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by

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Abstract. We consider the weighted Sobolev spaces associated with non-isotropic dilations of Calderón–Torchinsky and characterize the spaces by the square functions of Marcinkiewicz type including those defined with repeated uses of averaging operation.

1. Introduction. Let $B(x,t)$ be a ball in $\mathbb{R}^n$ with radius t centered at x. For $0<\alpha<2$ let

$$(1.1)\qquad V_\alpha(f)(x)=\left(\int_0^\infty \left|f(x)-\fint_{B(x,t)} f(y)\,dy\right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2},$$

where $\fint_{B(x,t)} f(y)\,dy$ denotes $|B(x,t)|^{-1}\int_{B(x,t)} f(y)\,dy$ and $|B(x,t)|$ the Lebesgue measure. In [1] the operator V_1 was used to characterize the Sobolev space $W^{1,p}(\mathbb{R}^n)$ as follows.

Theorem A. *Let $1<p<\infty$. Then f belongs to $W^{1,p}(\mathbb{R}^n)$ if and only if $f\in L^p(\mathbb{R}^n)$ and $V_1(f)\in L^p(\mathbb{R}^n)$; furthermore,*

$$\|V_1(f)\|_p \simeq \|\nabla f\|_p,$$

which means that there exist positive constants c_1, c_2 independent of f such that

$$c_1\|V_1(f)\|_p \le \|\nabla f\|_p \le c_2\|V_1(f)\|_p.$$

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing smooth functions on $\mathbb{R}^n$. Define

$$\mathcal{S}_0(\mathbb{R}^n)=\{f\in\mathcal{S}(\mathbb{R}^n) : \hat{f} \text{ vanishes near the origin}\},$$

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where the Fourier transform $\hat{f}$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle x,\xi\rangle}\,dx, \qquad \langle x,\xi\rangle = \sum_{k=1}^{n} x_k\xi_k.$$

We also write $\mathcal{F}(f)$ for $\hat{f}$. For $0 < \alpha < n$, $n \geq 2$, let I_α be the Riesz potential operator defined by

$$\mathcal{F}(I_\alpha(f))(\xi) = (2\pi|\xi|)^{-\alpha}\hat{f}(\xi), \qquad f \in \mathcal{S}_0 \tag{1.2}$$

(see [28, Chap. V]). Let

$$S_\alpha(f)(x) = \left(\int_0^\infty \left|I_\alpha(f)(x) - \fint_{B(x,t)} I_\alpha(f)(y)\,dy\right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2}, \tag{1.3}$$

Then we also find the following result in [1].

Theorem B. *Let $0 < \alpha < 2$ and $1 < p < \infty$. Then*

$$\|S_\alpha(f)\|_p \simeq \|f\|_p.$$

Theorem A can be derived from this result with $\alpha = 1$ when $n \geq 2$.

The operator S_α is a kind of Littlewood–Paley operator. Let $\psi \in L^1(\mathbb{R}^n)$ satisfy

$$\int_{\mathbb{R}^n} \psi(x)\,dx = 0. \tag{1.4}$$

Put $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. Then the Littlewood–Paley function on $\mathbb{R}^n$ is defined by

$$g_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t}\right)^{1/2}. \tag{1.5}$$

We can see that $S_\alpha(f) = g_{\psi^{(\alpha)}}(f)$, where

$$\psi^{(\alpha)}(x) = L_\alpha(x) - \Phi * L_\alpha(x), \tag{1.6}$$

with

$$L_\alpha(x) = \tau(\alpha)|x|^{\alpha-n}, \qquad \tau(\alpha) = \frac{\Gamma(n/2-\alpha/2)}{\pi^{n/2}2^\alpha\Gamma(\alpha/2)}$$

and $\Phi = \chi_0$, $\chi_0 = |B(0,1)|^{-1}\chi_{B(0,1)}$ (χ_E denotes the characteristic function of a set E). We note that $\mathcal{F}(L_\alpha)(\xi) = (2\pi|\xi|)^{-\alpha}$, $0 < \alpha < n$.

