}^{\infty} b^{q\epsilon|j|/2}\right]\sum_{\ell=1}^{M}\int_0^\infty (f*\varphi_t^{(\ell)})^{**}_{N,t^{-1}}(x)^q\,\frac{dt}{t},
\end{aligned}$$

where the inequality

$$(f*\varphi_t^{(\ell)})^{**}_{N,ut^{-1}}(x) \le u^{-N}(f*\varphi_t^{(\ell)})^{**}_{N,t^{-1}}(x)$$

has been used. The inequality (4.4) follows from this by taking δ sufficiently small, since the last sum of integrals is finite, which can be easily seen under the conditions that $f,\varphi^{(\ell)}\in\mathcal{S}$ and $\int\varphi^{(\ell)}\,dx=0$.

□

We have some vector-valued inequalities, which are stated in more general forms as weighted inequalities than needed in proving Theorem 1.1.

Theorem 4.6. *Let $N>0$, $\gamma/N<p,q<\infty, q\ge 1$ and $w\in A_{pN/\gamma}$. Let $\varphi^{(\ell)}\in\mathcal{S}$, $\int\varphi^{(\ell)}\,dx=0$, $1\le\ell\le M$. Suppose that there exist $\eta^{(\ell)}\in\mathcal{S}$, $1\le\ell\le M$, for which we have* (1.6). *Also, suppose that $(\eta^{(m)},X_k\varphi^{(\ell)})\in\mathcal{C}^{(1)}_{N+\epsilon,N}$ with some $\epsilon>0$ for $k=1,\dots,n$ and $\ell,m=1,\dots,M$. Let $\psi\in\mathcal{S}$ and $\int\psi\,dx=0$. Suppose that $(\eta^{(\ell)},\psi)\in\mathcal{C}^{(1)}_{N+\epsilon,N}$ for some $\epsilon>0$ for $1\le\ell\le M$. Let $f\in\mathcal{S}$. Then we have*

$$\left\|\left(\int_0^\infty\left((f*\psi_t)^{**}_{N,t^{-1}}\right)^q\,\frac{dt}{t}\right)^{1/q}\right\|_{L^p_w} \le C\sum_{\ell=1}^{M}\left\|\left(\int_0^\infty |f*\varphi_t^{(\ell)}|^q\,\frac{dt}{t}\right)^{1/q}\right\|_{L^p_w}$$

with a positive constant C independent of f.

Proof. Since, as in the proof of Theorem 4.5, $(\eta^{(\ell)},\psi)\in\mathcal{C}_{N+\epsilon,\epsilon,N}$, $1\le\ell\le M$, for some $\epsilon>0$, by (3.3) of Lemma 3.1 we have

$$\begin{aligned}
(f*\psi_t)^{**}_{N,t^{-1}}(x) &\le \sum_{\ell=1}^{M}\sum_{j=-\infty}^{\infty} C_N b^{-Nj_+}\int_b^1 C(\eta^{(\ell)},\psi,u^{-1}b^{-j},N)(f*\varphi^{(\ell)}_{ub^jt})^{**}_{N,b^{-j}t^{-1}}(x)\,\frac{du}{u}\\
&\le \sum_{\ell=1}^{M}\sum_{j=-\infty}^{\infty} C_{N,b}b^{\epsilon|j|}\int_b^1 (f*\varphi^{(\ell)}_{ub^jt})^{**}_{N,b^{-j}t^{-1}}(x)\,\frac{du}{u}.
\end{aligned}$$

Thus, as in the proof of Theorem 4.5, we can get

$$\int_0^\infty (f*\psi_t)^{**}_{N,t^{-1}}(x)^q\,\frac{dt}{t} \le C\sum_{\ell=1}^{M}\int_0^\infty (f*\varphi_t^{(\ell)})^{**}_{N,t^{-1}}(x)^q\,\frac{dt}{t}. \tag{4.7}$$

