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# SOME WEIGHTED ESTIMATES FOR LITTLEWOOD-PALEY FUNCTIONS AND RADIAL MULTIPLIERS

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ABSTRACT. We prove some weighted estimates for certain Littlewood-Paley operators on the weighted Hardy spaces  $H_w^p$  ( $0 < p \leq 1$ ) and on the weighted  $L^p$  spaces. We also prove some weighted estimates for the Bochner-Riesz operators and the spherical means.

## 1. INTRODUCTION

Let  $n \geq 2$  and  $\rho(x) \in C^\infty(\mathbb{R}^n \setminus \{0\})$  be positive and homogeneous of degree 1. We assume  $\nabla \rho \neq 0$  and the hypersurface

$$\Sigma = \{x \in \mathbb{R}^n : \rho(x) = 1\}$$

has non-vanishing Gaussian curvature. We define

$$\sigma_\delta(f)(x) = \left( \int_0^\infty |S_R^\delta(f)(x) - S_R^{\delta-1}(f)(x)|^2 \frac{dR}{R} \right)^{1/2}, \quad (1.1)$$

where

$$S_R^\delta(f)(x) = \int_{\mathbb{R}^n} (1 - R^{-2}\rho(\xi)^2)_+^\delta \hat{f}(\xi) e^{2\pi i x \xi} d\xi \quad (1.2)$$

is the Bochner-Riesz means of order  $\delta$  on  $\mathbb{R}^n$  with respect to  $\rho$ . By Sogge [18] we are motivated to consider  $S_R^\delta(f)$  with  $\rho(\xi)$  in place of the Euclidean norm  $|\xi|$ . We also define

$$\tau_\delta(f)(x) = \left( \int_0^\infty |\tilde{S}_R^{\delta-1}(f)(x)|^2 \frac{dR}{R} \right)^{1/2} \quad (1.3)$$

with

$$\tilde{S}_R^\delta(f)(x) = \int_{\mathbb{R}^n} \eta(\rho(\xi)/R) (1 - R^{-2}\rho(\xi)^2)_+^\delta \hat{f}(\xi) e^{2\pi i x \xi} d\xi, \quad (1.4)$$

where  $\eta \in C^\infty(\mathbb{R})$  is such that  $\eta(t) = 1$  if  $|t| \geq 1/4$  and  $\eta(t) = 0$  if  $|t| \leq 1/8$ .

Put  $\delta(p) = n|1/p - 1/2| + 1/2$ . We first study the behavior of  $\tau_\delta$ ,  $\delta \geq \delta(p)$ ,  $\delta > \delta(1)$ , on the weighted Hardy space  $H_w^p(\mathbb{R}^n)$ ,  $0 < p \leq 1$ . Under these conditions of  $\delta$  we can write  $\tau_\delta(f) = g_\psi(f)$ , where  $g_\psi(f)$  is the Littlewood-Paley function defined by

$$g_\psi(f)(x) = \left( \int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2};$$

here  $\psi_t(x) = t^{-n} \psi(t^{-1}x)$ , and  $\psi$  satisfies  $|\psi(x)| \leq c(1 + |x|)^{-n-\epsilon}$  with  $\epsilon = n(1/p - 1) + \delta - \delta(p) > 0$  and  $\int_{\mathbb{R}^n} \psi(x) dx = 0$ . So  $\tau_\delta$  is bounded on the weighted Lebesgue

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spaces  $L_w^r$  for all  $r \in (1, \infty)$  and all  $w \in A_r$  (see Sato [16] and Ding-Fan-Pan [7]), where we denote by  $A_r$  the weight class of Muckenhoupt.

*Remark 1.* We consider  $\tilde{S}_R^\delta$  to eliminate the singularity of  $\rho(\xi)$  at the origin. If  $\rho(\xi) = |\xi|$ , this is not needed. For example, we can treat  $\tau_\delta$  and  $\sigma_\delta$  in the same way in proving the estimates like those of Theorem 1 when  $\rho(\xi) = |\xi|$ .

Now we recall the definition of the weighted Hardy space  $H_w^p$ . We begin by defining the weight classes. Let  $B(x_0, s)$  be a closed ball of  $\mathbb{R}^n$  with center  $x_0$  and radius  $s > 0$ . Let  $w(x)$  be a positive measurable function on  $\mathbb{R}^n$ . Then we say  $w \in B_p$  ( $1 < p < \infty$ ) if

$$\int_{\mathbb{R}^n} |M(\chi_{B(x_0, s)})(x)|^p w(x) dx \leq C_{p, w} w(B(x_0, s)),$$

where  $M$  is the Hardy-Littlewood maximal operator,  $w(E) = \int_E w(x) dx$  and  $C_{p, w}$  is a constant independent of  $x_0$  and  $s$ ; and we say  $w \in B_1$  if

$$\sup_{\lambda > 0} \lambda w(\{x \in \mathbb{R}^n : M(\chi_{B(x_0, s)})(x) > \lambda\}) \leq C_{1, w} w(B(x_0, s)),$$

where  $C_{1, w}$  is independent of  $x_0$  and  $s$ . Note that  $M(\chi_{B(x_0, s)})(x) \approx s^n (s + |x - x_0|)^{-n}$ . It is easy to see that  $B_r \subset B_p$  for  $1 \leq r \leq p$  and  $A_p \subset B_p$  for  $1 \leq p < \infty$ . Also for any  $1 < p < \infty$  there exists  $w \in B_p$  which does not belong to  $A_\infty$  (see [10] and [23]). We observe that if  $w \in B_p$  and  $t \geq 1$ , then

$$w(B(x_0, ts)) \leq C t^{np} w(B(x_0, s)).$$

Put  $B_\infty = \cup_{p \geq 1} B_p$ . Choose  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  (the Schwartz space) which satisfies  $\int_{\mathbb{R}^n} \varphi(x) dx = 1$ . Let  $0 < p \leq 1$ ,  $w \in B_\infty$  and let  $f$  be a tempered distribution. We say that  $f \in H_w^p(\mathbb{R}^n)$  if

$$\|f\|_{H_w^p} = \left( \int_{\mathbb{R}^n} \sup_{t > 0} |\varphi_t * f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

It is convenient to consider a dense subspace of  $H_w^p$ . Let  $f \in \mathcal{S}(\mathbb{R}^n)$ ; we say  $f \in \mathcal{S}_0(\mathbb{R}^n)$  if its Fourier transform  $\hat{f}$  is compactly supported and vanishes in a neighborhood of the origin. It is known that if  $0 < p \leq 1$  and  $w \in B_\infty$ , the space  $\mathcal{S}_0$  is dense in  $H_w^p$  (see [24]).