The square function $S_1(f)$ is closely related to the Marcinkiewicz function on $\mathbb{R}^1$, which is defined by

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where $F(x) = \int_{-\infty}^{x} f(y)\,dy$ for $f \in \mathcal{S}(\mathbb{R})$. It is known that

$$\|\mu(f)\|_p \simeq \|f\|_p \tag{1.7}$$

for $1 < p < \infty$. Also, we consider a variant of $\mu(f)$ which can be regarded as an analogue of S_1 in the one-dimensional case:

$$\nu(f)(x) = \left(\int_0^\infty |F(x) - F * \Phi_t(x)|^2 \,\frac{dt}{t^3} \right)^{1/2},$$

where $\Phi = (1/2)\chi_{[-1,1]}$. It is known that

$$\mu(f) = g_\psi(f) \quad \text{with} \quad \psi(x) = \chi_{[-1,1]}(x)\operatorname{sgn}(x).$$

By inspection, we see that $\nu(f) = g_{\psi^{(0)}}(f)$, where $\psi^{(0)}(x) = (1/2)\psi(x) - (1/2)\psi^{(1)}(x)$ with $\psi^{(1)}(x) = x\chi_{[-1,1]}(x)$. This would indicate that the square functions $\mu(f)$ and $\nu(f)$ are intimately related. For the Marcinkiewicz function we refer to [14], Zygmund [32], Waterman [31].

An interesting feature of Theorem A is that it suggests the possibility of defining the Sobolev space analogous to $W^{1,p}(\mathbb{R}^n)$ in metric measure spaces in a reasonable way. In this note, we shall extend Theorem A to the case of weighted Sobolev spaces with parabolic metrics of Calderón–Torchinsky [3, 4].

Let P be an $n \times n$ real matrix, $n \geq 2$, such that

$$\langle Px, x\rangle \geq \langle x, x\rangle \quad \text{for all } x \in \mathbb{R}^n. \tag{1.8}$$

A dilation group $\{\delta_t\}_{t>0}$ on $\mathbb{R}^n$ is defined by $\delta_t = t^P = \exp((\log t)P)$.

It is known that $|\delta_t x| = \langle \delta_t x, \delta_t x\rangle^{1/2}$ is strictly increasing as a function of t on $(0, \infty)$ when $x \neq 0$. Let $\rho(x)$, $x \neq 0$, be the unique positive real number t such that $|\delta_{t^{-1}} x| = 1$, and let $\rho(0) = 0$. Then the norm function ρ is continuous on $\mathbb{R}^n$ and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$ and satisfies $\rho(A_t x) = t\rho(x)$, $t > 0$, $x \in \mathbb{R}^n$. We have the following properties of $\rho(x)$ (see [3, 5]):

(1) $\rho(-x) = \rho(x)$ for all $x \in \mathbb{R}^n$;
(2) $\rho(x + y) \leq \rho(x) + \rho(y)$ for all $x, y \in \mathbb{R}^n$;
(3) $\rho(x) \leq 1$ if and only if $|x| \leq 1$;
(4) $c_1\rho(x)^{\tau_1} \leq |x| \leq \rho(x)$ when $|x| \leq 1$ for some $c_1, \tau_1 > 0$;
(5) $\rho(x) \leq |x| \leq c_2\rho(x)^{\tau_2}$ when $|x| \geq 1$ for some $c_2, \tau_2 > 0$.

Moreover,

(a) $|\delta_t x| \geq t|x|$ for all $x \in \mathbb{R}^n$ and $t \geq 1$;
(b) $|\delta_t x| \leq t|x|$ for all $x \in \mathbb{R}^n$ and $0 < t \leq 1$.

Let δ_t^* denote the adjoint of δ_t. Then we can also consider a norm function $\rho^*(x)$ associated with the dilation group $\{\delta_t^*\}_{t>0}$, and we have properties of

$\rho^*(x)$ and δ_t^* analogous to those of $\rho(x)$ and δ_t above. It is known that a polar coordinates expression for the Lebesgue measure

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \int_{S^{n-1}} f(\delta_t\theta)t^{\gamma-1}s(\theta)\,d\sigma(\theta)\,dt \tag{1.9}$$

holds, where $\gamma = \operatorname{trace} P$ and s is a strictly positive C^∞ function on $S^{n-1} = \{|x| = 1\}$ and $d\sigma$ is the Lebesgue surface measure on S^{n-1} (see [7, 16, 29]). We note that the condition (1.8) implies that all eigenvalues of P have real parts greater than or equal to 1 (see [3, pp. 3–4], [13, p. 137]). So we have $\gamma \geq n$.