Let $r = \gamma/N < q, p < \infty$ and $w \in A_{pN/\gamma}$. Then by (4.7), Theorem 4.5 and Lemma 2.3, it follows that

$$\begin{aligned}
&\left(\int_{\mathbb{H}} \left(\int_0^\infty \left((f*\psi_t)^{**}_{N,t^{-1}}(x)\right)^q \frac{dt}{t}\right)^{p/q} w(x)\,dx\right)^{1/p} \\
&\qquad \le C\left(\int_{\mathbb{H}} \left(\sum_{\ell=1}^{M}\int_0^\infty (f*\varphi_t^{(\ell)})^{**}_{N,t^{-1}}(x)^q \frac{dt}{t}\right)^{p/q} w(x)\,dx\right)^{1/p} \\
&\qquad \le C\sum_{\ell=1}^{M}\left(\int_{\mathbb{H}} \left(\int_0^\infty M(|f*\varphi_t^{(\ell)}|^r)(x)^{q/r} \frac{dt}{t}\right)^{p/q} w(x)\ dx\right)^{1/p} \\
&\qquad \le C\sum_{\ell=1}^{M}\left(\int_{\mathbb{H}} \left(\int_0^\infty |(f*\varphi_t^{(\ell)})(x)|^q \frac{dt}{t}\right)^{p/q} w(x)\,dx\right)^{1/p}.
\end{aligned}$$

This completes the proof of Theorem 4.6. □

5. Proofs of Theorem 1.1 and Corollary 1.2

Suppose that $\Psi \in \mathcal{S}$ and $\int \Psi\,dx = 0$. Let $\epsilon \in (0,1)$ and

$$S_{\Psi,\epsilon}(h)(x) = \int_\epsilon^{\epsilon^{-1}} h(\cdot,t)*\Psi_t(x)\,\frac{dt}{t},$$

for appropriate functions h on $\mathbb{H}\times(0,\infty)$. Let $\mathcal{H}$ be the Hilbert space of functions $\ell(t)$ on $(0,\infty)$ such that $\|\ell\|_{\mathcal{H}} = \left(\int_0^\infty |\ell(t)|^2\,dt/t\right)^{1/2} < \infty$. Let $H^p_{\mathcal{H}}$ be the Hardy spaces of distributions on $\mathbb{H}$ with values in $\mathcal{H}$ and let $L^2_{\mathcal{H}}$ be the L^2 space of functions on $\mathbb{H}$ with values in $\mathcal{H}$.

We state some lemmas for the proof of Theorem 1.1.

Lemma 5.1. *Let $0 < p \le 1$. If $h \in H^p_{\mathcal{H}} \cap L^2_{\mathcal{H}}$, then*

$$\sup_{\epsilon\in(0,1)} \|S_{\Psi,\epsilon}(h)\|_{H^p} \le C\|h\|_{H^p_{\mathcal{H}}},$$

where C is a constant independent of h.

Proof. First we show that

$$\|S_{\Psi,\epsilon}(h)\|_2 \le C\|h\|_{L^2_{\mathcal{H}}}.$$

To see this, we note that

$$\int S_{\Psi,\epsilon}(h)(x)g(x)\,dx = \int_\epsilon^{\epsilon^{-1}} \int h(y,t)g*\tilde{\Psi}_t(y)\,dy\,\frac{dt}{t}.$$

So, Schwarz's inequality implies that

$$\left|\int S_{\Psi,\epsilon}(h)(x)g(x)\,dx\right| \le \|h\|_{L^2_{\mathcal{H}}}\left(\int\int_0^\infty |g*\tilde{\Psi}_t(y)|^2\,dy\,\frac{dt}{t}\right)^{1/2}.$$

It is known that

$$\left(\int\int_0^\infty |g*\tilde{\Psi}_t(y)|^2\,dy\,\frac{dt}{t}\right)^{1/2} \le C\|g\|_2.$$

(See [7, pp. 223–224].) Thus the result follows from the converse of Hölder's inequality.

Since $Y^I\Psi_t = t^{-a(I)}(Y^I\Psi)_t$, we easily see that

$$\int_0^\infty \left|Y^I\Psi_t(x)\ell(t)\right| \frac{dt}{t} \le \left\|Y^I\Psi_t(x)\right\|_{\mathcal{H}} \|\ell\|_{\mathcal{H}} \le C\|\ell\|_{\mathcal{H}}\rho(x)^{-\gamma-a(I)}.$$

Thus if we define $K : \mathbb{H} \to \mathcal{B} = \mathcal{B}(\mathcal{H}, \mathbb{C})$ (the space of bounded linear operators from $\mathcal{H}$ to $\mathbb{C}$) by

$$K(x)\ell = \int_\epsilon^{\epsilon^{-1}} \Psi_t(x)\ell(t)\,\frac{dt}{t},$$

then

$$\|K(x)\|_{\mathcal{B}} \le C\rho(x)^{-\gamma}, \quad \|Y^I K(x)\|_{\mathcal{B}} \le C\rho(x)^{-\gamma-a(I)}.$$

Therefore the conclusion of the lemma follows from a vector-valued version of [7, Theorem 6.10] (see [7, Theorem 6.20]). □

Also, we need the following result in proving the theorem.