Also let  $L_w^{p, \infty}(\mathbb{R}^n)$  denotes the weighted weak  $L^p$  space of all those measurable functions  $f$  which satisfy

$$\sup_{\lambda > 0} \lambda^p w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) = \|f\|_{L_w^{p, \infty}}^p < \infty.$$

Then we prove the following:

**Theorem 1.** *Let  $\tau_\delta$  be as in (1.3).*

(1) *Let  $0 < p < 1$ . Suppose  $w \in B_1$  and  $w \in A_\infty$ . Then*

$$\|\tau_{\delta(p)}(f)\|_{L_w^{p, \infty}} \leq C_{p, w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(2) *Let  $0 < p \leq 1$  and  $\delta > \delta(p)$ . Suppose  $w \in B_{1+n^{-1}p(\delta-\delta(p))}$  and  $w \in A_\infty$ . Then*

$$\|\tau_\delta(f)\|_{L_w^p} \leq C_{p, \delta, w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

When  $\rho(\xi) = |\xi|$ , these results also hold for  $\sigma_\delta$  in place of  $\tau_\delta$ , as we mentioned in Remark 1. We note that when  $\rho(\xi) = |\xi|$  and  $w(x) \equiv 1$ , Theorem 1 (with  $\sigma_\delta$  in place of  $\tau_\delta$ ) is due to Kaneko-Sunouchi [12]. By part (1) the Littlewood-Paley operator  $\tau_{\delta(p)}$ , initially defined on  $\mathcal{S}_0$ , has a unique sublinear extension which is bounded from  $H_w^p$  to  $L_w^{p,\infty}$ ; and by part (2)  $\tau_\delta$  extends likewise to a bounded operator from  $H_w^p$  to  $L_w^p$ . As for a recent article dealing with the boundedness on the Hardy spaces for the Littlewood-Paley functions, see also Ding-Lu-Xue [8], where they study the Marcinkiewicz integrals.

*Remark 2.* For a bounded function  $m$  define a multiplier operator  $T_m$  by  $\widehat{(T_m f)}(\xi) = m(\rho(\xi))\hat{f}(\xi)$  and a maximal function  $T_m^* f(x) = \sup_{t>0} |T_t f(x)|$ , where  $\widehat{T_t f}(\xi) = m(t\rho(\xi))\hat{f}(\xi)$ . Then by the methods of Carbery [3] (see also [5]) and essentially by Theorem 1 we can prove some estimates for  $T_m$  and  $T_m^*$  on  $H_w^p$  under certain, suitable conditions on  $m$ .

We also prove the following weighted  $L^2$  estimates for  $\sigma_\delta$  defined in (1.1).

**Theorem 2.** *If  $\delta > 1/2$  and  $0 \leq \alpha < 1$ , then*

$$\int_{\mathbb{R}^n} |\sigma_\delta(f)(x)|^2 |x|^{-\alpha} dx \leq C_{\delta,\alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx.$$

In Carbery-Rubio de Francia-Vega [6] this is proved for the case  $\rho(\xi) = |\xi|$  (see also Rubio de Francia [14] for another proof). We prove Theorem 2 for the general  $\rho(\xi)$  by applying the method of Rubio de Francia [14]. Let  $S_R^\delta$  be as in (1.2) and define

$$S_*^\delta(f)(x) = \sup_{R>0} |S_R^\delta(f)(x)|. \quad (1.5)$$

Then Theorem 2 implies, as in the case  $\rho(\xi) = |\xi|$ , the following (see [6], [14]):

**Corollary 1.** *Let  $0 < \lambda \leq (n-1)/2$ . If  $-2\lambda - 1 < \alpha < 2n\lambda/(n-1)$ , then*

$$\int_{\mathbb{R}^n} |S_*^\lambda(f)(x)|^2 |x|^\alpha dx \leq C_{\lambda,\alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^\alpha dx.$$

As in [6], by Corollary 1 we see that  $\lim_{R \rightarrow \infty} S_R^\lambda(f)(x) = f(x)$  a.e. for all  $\lambda > 0$  and  $f \in L^p(\mathbb{R}^n)$  provided  $2 \leq p < 2n/(n-1-2\lambda)$  (for the case  $p < 2$  see Tao [25]).

We can also consider the spherical means with respect to  $\rho$ . For  $\beta > 0$  let

$$M_t^\beta(f)(x) = c_\beta t^{-n} \int_{\rho(y)<t} (1-t^{-2}\rho(y)^2)^{\beta-1} f(x-y) dy \quad (f \in \mathcal{S}), \quad (1.6)$$

where  $c_\beta = \Gamma(\beta + n/2)/(\pi^{n/2}\Gamma(\beta))$ . In Section 4 we shall prove some weighted estimates for a modified version of  $M_t^\beta(f)$ .