Let

$$B(x,t) = \{y \in \mathbb{R}^n : \rho(x-y) < t\} \tag{1.10}$$

be a ball with respect to ρ (a ρ-ball) in $\mathbb{R}^n$ with radius t centered at x. We say that a weight function w belongs to the Muckenhoupt class A_p, $1 < p < \infty$, if

$$[w]_{A_p} = \sup_B \Big(|B|^{-1}\int_B w(x)\,dx\Big)\Big(|B|^{-1}\int_B w(x)^{-1/(p-1)}\,dx\Big)^{p-1} < \infty,$$

where the supremum is taken over all ρ-balls B in $\mathbb{R}^n$. The Hardy–Littlewood maximal operator M is defined as

$$M(f)(x) = \sup_{x\in B} |B|^{-1}\int_B |f(y)|\,dy,$$

where the supremum is taken over all ρ-balls B in $\mathbb{R}^n$ containing x. The class A_1 is defined to be the family of weight functions w such that $M(w) \leq Cw$ almost everywhere; the infimum of all such C will be denoted by $[w]_{A_1}$. We denote by L^p_w (or $L^p(w)$) the weighted L^p space with the norm defined as

$$\|f\|_{L^p_w} = \|f\|_{L^p(w)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\Big)^{1/p}.$$

See [2, 6, 9, 30] for results related to the weight class A_p. The following results are known and useful.

Proposition 1.1. *Let $1 < p < \infty$ and $w \in A_p$.*

(i) *The space $\mathcal{S}_0$ is dense in L^p_w.*
(ii) *The maximal operator M is bounded on L^p_w.*
(iii) *If $\varphi \in \mathcal{S}$, then $\sup_{t>0}|f * \varphi_t| \leq CM(f)$. Here and in what follows $\varphi_t(x) = t^{-\gamma}\varphi(\delta_t^{-1}x)$.*
(iv) *$\mathcal{F}(g * \varphi_t)(\xi) = \hat{g}(\xi)\hat{\varphi}(\delta_t^*\xi)$ for $g, \varphi \in \mathcal{S}$.*

Let $\beta \in \mathbb{R}$ and define the Riesz potential operator $\mathcal{I}_\beta$ associated with the dilations δ_t^* by

$$\mathcal{F}(\mathcal{I}_\beta(f))(\xi) = \rho^*(\xi)^{-\beta}\hat{f}(\xi) \tag{1.11}$$

for $f \in \mathcal{S}_0$. Let $1 < p < \infty$, $\alpha > 0$ and $w \in A_p$. Define the weighted parabolic Sobolev space $W_w^{\alpha,p}$ by

$$W_w^{\alpha,p} = \{f \in L_w^p : f = \mathcal{I}_\alpha(g) \text{ for some } g \in L_w^p\}, \tag{1.12}$$

where $f = \mathcal{I}_\alpha(g)$ means that

$$\int_{\mathbb{R}^n} f(x)h(x)\,dx = \int_{\mathbb{R}^n} g(x)\mathcal{I}_\alpha(h)\,dx \quad \text{ for all } h \in \mathcal{S}_0.$$

We note that the function $g \in L_w^p$ is uniquely determined by f, since $\mathcal{I}_\alpha$ is a bijection on $\mathcal{S}_0$ and $\mathcal{S}_0$ is dense in $L^{p'}(w^{-p'/p})$, the dual space of $L^p(w)$, where $1/p + 1/p' = 1$. We write $g = \mathcal{I}_{-\alpha}(f)$. For $f \in W_w^{\alpha,p}$ we define

$$\|f\|_{p,\alpha,w} = \|f\|_{p,w} + \|\mathcal{I}_{-\alpha}(f)\|_{p,w}. \tag{1.13}$$