Lemma 5.2. *Let A be a non-negative integer. We can find functions $U^{(\ell)}, V^{(\ell)} \in \mathcal{S}$, $1 \le \ell \le M$, such that*

(1)
$$\int U^{(\ell)} P\,dx = \int V^{(\ell)} P\,dx = 0$$
for all $P \in \mathcal{P}_A$;

(2) $U^{(\ell)} = u^{(\ell)} * v^{(\ell)}$, *with $u^{(\ell)}, v^{(\ell)} \in \mathcal{S}(\mathbb{R}^n)$ satisfying*
$$\int u^{(\ell)} P\,dx = \int v^{(\ell)} P\,dx = 0$$
for all $P \in \mathcal{P}_A$;

(3) $\operatorname{supp}(V^{(\ell)}) \subset B(0,1)$;

(4)
$$\sum_{\ell=1}^M \int_0^\infty \left(U_t^{(\ell)} * V_t^{(\ell)}\right) \frac{dt}{t} = \lim_{\substack{\epsilon\to 0,\\ B\to\infty}} \sum_{\ell=1}^M \int_\epsilon^B \left(U_t^{(\ell)} * V_t^{(\ell)}\right) \frac{dt}{t} = \delta \quad \textit{in } \mathcal{S}'.$$

Proof. This follows from [7, Theorem 1.62] except for the vanishing moment property of $v^{(\ell)}$ in (2), which can be easily shown as follows by using Lemma 1.60 of [7]. Let L be a sufficiently large number with $L \ge A$. By the remark above, for this L we have functions $\Phi^{(\ell)}, \Psi^{(\ell)} \in \mathcal{S}$, $1 \le \ell \le H$, such that

- $$\int \Phi^{(\ell)} P\,dx = \int \Psi^{(\ell)} P\,dx = 0 \quad \text{for all } P \in \mathcal{P}_L;$$
- $\Phi^{(\ell)} = \varphi^{(\ell)} * \alpha^{(\ell)}$, with $\alpha^{(\ell)} \in \mathcal{S}$ and $\varphi^{(\ell)} \in \mathcal{S}$ satisfying
$$\int \varphi^{(\ell)} P\,dx = 0 \quad \text{for all } P \in \mathcal{P}_L;$$
- $\Psi^{(\ell)}$ is supported on $B(0,1)$;
- $$\sum_{\ell=1}^H \int_0^\infty \Phi_t^{(\ell)} * \Psi_t^{(\ell)} = \delta \quad \text{in } \mathcal{S}'.$$

If L is sufficiently large, then by Lemma 1.60 of [7] $\varphi^{(\ell)}$ can be written as

$$\varphi^{(\ell)} = \sum_{A+1\le a(J)\le a_n(A+1)} X^J \phi_{J,\ell},$$

with $\phi_{J,\ell} \in \mathcal{S}$ satisfying

$$\int \phi_{J,\ell} P\, dx = 0 \quad \text{for all } P \in \mathcal{P}_A.$$

Thus, using a result of Section 2.1, we see that

$$\Phi^{(\ell)} = \sum_{A+1\le a(J)\le a_n(A+1)} X^J \phi_{J,\ell} * \alpha^{(\ell)} = \sum_{A+1\le a(J)\le a_n(A+1)} \phi_{J,\ell} * Y^{J'} \alpha^{(\ell)}. \tag{5.1}$$

Since $a(J) \ge A+1$, we have

$$\int (Y^{J'}\alpha^{(\ell)}) P\, dx = (-1)^{|J|} \int \alpha^{(\ell)} Y^J P\, dx = 0 \quad \text{for all } P \in \mathcal{P}_A.$$

We rewrite the expression of $\Phi^{(\ell)}$ in (5.1) as

$$\Phi^{(\ell)} = \sum_{k=1}^{K} \phi_{k,\ell} * \alpha_{k,\ell}$$

with

$$\int \phi_{k,\ell} P\, dx = \int \alpha_{k,\ell} P\, dx = 0 \quad \text{for all } P \in \mathcal{P}_A.$$

Then

$$\Phi^{(\ell)} * \Psi^{(\ell)} = \sum_{k=1}^{K} (\phi_{k,\ell} * \alpha_{k,\ell}) * \psi_{k,\ell},$$

where $\psi_{k,\ell} = \Psi^{(\ell)}$ for $1 \le k \le K$. By this decomposition, obviously we obtain the desired result. □

By Lemma 5.1 we have the following.