We assume  $\rho(x) = |x|$  in (1.6) for the rest of this section. By taking the Fourier transform, we can embed these operators in an analytic family of operators in  $\beta$  in such a way that

$$M_t^0(f)(x) = c \int_{S^{n-1}} f(x-ty) d\sigma(y),$$

where  $d\sigma$  denotes the Lebesgue surface measure on the unit sphere  $S^{n-1}$ . We also define  $M_*^\beta(f)(x) = \sup_{t>0} |M_t^\beta(f)(x)|$ . The operator  $M_t^\beta$  was studied in Stein [19] (see also Stein-Wainger [21] and Kaneko-Sunouchi [12]).

Now we see some applications of Theorems 1 and 2 to the spherical means.

*Remark 3.* Define, when  $\beta + n/2 - 1 > 0$ ,

$$\begin{aligned} \nu_\beta(f)(x) &= \left( \int_0^\infty \left| \frac{\partial}{\partial t} M_t^\beta(f)(x) \right|^2 t dt \right)^{1/2} \\ &= 2 \left| \beta + \frac{n}{2} - 1 \right| \left( \int_0^\infty |M_t^\beta(f)(x) - M_t^{\beta-1}(f)(x)|^2 \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

If  $\delta = \beta + n/2 - 1 > 0$ , then  $\sigma_\delta(f)$  and  $\nu_\beta(f)$  ( $f \in \mathcal{S}$ ) are pointwise equivalent; that is, there are two positive constants  $A$  and  $B$  such that

$$\sigma_\delta(f)(x) \leq A \nu_\beta(f)(x) \leq B \sigma_\delta(f)(x). \quad (1.7)$$

This was proved by [12]. By (1.7) we immediately get the  $\nu_\beta(f)$  analogue of Theorem 1 (see the remark below Theorem 1).

*Remark 4.* Let  $\beta > 3/2 - n/2$  and  $0 \leq \alpha < 1$ . By Theorem 2 for  $\rho(\xi) = |\xi|$  (a result of Carbery-Rubio de Francia-Vega [6]) and (1.7) we have

$$\int_{\mathbb{R}^n} |\nu_\beta(f)(x)|^2 |x|^{-\alpha} dx \leq C_{\beta, \alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx.$$

*Remark 5.* We write

$$\mathcal{M}(f)(x) = \sup_{t>0} \left| \int_{S^{n-1}} f(x - ty) d\sigma(y) \right|.$$

Note that  $\mathcal{M}(f)(x) = cM_*^0(f)(x)$ . Let  $n \geq 2$ ,  $n/(n-1) < p$ . Then Duoandikoetxea-Vega [9] proved that the inequality

$$\int_{\mathbb{R}^n} |\mathcal{M}(f)(x)|^p |x|^{-\alpha} dx \leq C \int_{\mathbb{R}^n} |f(x)|^p |x|^{-\alpha} dx \quad (1.8)$$

holds for  $n - p(n-1) < \alpha < n-1$  (this was partly proved in Rubio de Francia [13]) and does not hold for  $\alpha > n-1$ . Stein [19] proved (1.8) when  $n \geq 3$ ,  $\alpha = 0$ ; the result for  $\alpha = 0$  and  $n = 2$  is due to Bourgain [1] (see also [18]). By Remark 4 and a well-known argument (see [19] and also [21]) we can give another proof of the inequality (1.8) when  $n \geq 3$ ,  $0 \leq \alpha < n-1$  and  $n/(n-1) < p$ .

In the following sections we shall give the proofs of the theorems and the corollary stated above.

## 2. PROOF OF THEOREM 1

To show Theorem 1 we prove a more general result. For a locally integrable function  $f$ , a non-negative integer  $m$  and  $\sigma \geq 0$ , we define

$$|f|_{m, \sigma} = \sup_{z \in \mathbb{R}^n, s \in (0, 1]} \inf_{Q \in \mathcal{P}_m} s^{-\sigma-n} \int_{B(z, s)} |f(y) - Q(y)| dy,$$

where  $\mathcal{P}_m$  denotes the collection of polynomials of degree less than or equal to  $m$ . We also write  $|f|_{m, \sigma} = |f : m, \sigma|$ .

Let  $\theta > n$  and let  $\psi$  be a measurable function on  $\mathbb{R}^n$  satisfying the following properties:

$$|\psi(x)| \leq C(1 + |x|)^{-\theta}, \quad (2.1)$$

$$\int_{\mathbb{R}^n} \psi(x) dx = 0; \quad (2.2)$$

furthermore,  $\psi$  can be written as

$$\psi(x) = \sum_{k=0}^{\infty} 2^{-k\theta} \eta_k(x), \quad (2.3)$$

where  $\{\eta_k\}_{k \geq 0}$  is a sequence of integrable functions satisfying the following:

$$\text{supp}(\eta_k) \subset \{2^{k-2} \leq |x| \leq 2^{k+2}\} \quad (k \geq 1), \quad \text{supp}(\eta_0) \subset \{|x| \leq 1\}, \quad (2.4)$$

$$\sup_{j \geq 1} |\eta_j : [\theta - n], \theta - n + \kappa| < \infty \quad \text{for some } \kappa > 0, \quad (2.5)$$

$$|\eta_0 : [\theta - n], \theta - n| < \infty. \quad (2.6)$$

Here  $[a]$  denotes the greatest integer less than or equal to  $a$ . Then we shall prove the following:

**Proposition 1.** *Let  $g_\psi$  be the Littlewood-Paley operator with  $\psi$  satisfying (2.1) to (2.6).*

(1) *Let  $0 < p < 1$ . Suppose  $\theta = n/p$ ,  $w \in B_1$  and  $w \in A_\infty$ . Then*

$$\|g_\psi(f)\|_{L_w^{p,\infty}} \leq C_{p,w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(2) *Let  $0 < p \leq 1$ . Suppose  $\theta > n/p$ ,  $w \in B_{p\theta/n}$  and  $w \in A_\infty$ . Then*

$$\|g_\psi(f)\|_{L_w^p} \leq C_{p,\theta,w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

To prove Proposition 1 we use the following result:

**Proposition 2.** *Let  $\Psi \in L^1(\mathbb{R}^n)$  satisfy  $\int_{\mathbb{R}^n} \Psi(x) dx = 0$  and let  $\theta > n$ . Suppose that*

$$\left( \int_0^\infty \inf_{P \in \mathcal{P}_{[\theta-n]}} \left( \int_{|y|<1} |r^n \Psi(r(x-y)) - P(y)| dy \right)^2 \frac{dr}{r} \right)^{1/2} \leq C|x|^{-\theta} \quad (2.7)$$

for  $|x| > 2$ . Then we have the following:

(1) *Let  $0 < p < 1$ . Suppose  $\theta = n/p$  and  $w \in B_1$ . If the operator  $g_\Psi$  is bounded on  $L_w^{p_0}$  for some  $p_0 \in (p, \infty)$ , then*

$$\|g_\Psi(f)\|_{L_w^{p,\infty}} \leq C_{p,w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(2) *Let  $0 < p \leq 1$ . Suppose  $\theta > n/p$  and  $w \in B_{p\theta/n}$ . If the operator  $g_\Psi$  is bounded on  $L_w^{p_0}$  for some  $p_0 \in (p, \infty)$ , then*

$$\|g_\Psi(f)\|_{L_w^p} \leq C_{p,\theta,w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

We use the atomic decomposition to prove Proposition 2. Let  $N$  be a non-negative integer and  $w$  be a locally integrable positive function on  $\mathbb{R}^n$ . Then a measurable function  $a$  on  $\mathbb{R}^n$  is called a  $(p, N, w)$  atom ( $0 < p \leq 1$ ) if for some  $x_0$  and  $s$  we have

$$\text{supp}(a) \subset B(x_0, s), \quad (2.8)$$

$$\|a\|_\infty \leq w(B(x_0, s))^{-1/p}; \quad (2.9)$$

and

$$\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0 \quad \text{for all } |\alpha| \leq N, \quad (2.10)$$

where  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multi-index and  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_n$ .

**Lemma 1.** *Let  $\Psi \in L^1(\mathbb{R}^n)$  satisfy  $\int_{\mathbb{R}^n} \Psi(x) dx = 0$  and (2.7).*

- (1) *Let  $0 < p < 1$ . Suppose  $\theta = n/p$  and  $w \in B_1$ . If the operator  $g_\Psi$  is bounded on  $L_w^{p_0}$  for some  $p_0 \in (p, \infty)$ , then for a  $(p, [n/p - n], w)$  atom  $a$  we have*

$$w(\{x \in \mathbb{R}^n : g_\Psi(a)(x) > \lambda\}) \leq C\lambda^{-p},$$

where  $C$  is independent of  $a$  and  $\lambda$ .

- (2) *Let  $0 < p \leq 1$ . Suppose  $\theta > n/p$  and  $w \in B_{p\theta/n}$ . If the operator  $g_\Psi$  is bounded on  $L_w^{p_0}$  for some  $p_0 \in (p, \infty)$ , then for a  $(p, [\theta - n], w)$  atom  $a$  we have*

$$\|g_\Psi(a)\|_{L_w^p} \leq C,$$

where  $C$  is independent of  $a$ .

This follows from the following result:

**Lemma 2.** *Let  $\Psi \in L^1(\mathbb{R}^n)$  satisfy  $\int_{\mathbb{R}^n} \Psi(x) dx = 0$  and (2.7). Let  $a$  be a  $(p, [\theta - n], w)$  atom supported in  $B(x_0, s)$  with (2.9). Then we have*

$$g_\Psi(a)(x) \leq C (|B(x_0, s)|/w(B(x_0, s)))^{1/p} s^{(\theta - n/p)} (s + |x - x_0|)^{-\theta}$$

for  $x$  with  $|x - x_0| > 2s$ .

*Proof.* We first give a proof for the case  $w(x) \equiv 1$ . By (2.7)–(2.10) with  $N = [\theta - n]$  we have, if  $|x - x_0| > 2s$ ,

$$\begin{aligned} g_\Psi(a)(x)^2 &= \int_0^\infty \left| \int_{\mathbb{R}^n} a(y) r^n \Psi(r(x - y)) dy \right|^2 \frac{dr}{r} \\ &= \int_0^\infty \inf_{P \in \mathcal{P}_{[\theta - n]}} \left| \int_{B(x_0, s)} a(y) (r^n \Psi(r(x - y)) - P(y)) dy \right|^2 \frac{dr}{r} \\ &\leq \|a\|_\infty^2 \int_0^\infty \inf_{P \in \mathcal{P}_{[\theta - n]}} \left( \int_{B(x_0, s)} |r^n \Psi(r(x - y)) - P(y)| dy \right)^2 \frac{dr}{r} \\ &= \|a\|_\infty^2 \int_0^\infty \inf_{P \in \mathcal{P}_{[\theta - n]}} \left( \int_{|y| < 1} |(rs)^n \Psi(rs(s^{-1}x - s^{-1}x_0 - y)) - P(y)| dy \right)^2 \frac{dr}{r} \\ &\leq C \|a\|_\infty^2 (s^{-1}|x - x_0|)^{-2\theta} \\ &\leq C s^{-2n/p + 2\theta} |x - x_0|^{-2\theta} \leq C s^{2(\theta - n/p)} (s + |x - x_0|)^{-2\theta}. \end{aligned}$$

Next, let  $a$  be a  $(p, [\theta - n], w)$  atom supported in  $B(x_0, s)$  with (2.9). Then applying the above estimate to

$$(w(B(x_0, s))/|B(x_0, s)|)^{1/p} a,$$

we get the conclusion.  $\square$

Now we give the proof of Lemma 1. We first prove part (1). Let  $a$  be a  $(p, [n/p - n], w)$  atom supported in  $B(x_0, s)$  with (2.9). Then

$$\begin{aligned} w(\{x \in \mathbb{R}^n : g_\Psi(a)(x) > \lambda\}) &\leq w(\{x \in B(x_0, 2s) : g_\Psi(a)(x) > \lambda\}) \\ &\quad + w(\{x \in \mathbb{R}^n \setminus B(x_0, 2s) : g_\Psi(a)(x) > \lambda\}) \\ &= I + II, \quad \text{say.} \end{aligned}$$