We have analogues of Theorems A and B in the case of non-isotropic dilations δ_t with weights. Let $B(x,t)$ be as in (1.10) and

$$B_\alpha(f)(x) = \left(\int_0^\infty \left|f(x) - \fint_{B(x,t)} f(y)\,dy\right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2}, \quad \alpha > 0. \tag{1.14}$$

THEOREM 1.2. *Suppose that $1 < p < \infty$, $w \in A_p$ and $0 < \alpha < 2$. Then $f \in W_w^{\alpha,p}$ if and only if $f \in L_w^p$ and $B_\alpha(f) \in L_w^p$; moreover,*

$$\|\mathcal{I}_{-\alpha}(f)\|_{p,w} \simeq \|B_\alpha(f)\|_{p,w}.$$

Let

$$C_\alpha(f)(x) = \left(\int_0^\infty \left|\mathcal{I}_\alpha(f)(x) - \fint_{B(x,t)} \mathcal{I}_\alpha(f)(y)\,dy\right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2}. \tag{1.15}$$

Then Theorem 1.2 can be derived from the following result.

THEOREM 1.3. *Let $1 < p < \infty$, $w \in A_p$, $0 < \alpha < 2$ and let C_α be as in* (1.15). *Then*

$$\|C_\alpha(f)\|_{p,w} \simeq \|f\|_{p,w}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

The range of α in Theorem 1.2 will be extended in Theorem 4.2 by considering square functions with repeated uses of averaging operation $\fint_B f$.

We consider square functions generalizing B_α and C_α in (1.14) and (1.15). Let Φ be a bounded function on $\mathbb{R}^n$ with compact support. We say that $\Phi \in \mathcal{M}^\alpha$, $\alpha \geq 0$, if Φ satisfies

(i) $\int_{\mathbb{R}^n} \Phi(x)\,dx = 1$;

(ii) if $\alpha \geq 1$, then

$$\int_{\mathbb{R}^n} \Phi(x)x^a\,dx = 0 \quad \text{ for all multi-indices } a \text{ with } 1 \leq |a| \leq [\alpha], \tag{1.16}$$

where $x^a = x_1^{a_1} \dots x_n^{a_n}$ with $a = (a_1, \dots, a_n)$, $|a| = a_1 + \dots + a_n$, $a_j \in \mathbb{Z}$, $a_j \geq 0$, $1 \leq j \leq n$, and $[\alpha] = \max\{k \in \mathbb{Z} : k \leq \alpha\}$.

We note that $\mathcal{M}^\alpha \subset \mathcal{M}^\beta$ if $\alpha \geq \beta$ and $\mathcal{M}^\alpha = \mathcal{M}^j$ if $j \leq \alpha < j+1$, $j \geq 0$, $j \in \mathbb{Z}$. If Φ is even and $1 \leq \alpha < 2$, we have (1.16). In particular, $\chi_0 = |B(0,1)|^{-1}\chi_{B(0,1)} \in \mathcal{M}^\alpha$ for $0 \leq \alpha < 2$.

Let $\Phi \in \mathcal{M}^\alpha$ and

$$G_\alpha(f)(x) = \left(\int_0^\infty |f(x) - \Phi_t * f(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \qquad \alpha > 0. \tag{1.17}$$

We note that if $\Phi = \chi_0$ in (1.17), we get B_α of (1.14). Also, let $\Phi \in \mathcal{M}^\alpha$ and

$$H_\alpha(f)(x) = \left(\int_0^\infty |\mathcal{I}_\alpha(f)(x) - \Phi_t * \mathcal{I}_\alpha(f)(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \qquad 0 < \alpha < \gamma. \tag{1.18}$$

If we set $\Phi = \chi_0$ in (1.18), we get C_α of (1.15) for $0 < \alpha < 2$.

We prove the following.

Theorem 1.4. *Let H_α be as in* (1.18) *and $0 < \alpha < \gamma$, $1 < p < \infty$, $w \in A_p$. Then*

$$\|H_\alpha(f)\|_{p,w} \simeq \|f\|_{p,w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n).$$

Applying Theorem 1.4, we obtain the following.