Lemma 5.3. *Let $U^{(\ell)}$, $1 \le \ell \le M$, be functions in $\mathcal{S}$ with $\int U^{(\ell)}\, dx = 0$ for which there exist $V^{(\ell)} \in \mathcal{S}$, $1 \le \ell \le M$, such that $\int V^{(\ell)}\, dx = 0$ and*

$$\sum_{\ell=1}^{M} \int_0^\infty \left(U_t^{(\ell)} * V_t^{(\ell)}\right) \frac{dt}{t} = \lim_{\substack{\epsilon\to 0,\\ B\to\infty}} \sum_{\ell=1}^{M} \int_\epsilon^B \left(U_t^{(\ell)} * V_t^{(\ell)}\right) \frac{dt}{t} = \delta \quad \text{in } \mathcal{S}'.$$

Suppose that $f \in \mathcal{S} \cap H^p$, $0 < p \le 1$. Put $h^{(\ell)}(y,t) = f * U_t^{(\ell)}(y)$. Then, $h^{(\ell)} \in H^p_{\mathcal{H}}$ and

$$\|f\|_{H^p} \le C \sum_{\ell=1}^{M} \|h^{(\ell)}\|_{H^p_{\mathcal{H}}}.$$

Proof. The fact that $h^{(\ell)} \in H^p_{\mathcal{H}}$ can be shown as in the proof of Lemma 5.1 by a theory of vector-valued singular integrals (see [7, Chap. 7]). Let $\psi \in \mathcal{S}$. If $f \in \mathcal{S} \cap H^p$, by Theorem 1.64 of [7]

$$\sum_{\ell=1}^{M} \int_\epsilon^{\epsilon^{-1}} f * (U^{(\ell)} * V^{(\ell)})_t \frac{dt}{t} * \psi_s \to f * \psi_s \quad \text{as } \epsilon \to 0.$$

It follows that

$$|f * \psi_s| \leq \liminf_{\epsilon\to 0} \sum_{\ell=1}^{M} \sup_{s>0} \left| \int_\epsilon^{\epsilon^{-1}} f * U_t^{(\ell)} * V_t^{(\ell)} \frac{dt}{t} * \psi_s \right|.$$

Taking $h(y,t) = f * U_t^{(\ell)}(y)$ and $\Psi = V^{(\ell)}$ in Lemma 5.1, we see that

$$\int \sup_{s>0, \psi\in B_{N_p}} \left| \int_\epsilon^{\epsilon^{-1}} f * U_t^{(\ell)} * V_t^{(\ell)} \frac{dt}{t} * \psi_s \right|^p dx$$
$$\leq C \int \sup_{s>0,\phi\in B_{N_p}} \left(\int_0^\infty |f * U_t^{(\ell)} * \phi_s|^2 \frac{dt}{t} \right)^{p/2} dx$$

for sufficiently large N_p. Therefore by Fatou's lemma we have

$$\int \sup_{s>0,\psi\in B_{N_p}} |f * \psi_s|^p \, dx \leq C \liminf_{\epsilon\to 0} \sum_{\ell=1}^{M} \int \sup_{s>0,\psi\in B_{N_p}} \left| \int_\epsilon^{\epsilon^{-1}} f * *U_t^{(\ell)} * V_t^{(\ell)} \frac{dt}{t} * \psi_s \right|^p dx$$
$$\leq C \sum_{\ell=1}^{M} \int \sup_{s>0,\phi\in B_{N_p}} \left(\int_0^\infty |f * U_t^{(\ell)} * \phi_s|^2 \frac{dt}{t} \right)^{p/2} dx,$$

which implies the conclusion, if N_p is sufficiently large.