Since  $g_\Psi$  is bounded on  $L_w^{p_0}$ , by Chebyshev's inequality and Hölder's inequality we have

$$\begin{aligned}
I &\leq \lambda^{-p} \int_{B(x_0, 2s)} |g_\Psi(a)(x)|^p w(x) dx \\
&\leq \lambda^{-p} w(B(x_0, 2s))^{(p_0-p)/p_0} \left( \int |g_\Psi(a)(x)|^{p_0} w(x) dx \right)^{p/p_0} \\
&\leq C \lambda^{-p} w(B(x_0, 2s))^{(p_0-p)/p_0} \left( \int |a(x)|^{p_0} w(x) dx \right)^{p/p_0} \\
&\leq C \lambda^{-p} w(B(x_0, 2s))^{(p_0-p)/p_0} w(B(x_0, 2s))^{-1+p/p_0} \\
&= C \lambda^{-p},
\end{aligned} \tag{2.11}$$

where to get the last inequality we have used the doubling condition.

Next, by Lemma 2 we see that

$$\begin{aligned}
II &\leq w \left( \left\{ x \in \mathbb{R}^n : C (|B(x_0, s)|/w(B(x_0, s)))^{1/p} (s + |x - x_0|)^{-n/p} > \lambda \right\} \right) \\
&= w \left( \left\{ x \in \mathbb{R}^n : C s^n (s + |x - x_0|)^{-n} > w(B(x_0, s)) \lambda^p \right\} \right) \\
&= III, \quad \text{say.}
\end{aligned}$$

Since  $w \in B_1$ , recalling that  $s^n (s + |x - x_0|)^{-n} \approx M(\chi_{B(x_0, s)})(x)$ , we have

$$III \leq w \left( \left\{ x \in \mathbb{R}^n : M(\chi_{B(x_0, s)})(x) > w(B(x_0, s)) \lambda^p \right\} \right) \leq C \lambda^{-p}.$$

Combining the estimates for  $I$  and  $II$ , we conclude the proof of part (1).

Next we turn to the proof of part (2). Let  $a$  be a  $(p, [\theta - n], w)$  atom supported in  $B(x_0, s)$  with (2.9). Then by Lemma 2 we have

$$g_\Psi(a)(x) \leq C w(B(x_0, s))^{-1/p} M(\chi_{B(x_0, s)})(x)^{\theta/n} \quad \text{for } |x - x_0| > 2s.$$

Since  $w \in B_{p\theta/n}$ , we find

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B(x_0, 2s)} g_\Psi(a)(x)^p w(x) dx &\leq C w(B(x_0, s))^{-1} \int_{\mathbb{R}^n} M(\chi_{B(x_0, s)})(x)^{p\theta/n} w(x) dx \\
&\leq C.
\end{aligned}$$

Combining this with the estimate appearing in (2.11), we get the conclusion.

To prove Proposition 2 (1) we need the following result (see [20]):

**Lemma 3.** *Let  $0 < p < 1$ . Suppose  $\{f_k\}$  is a sequence of measurable functions on  $\mathbb{R}^n$  such that*

$$\sup_{\lambda > 0} \lambda^p w(\{x : |f_k(x)| > \lambda\}) \leq 1 \quad \text{for all } k,$$

*and suppose  $\{c_k\}$  is a sequence of complex numbers satisfying  $\sum |c_k|^p \leq 1$ . Then we have*

$$\sup_{\lambda > 0} \lambda^p w \left( \left\{ x \in \mathbb{R}^n : \sum |c_k f_k(x)| > \lambda \right\} \right) \leq \frac{2-p}{1-p}.$$

Now we can prove Proposition 2. We note that  $f \in \mathcal{S}_0(\mathbb{R}^n)$  can be decomposed as  $f = \sum \lambda_k a_k$  by  $(p, [\theta - n], w)$ -atoms ( $w \in B_{p\theta/n}$ ), where we have  $\sum \lambda_k^p \leq C \|f\|_{H_w^p}^p$ ,  $\sum \lambda_k a_k = f$  a.e. and  $\sum \lambda_k |a_k| \leq C f^*$ , with  $f^*$  denoting the grand maximal function (see [24]). Using this decomposition, we first prove part (1). Since  $f^*$  is

bounded, by the dominated convergence theorem we have  $\Psi_t * f = \sum \lambda_k \Psi_t * a_k$  a.e. and so  $g_\Psi(f) \leq \sum_k |\lambda_k| g_\Psi(a_k)$ . Thus by Lemma 1 (1) and Lemma 3 we see that

$$\sup_{\lambda > 0} \lambda^p w(\{x \in \mathbb{R}^n : g_\Psi(f)(x) > \lambda\}) \leq C \sum \lambda_k^p \leq C \|f\|_{H_w^p}^p.$$

This completes the proof of Proposition 2 (1). Part (2) can be proved in the same way by using Lemma 1 (2).

Now we turn to the proof of Proposition 1. First we see that if  $\psi$  satisfies the conditions (2.1)–(2.6), then  $\psi$  satisfies the condition (2.7) of Proposition 2. Let  $|x| > 2$ . Then by (2.1) we have

$$\begin{aligned} \int_1^\infty \left( \int_{|y|<1} |r^n \psi(r(x-y))| dy \right)^2 \frac{dr}{r} &\leq C \int_1^\infty r^{2n} (1+r|x|)^{-2\theta} \frac{dr}{r} \\ &\leq C |x|^{-2\theta} \int_1^\infty r^{2n-2\theta} \frac{dr}{r} \leq C |x|^{-2\theta}. \end{aligned} \quad (2.12)$$

Let  $r \leq 1$ . Suppose  $2^m |x|^{-1} \leq r < 2^{m+1} |x|^{-1}$  for  $m \leq m_x := [(\log 2)^{-1} \log |x|]$ . If  $|y| \leq 1$ , then  $r|x|/2 \leq r|x-y| \leq 3r|x|/2$ . Therefore, if  $m \geq 5$ , by (2.3) and (2.4) we have

$$\psi(r(x-y)) = \sum_{k=m-3}^{m+5} 2^{-k\theta} \eta_k(r(x-y)).$$

This expression of  $\psi$  and (2.5) imply that there exists a polynomial  $P = P_{r,x} \in \mathcal{P}_{[\theta-n]}$  such that

$$\int_{|y|<1} |r^n \psi(r(x-y)) - P(y)| dy \leq C r^{\kappa+\theta} 2^{-m\theta} \leq C |x|^{-\kappa-\theta} 2^{m\kappa}. \quad (2.13)$$

If  $m \leq 4$ , then

$$\psi(r(x-y)) = \sum_{k=0}^8 2^{-k\theta} \eta_k(r(x-y)).$$

Therefore, by (2.5) and (2.6) there exists a polynomial  $P = P_{r,x} \in \mathcal{P}_{[\theta-n]}$  such that

$$\int_{|y|<1} |r^n \psi(r(x-y)) - P(y)| dy \leq C r^\theta \leq C |x|^{-\theta} 2^{m\theta}. \quad (2.14)$$

By (2.13) and (2.14) we have

$$\begin{aligned} &\int_0^1 \inf_{P \in \mathcal{P}_{[\theta-n]}} \left( \int_{|y|<1} |r^n \psi(r(x-y)) - P(y)| dy \right)^2 \frac{dr}{r} \\ &\leq \sum_{m \leq m_x} \int_{2^m |x|^{-1}}^{2^{m+1} |x|^{-1}} \inf_{P \in \mathcal{P}_{[\theta-n]}} \left( \int_{|y|<1} |r^n \psi(r(x-y)) - P(y)| dy \right)^2 \frac{dr}{r} \\ &\leq \sum_{m \leq 4} C |x|^{-2\theta} 2^{2m\theta} + \sum_{5 \leq m \leq m_x} C |x|^{-2(\kappa+\theta)} 2^{2m\kappa} \leq C |x|^{-2\theta}. \end{aligned} \quad (2.15)$$

Now the condition (2.7) of Proposition 2 follows from (2.12) and (2.15).

Also by [16] we see that the conditions (2.1) and (2.2) imply the  $L_w^p$ -boundedness of  $g_\psi$  for all  $p \in (1, \infty)$  and all  $w \in A_p$ . So Proposition 1 follows from Proposition 2.

Now we give the proof of Theorem 1. Let

$$K^\delta(x) = \int_{\mathbb{R}^n} \eta(\rho(\xi)) (1 - \rho(\xi)^2)_+^\delta e^{2\pi i x \xi} d\xi.$$

Then

$$|D^\alpha K^{\delta-1}(x)| \leq C_\alpha (1 + |x|)^{-\delta-(n-1)/2} \quad (2.16)$$

for all  $\alpha$ , where  $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$  (see [18]). Therefore, by [15] we see that  $K^{\delta-1}$  satisfies the conditions (2.1)–(2.6) for  $\psi$  with  $\theta = \delta + (n-1)/2$  and  $0 < \kappa \leq [\delta - (n+1)/2] + 1 - \delta + (n+1)/2$  in (2.5). Thus Theorem 1 follows from Proposition 1.

### 3. PROOFS OF THEOREM 2 AND COROLLARY 1

The following result can be used to prove Theorem 2.

**Proposition 3.** *Let  $0 < \delta < 1$  and suppose that  $m_\delta(r) = \chi_{[1-\delta, 1]}(r)$  or  $m_\delta(r)$  is a continuously differentiable function supported in the interval  $[1-\delta, 1]$  and satisfying  $\|(d/dr)m_\delta\|_{L^1(\mathbb{R})} \leq 1$ . Define*

$$\widehat{(U_t^\delta f)}(\xi) = \hat{f}(\xi) m_\delta(t\rho(\xi)).$$

Then for  $0 \leq \alpha < 1$  we have

$$\int_{\mathbb{R}^n} \int_0^\infty |U_t^\delta f(x)|^2 |x|^{-\alpha} \frac{dt}{t} dx \leq C_\alpha \delta \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx,$$

where  $C_\alpha$  is independent of  $\delta$ .

This was proved in Carbery-Rubio de Francia-Vega [6] and Rubio de Francia [14] when  $\rho(\xi) = |\xi|$ . To prove the general case we use the method of [14], which is based on an application of Hirschman's method in [11] and the weighted estimates for the one dimensional square functions. To apply that method to our case we only need to observe that  $\Lambda(x) = (|x|/\rho(x))x$  is bi-Lipschitz, with  $\|x\| = \max(|x_1|, \dots, |x_n|)$ , that is

$$A|x - y| \leq |\Lambda(x) - \Lambda(y)| \leq B|x - y|$$

for some constants  $A, B > 0$ ; but this is an easy consequence of the fact that  $\rho(x)$  is positive, homogeneous of degree one and  $C^\infty$  in  $\mathbb{R}^n \setminus \{0\}$ .