□

In proving the theorem we combine Lemmas 5.2 and 5.3 with the following result.

Lemma 5.4. *Let $f \in \mathcal{S}'$ and $N > 0$. Then there exist $L > 0$ and $a \in \Delta$ such that if $\Phi = \psi * \alpha$, $\psi, \alpha \in \mathcal{S}$ with $\int \alpha P \, dx = 0$ for all $P \in \mathcal{P}_a$, then*

$$\sup_{s>0,\phi\in B_L} |f * \Phi_t * \phi_s| \leq C (f * \psi_t)^{**}_{N,t^{-1}}$$

with some constant C depending only on $\|\alpha\|_{(L)}, \|\phi\|_{(L)}, N$.

Proof. To prove this we first note that

$$C(\alpha,\phi,u,N) = \int (1+\rho(y))^N |\alpha * \phi_u(y)| \, dy \leq C \quad \text{for } u > 0, \tag{5.2}$$

if $\phi, \alpha \in B_L$ and $\int \alpha P \, dx = 0$ when $P \in \mathcal{P}_a$ for some suitable L, a. This can be seen as follows. If $u \in (0,1]$,

$$(1+\rho(y))^M |\alpha * \phi_u(y)| \leq C \int (1+\rho(yz^{-1}))^M |\alpha(yz^{-1})| (1+\rho(z))^M |\phi_u(z)| \, dz$$
$$\leq C \|\alpha\|_{(M/\gamma)} \int (1+u\rho(z))^M |\phi(z)| \, dz$$
$$\leq C \|\alpha\|_{(M/\gamma)} \int (1+\rho(z))^M |\phi(z)| \, dz$$
$$\leq C \|\alpha\|_{(M/\gamma)} \int (1+\rho(z))^{-\gamma-1} (1+\rho(z))^{(\gamma^{-1}M+1)(\gamma+1)} |\phi(z)| \, dz$$
$$\leq C \|\alpha\|_{(M/\gamma)} \|\phi\|_{(M/\gamma)}.$$

This implies (5.2) for $u \in (0,1]$ if $M \geq N + \gamma + 1$.

Next, let $u > 1$. Then, (5.2) follows from (1) of Remark 4.4 and its proof along with Remark 3.3.

Using (5.2), we see that

$$\begin{aligned}|f * \Phi_t * \phi_s(x)| = |f * \psi_t * \alpha_t * \phi_s(x)| &= \left|\int (f * \psi_t)(xy^{-1})(\alpha_t * \phi_s)(y)\, dy\right| \\ &\le (f * \psi_t)^{**}_{N,t^{-1}}(x) \int (1 + t^{-1}\rho(y))^N |\alpha_t * \phi_s(y)|\, dy \\ &= (f * \psi_t)^{**}_{N,t^{-1}}(x) \int (1 + \rho(y))^N |\alpha * \phi_{s/t}(y)|\, dy \\ &\le C(f * \psi_t)^{**}_{N,t^{-1}}(x).\end{aligned}$$

This implies the conclusion. □

Proof of Theorem 1.1. Since the second inequality of the conclusion (1.7) is shown in [7], it remains only to prove the first inequality for some $d \in \Delta$. Let $f \in \mathcal{S} \cap H^p$, $0 < p \le 1$. Let $N > \gamma/p$. Let $U^{(\ell)}$, $1 \le \ell \le M'$, be as in Lemma 5.3 with M' in place of M. Then, by Lemma 5.3 we have, for sufficiently large N_p,

$$\|f\|^p_{H^p} \le C \sum_{\ell=1}^{M'} \int \sup_{s>0, \phi \in B_{N_p}} \left(\int_0^\infty |f * U^{(\ell)}_t * \phi_s|^2 \frac{dt}{t}\right)^{p/2} dx.$$

By Lemma 5.2 we can find such $U^{(\ell)}$ and we may assume that $U^{(\ell)} = u^{(\ell)} * v^{(\ell)}$ as in Lemma 5.2 (2). For $v^{(\ell)}$, we use Lemma 5.2 (2) with a number A large enough and we apply the property $\int u^{(\ell)}\, dx = 0$ for $u^{(\ell)}$. If A of Lemma 5.2 (2) and N_p are sufficiently large, by Lemma 5.4 we have

$$\begin{aligned}&\sum_{\ell=1}^{M'} \int \sup_{s>0, \phi \in B_{N_p}} \left(\int_0^\infty |f * U^{(\ell)}_t * \phi_s|^2 \frac{dt}{t}\right)^{p/2} dx \\ &\qquad\qquad \le C \sum_{\ell=1}^{M'} \left\| \left(\int_0^\infty \left((f * u^{(\ell)}_t)^{**}_{N,t^{-1}}\right)^2 \frac{dt}{t}\right)^{1/2} \right\|_p^p,\end{aligned}$$

which implies

$$\|f\|^p_{H^p} \le C \sum_{\ell=1}^{M'} \left\| \left(\int_0^\infty \left((f * u^{(\ell)}_t)^{**}_{N,t^{-1}}\right)^2 \frac{dt}{t}\right)^{1/2} \right\|_p^p.$$

Combining this with Theorem 4.6 with $\psi = u^{(\ell)}$, $q = 2$ and $w = 1$ and recalling Remark 4.4, if d of Theorem 1.1 is sufficiently large, we get the first inequality of (1.7) for $f \in \mathcal{S} \cap H^p$. So we have (1.7) for $f \in \mathcal{S} \cap H^p$, from which we can deduce (1.7) for general $f \in H^p$, since $\mathcal{S} \cap H^p$ is dense in H^p (see [7]). This completes the proof of Theorem 1.1. □

Proof of Corollary 1.2. Let $\phi^{(j)}$, $j = 1, 2, \dots$, be as in the corollary. Then it is known that

$$\int_{\mathbb{H}} \phi^{(j)}(x) P(x)\, dx = 0 \quad \text{for all } P \in \mathcal{P}_{2j-1}$$

and

$$c_{jk} \int_0^\infty \phi^{(j)}_t * \phi^{(k)}_t \frac{dt}{t} = \delta$$

for all positive integers j, k with some non-zero constant c_{jk} (see [7, Chap. 7]). Thus we can apply Theorem 1.1 with $M = 1$, $\varphi^{(1)} = \phi^{(j)}$ and $\eta^{(1)} = \phi^{(k)}$, taking a sufficiently large number k, to get the desired result.

□

6. Another formulation for non-degeneracy

In this section we employ a version of (1.4) as a non-degeneracy condition. We first state results analogous to Theorems 1.1 and 4.6.

Theorem 6.1. *Let $0 < p \le 1$. There exists $d \in \Delta$ with the following property. If $\{\varphi^{(\ell)} \in \mathcal{S} : 1 \le \ell \le M\}$ is a family of functions such that*

(1)

$$\int \varphi^{(\ell)}\, dx = 0, \quad 1 \le \ell \le M;$$

(2)

$$\sum_{j=-\infty}^{\infty} \sum_{\ell=1}^{M} \varphi^{(\ell)}_{b^j} * \eta^{(\ell)}_{b^j} = \lim_{\substack{k\to\infty,\\ m\to\infty}} \sum_{j=-k}^{m} \sum_{\ell=1}^{M} \varphi^{(\ell)}_{b^j} * \eta^{(\ell)}_{b^j} = \delta \quad \text{in } \mathcal{S}' \tag{6.1}$$

for some $b \in (0,1)$ with some $\eta^{(\ell)} \in \mathcal{S}$, $1 \le \ell \le M$, satisfying that

$$\int \eta^{(\ell)} P\, dx = 0 \quad \text{for all } P \in \mathcal{P}_d, \quad 1 \le \ell \le M.$$

Then we have

$$c_p \|f\|_{H^p} \le \sum_{\ell=1}^{M} \|g_{\varphi^{(\ell)}}(f)\|_p \le C_p \|f\|_{H^p} \quad \text{for } f \in H^p,$$

where c_p and C_p are positive constants independent of f.