Now we prove Theorem 2. We decompose

$$\rho(\xi)^2 (1 - \rho(\xi)^2)_+^{\delta-1} = \sum_{k=0}^{\infty} 2^{-(\delta-1)k} m_k(\rho(\xi)),$$

where  $m_k(t) \in C_0^\infty(\mathbb{R})$ ,  $\text{supp}(m_k) \subset [1 - 2^{-k}, 1]$  and  $|(d/dr)m_k(r)| \leq C2^k$ , for  $k \geq 1$ . Put  $\psi_k(x) = \mathcal{F}^{-1}(m_k(\rho(\xi)))(x)$  and  $g_k(f) = g_{\psi_k}(f)$ , where  $\mathcal{F}^{-1}$  denotes the inverse Fourier transform. We can take  $m_0(t)$  so that  $g_0$  is bounded on  $L_w^2$  for any  $w \in A_2$ . Now by Proposition 3 for  $k \geq 1$  we have

$$\|g_k(f)\|_{L^2(|x|^{-\alpha})} \leq C2^{-k/2} \|f\|_{L^2(|x|^{-\alpha})} \quad \text{for } 0 \leq \alpha < 1.$$

Thus if  $\delta > 1/2$  we have

$$\begin{aligned} \|\sigma_\delta(f)\|_{L^2(|x|^{-\alpha})} &\leq \sum_{k=0}^{\infty} 2^{-(\delta-1)k} \|g_k(f)\|_{L^2(|x|^{-\alpha})} \\ &\leq \sum_{k=0}^{\infty} C 2^{-(\delta-1)k} 2^{-k/2} \|f\|_{L^2(|x|^{-\alpha})} \\ &\leq C_\delta \|f\|_{L^2(|x|^{-\alpha})}. \end{aligned}$$

This completes the proof.

To apply the result to the maximal operator  $S_*^\delta$  defined in (1.5) we use the following, which can be proved as in the case  $\rho(\xi) = |\xi|$  (see Stein-Weiss [22, Chap. VII]).

**Lemma 4.** *Let  $S_R^\delta$  be as in (1.2). If  $\beta > 0$  and  $\delta > -1$ , then we have*

$$S_R^{\delta+\beta}(f)(x) = \frac{2\Gamma(\delta + \beta + 1)}{\Gamma(\delta + 1)\Gamma(\beta)} \int_0^1 (1-t^2)^{\beta-1} t^{2\delta+1} S_{Rt}^\delta(f)(x) dt,$$

for a suitable function  $f$ .

Here we give the proof of Corollary 1. Using Lemma 4 and Theorem 2 and arguing as in the proof of [22, Lemma 5.10] we have

$$\|S_*^\lambda(f)\|_{L^2(|x|^\alpha)} \leq C_{\lambda,\alpha} \|f\|_{L^2(|x|^\alpha)} \quad (3.1)$$

for all  $\lambda > 0$  and  $-1 < \alpha \leq 0$ . It is known that if  $\lambda \geq (n-1)/2$ , then

$$\|S_*^\lambda(f)\|_{L^2(|x|^\beta)} \leq C_{\lambda,\beta} \|f\|_{L^2(|x|^\beta)} \quad (3.2)$$

for  $-n < \beta < n$ . We extend the estimates (3.1) and (3.2) to complex  $\lambda$  and interpolating between them, we get the conclusion.

*Remark 6.* Let

$$H_\delta(f)(x) = \left( \int_0^\infty |s_R^\delta(f)(x) - s_R^{\delta-1}(f)(x)|^2 \frac{dR}{R} \right)^{1/2},$$

where

$$s_R^\delta(f)(x) = \int_{\mathbb{R}^n} (1 - R^{-1}\rho(\xi))_+^\delta \hat{f}(\xi) e^{2\pi i x \xi} d\xi.$$

Then we can prove the weighted estimates of Theorem 2 for  $H_\delta$  in place of  $\sigma_\delta$  by the same argument as in the case of  $\sigma_\delta$ . This result also can be used to prove the estimate (3.1) (see [2, 3, 4]).

#### 4. FURTHER RESULTS

For a locally integrable function  $f$ , a non-negative integer  $m$  and  $\sigma \geq 0$ , we define

$$|f|_{m,\sigma}^* = \sup_{z \in \mathbb{R}^n, s > 0} \inf_{Q \in \mathcal{P}_m} s^{-\sigma-n} \int_{B(z,s)} |f(y) - Q(y)| dy.$$

Let  $\psi \in L^1(\mathbb{R}^n)$  and  $\theta \geq 0$ . We say  $\psi \in \mathcal{F}(m, \sigma, \theta)$  if  $\psi$  can be written as in (2.3) with  $\{\eta_k\}_{k \geq 0}$  satisfying (2.4) and the condition  $\sup_{k \geq 0} |\eta_k|_{m,\sigma}^* < \infty$ . This function class was introduced by Sato [15] to make a unified approach to the studies of maximal Bochner-Riesz means and maximal spherical means in certain problems. By the methods in the proof of Theorem 1 we can prove the following:

**Proposition 4.** *Let  $\theta > n$  and  $L \in \mathcal{F}([\theta - n], \theta - n, \theta)$ . Define  $T^*(f)(x) = \sup_{t>0} |L_t * f(x)|$ .*

(1) *Let  $0 < p < 1$ . Suppose  $\theta = n/p$  and  $w \in B_1$ . Then*

$$\|T^*(f)\|_{L_w^{p,\infty}} \leq C_{p,w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(2) *Let  $0 < p \leq 1$ . Suppose  $\theta > n/p$  and  $w \in B_{p\theta/n}$ . Then*

$$\|T^*(f)\|_{L_w^p} \leq C_{p,\theta,w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(3) *Let  $0 < p \leq 1$ . Suppose  $\theta > n/p$ ,  $w \in B_{p\theta/n}$  and  $w \in A_\infty$ . Then*

$$\|L_t * f\|_{H_w^p} \leq C_{p,\theta,w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n),$$

where the constant  $C_{p,\theta,w}$  is independent of  $t > 0$ .