Theorem 6.2. *Suppose that $\varphi^{(\ell)} \in \mathcal{S}$, $\int \varphi^{(\ell)}\, dx = 0$, $1 \le \ell \le M$, and that we can find $\eta^{(\ell)} \in \mathcal{S}$ for which (6.1) holds with $b \in (0,1)$. Let $N > 0$, $\gamma/N < p, q < \infty$ and $w \in A_{pN/\gamma}$. Let $\varphi^{(\ell)}, \eta^{(\ell)}$ satisfy that $(\eta^{(m)}, X_k\varphi^{(\ell)}) \in \mathcal{C}^{(1)}_{N+\epsilon,N}$ for some $\epsilon > 0$, $1 \le k \le n$, $1 \le \ell, m \le M$. Let $\psi \in \mathcal{S}$ with $\int \psi\, dx = 0$. If $(\eta^{(\ell)}, \psi) \in \mathcal{C}^{(1)}_{N+\epsilon,N}$ for $1 \le \ell \le M$ with some $\epsilon > 0$, then for $f \in \mathcal{S}$ we have*

$$\left\| \left(\int_0^\infty \left((f * \psi_t)^{**}_{N,t^{-1}} \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p_w} \le C \sum_{\ell=1}^{M} \left\| \left(\int_0^\infty |f * \varphi^{(\ell)}_t|^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p_w}$$

for some positive constant C independent of f.

For $q > 0$ and $b \in (0,1)$, let

$$\Delta^{(q)}_{\varphi,b}(f)(x) = \left(\sum_{j=-\infty}^{\infty} |f * \varphi_{b^j}(x)|^q \right)^{1/q}.$$

Put $\Delta_{\varphi,b}(f) = \Delta^{(2)}_{\varphi,b}(f)$. Then we can regard $\Delta_{\varphi,b}(f)$ as a discrete parameter analogue of $g_\varphi(f)$. We have discrete parameter versions of Theorems 6.1 and 6.2 as follows.

Theorem 6.3. *Let $0 < p \le 1$. There exists $d \in \Delta$ such that if functions $\varphi^{(\ell)} \in \mathcal{S}$, $1 \le \ell \le M$, satisfy the conditions* (1) *and* (2) *of Theorem* 6.1, *then we have*

$$c_p\|f\|_{H^p} \le \sum_{\ell=1}^{M} \|\Delta_{\varphi^{(\ell)},b}(f)\|_p \le C_p\|f\|_{H^p} \quad \text{for } f \in H^p.$$

Theorem 6.4. *Let $N > 0$, $\gamma/N < p, q < \infty$ and $w \in A_{pN/\gamma}$. Let $\varphi^{(\ell)} \in \mathcal{S}$, $\int \varphi^{(\ell)}\, dx = 0$, $1 \le \ell \le M$. Suppose that* (6.1) *holds with some $\eta^{(\ell)} \in \mathcal{S}$, $1 \le \ell \le M$. We also assume that $(\eta^{(m)}, X_k\varphi^{(\ell)}) \in \mathcal{C}^{(1)}_{N+\epsilon,N}$ for some $\epsilon > 0$, $1 \le k \le n$, $1 \le \ell, m \le M$. Let $\psi \in \mathcal{S}$ and $\int \psi\, dx = 0$. If $(\eta^{(\ell)}, \psi) \in \mathcal{C}^{(1)}_{N+\epsilon,N}$ for $1 \le \ell \le M$ with some $\epsilon > 0$, then we have*

$$\left\| \left(\sum_{j=-\infty}^{\infty} \left((f * \psi_{b^j})^{**}_{N,b^{-j}} \right)^q \right)^{1/q} \right\|_{L^p_w} \le C \sum_{\ell=1}^{M} \left\| \Delta^{(q)}_{\varphi^{(\ell)},b}(f) \right\|_{L^p_w}$$

for $f \in \mathcal{S}$ with a positive constant C independent of f.

We can prove Theorems 6.1 and 6.2 similarly to Theorems 1.1 and 4.6, respectively. Let

$$\zeta = \sum_{j=-\infty}^{0} \sum_{\ell=1}^{M} \varphi^{(\ell)}_{b^j} * \eta^{(\ell)}_{b^j},$$

where $\varphi^{(\ell)}$, $\eta^{(\ell)}$ are as in Theorem 6.1. To prove Theorem 6.1 similarly to Theorem 1.1, it will be useful to note that $\zeta \in \mathcal{S}$. Also, methods which prove Theorems 6.1 and 6.2 can be applied to show Theorems 6.3 and 6.4, respectively. The restriction $q \ge 1$ is not needed in Theorems 6.2 and 6.4, which is assumed in Theorem 4.6, since estimates like (4.5) are not needed in the situation under the non-degeneracy condition (6.1). We can find relevant arguments in [16, 17]. We omit the details for the proofs of the results stated in this section.

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