*Proof.* Since  $L \in \mathcal{F}([\theta - n], \theta - n, \theta)$ , arguing as in [15] we have

$$\begin{aligned} T^*(a)(x) &\leq C (|B(x_0, s)|/w(B(x_0, s)))^{1/p} s^{(\theta-n/p)} (s + |x - x_0|)^{-\theta} \\ &\leq C w(B(x_0, s))^{-1/p} M(\chi_{B(x_0, s)})(x)^{\theta/n}, \end{aligned} \quad (4.1)$$

where  $a$  is a  $(p, [\theta - n], w)$  atom supported in  $B(x_0, s)$  with (2.9). As in the case of the proof of Proposition 2, this implies parts (1) and (2). Part (3) follows from this estimate along with the multiplier characterization of the weighted Hardy spaces (see [24, Chap. VI, Theorem 4]), which requires the condition  $w \in A_\infty$ . This completes the proof.  $\square$

When  $w \in A_1$ , part (1) of Proposition 4 is in [15]. Also, if  $0 < p < 1$ ,  $w \in A_1$  and  $\rho(\xi) = |\xi|$ , it is known that  $S_*^{\delta(p)-1}$  extends to a bounded operator from  $H_w^p$  to  $L_w^{p,\infty}$  (see [15]). Let  $\theta = \delta + (n-1)/2$ ,  $\delta \geq \delta(p)$ ,  $0 < p \leq 1$ ,  $\delta > \delta(1)$ . Then the estimate (2.16) implies that  $K^{\delta-1} \in \mathcal{F}([\theta - n], \theta - n, \theta)$  (see [15]). Thus by Proposition 4 we have the following:

**Corollary 2.** *Let  $\tilde{S}_*^\delta(f)(x) = \sup_{R>0} |\tilde{S}_R^\delta(f)(x)|$ , where  $\tilde{S}_R^\delta(f)(x)$  is as in (1.4).*

(1) *Let  $0 < p < 1$  and  $w \in B_1$ . Then*

$$\|\tilde{S}_*^{\delta(p)-1}(f)\|_{L_w^{p,\infty}} \leq C_{p,w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(2) *Let  $0 < p \leq 1$ ,  $\delta > \delta(p)$  and  $w \in B_{1+n^{-1}p(\delta-\delta(p))}$ . Then*

$$\left\| \tilde{S}_*^{\delta-1}(f) \right\|_{L_w^p} \leq C_{p,\delta,w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(3) *Let  $0 < p \leq 1$ ,  $\delta > \delta(p)$ ,  $w \in B_{1+n^{-1}p(\delta-\delta(p))}$  and  $w \in A_\infty$ . Then*

$$\left\| \tilde{S}_R^{\delta-1}(f) \right\|_{H_w^p} \leq C_{p,\delta,w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n),$$

where the constant  $C_{p,\delta,w}$  is independent of  $R > 0$ .

Part (3) of Corollary 2 extends a result of Sjölin [17] to the weighted Hardy spaces. When  $\rho(\xi) = |\xi|$  and  $w(x) \equiv 1$ , part (1) (with  $S_*^\delta(f)$  in place of  $\tilde{S}_*^\delta(f)$ ) is proved in Stein-Taibleson-Weiss [20]. The estimate for  $\tilde{S}_*^\delta$  similar to [20, (2.9)] immediately follows from (2.16), as we can see from the proof of [20, (2.9)]. We can also have the estimate (4.1) for  $\tilde{S}_*^{\delta-1}$  in place of  $T^*$  as an application of that estimate.

If  $0 < p < 1$ ,  $w \in A_1$  and  $\rho(x) = |x|$ , then it is known that  $M_*^{\beta(p)-1/2}$  is bounded from  $H_w^p$  to  $L_w^{p,\infty}$ , where  $\beta(p) = n(1/p - 1) + 3/2$  (see [15]). For  $\beta > 0$  let

$$\tilde{M}_t^\beta(f)(x) = c_\beta t^{-n} \int_{\rho(y) < t} \eta(\rho(y)/t) (1 - t^{-2}\rho(y)^2)^{\beta-1} f(x-y) dy,$$

where  $c_\beta$  is as in (1.6) and  $\eta$  is as in (1.4). Then  $\eta(\rho(y))(1 - \rho(y)^2)_+^{\beta-1} \in \mathcal{F}([\theta - n], \theta - n, \theta)$ , where  $\beta > 1$  and  $\theta = \beta + n - 1$ , and hence by Proposition 4 we also have the following:

**Corollary 3.** *Let  $\tilde{M}_*^\beta(f)(x) = \sup_{t>0} |\tilde{M}_t^\beta(f)(x)|$  and write  $\beta(p) = n(1/p - 1) + 3/2$ .*

(1) *Let  $0 < p < 1$  and  $w \in B_1$ . Then*

$$\|\tilde{M}_*^{\beta(p)-1/2}(f)\|_{L_w^{p,\infty}} \leq C_{p,w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(2) *Let  $0 < p \leq 1$ ,  $\beta > \beta(p)$  and  $w \in B_{1+n^{-1}p(\beta-\beta(p))}$ . Then*

$$\|\tilde{M}_*^{\beta-1/2}(f)\|_{L_w^p} \leq C_{p,\beta,w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(3) *Let  $0 < p \leq 1$ ,  $\beta > \beta(p)$ ,  $w \in B_{1+n^{-1}p(\beta-\beta(p))}$  and  $w \in A_\infty$ . Then*

$$\|\tilde{M}_t^{\beta-1/2}(f)\|_{H_w^p} \leq C_{p,\beta,w} \|f\|_{H_w^p}, \quad f \in \mathcal{S}_0(\mathbb{R}^n),$$

where the constant  $C_{p,\beta,w}$  is independent of  $t > 0$ .

When  $\rho(\xi) = |\xi|$  and  $w(x) \equiv 1$ , part (1) of Corollary 3 with  $M_*^\beta(f)$  in place of  $\tilde{M}_*^\beta(f)$  is proved in Stein-Taibleson-Weiss [20]. The estimate (4.1) for  $\tilde{M}_*^{\beta-1/2}$  in place of  $T^*$  also follows from an application of the argument in [20].

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