Theorem 1.5. *Suppose that $1 < p < \infty$, $w \in A_p$ and $0 < \alpha < \gamma$. Let G_α be as in* (1.17). *Then $f \in W^{\alpha,p}_w$ if and only if $f \in L^p_w$ and $G_\alpha(f) \in L^p_w$; furthermore,*

$$\|\mathcal{I}_{-\alpha}(f)\|_{p,w} \simeq \|G_\alpha(f)\|_{p,w}.$$

Theorems 1.2 and 1.3 follow from Theorems 1.5 and 1.4, respectively. The proofs of Theorems 1.4 and 1.5 will be given in Section 3. To prove Theorem 1.4, we consider the Littlewood–Paley functions

$$g_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t} \right)^{1/2}, \tag{1.19}$$

where $\psi_t(x) = t^{-\gamma}\psi(\delta_t^{-1}x)$ with $\psi \in L^1(\mathbb{R}^n)$ satisfying (1.4). Then we can see that $H_\alpha(f) = g_{\psi^{(\alpha)}}$ for some $\psi^{(\alpha)}$ analogous to the one in (1.6). We shall prove Theorem 1.4 by applying Theorem 2.1 below in Section 2, which is a result for parabolic Littlewood–Paley functions complementing the boundedness result given in [25] and generalizing [22, Corollary 2.11] to the case of non-isotropic dilations.

The proof of Theorem 2.1 will be completed by applying Theorem 2.8, which provides the estimates

$$\|f\|_{p,w} \leq C\|g_\psi(f)\|_{p,w} \tag{1.20}$$

under certain conditions. Theorem 2.8 is deduced from Corollary 2.7, which is a result on the invertibility of Fourier multipliers homogeneous of degree 0 with respect to δ_t^* generalizing [22, Corollary 2.6] to the case of general homogeneity. Corollary 2.7 will follow from a more general result (Theorem 2.3).

Here we review some recent developments of the theory related to the results given in this note after the article [1] (see also the remarks at the end of this note).

Theorem A was generalized to the weighted Sobolev spaces in [10]. Also, Theorems A and B were extended to the weighted Sobolev spaces in [19] by applying a theorem of [17] for the boundedness of Littlewood–Paley functions g_ψ in (1.5) on the weighted L^p spaces, which is partly a special case of Theorem 2.1.

In [19] it was shown that the theorem of [17] is particularly suitable for handling the square functions in Theorem 1.4 for the case of the Euclidean structures (with the Euclidean norm and the ordinary dilation). Some results of [19] were generalized in [22] by introducing the function class $\mathcal{M}^\alpha$ and by proving the weighted L^p norm equivalence between $g_\psi(f)$ in (1.5) and f, part of which was not included in [17]; the estimates in (1.20) in the case of the Euclidean structures for a sufficiently large class of ψ and $p \in (1, \infty)$, $w \in A_p$ were absent from [17].

In [20] and [22], discrete parameter versions of Littlewood–Paley functions $g_\psi(f)$ in (1.5) of the form

$$\varDelta_\psi(f)(x) = \Big(\sum_{k=-\infty}^{\infty} |f * \psi_{2^k}(x)|^2\Big)^{1/2}$$

are also applied to characterize Sobolev spaces. See also [10] and [21] for applications of the square function

$$D^\alpha(f)(x) = \bigg(\int_0^\infty \Big|t^{-\alpha} \int_{S^{n-1}} (f(x - t\theta) - f(x))\, d\sigma(\theta)\Big|^2 \frac{dt}{t}\bigg)^{1/2}$$

in the theory of Sobolev spaces.

In Section 4, we shall establish another characterization of the Sobolev spaces $W_w^{\alpha,p}$ similar to Theorem 1.2 (Theorem 4.2), which is novel even in the case of the Euclidean structures. In Theorem 1.2, the averaging operator $⨍_B f$ is used to define the square function $B_\alpha(f)$ in (1.14), which is applied to characterize $W_w^{\alpha,p}$ for $\alpha \in (0, 2)$. In Theorem 4.2 we shall extend the range of α by introducing square functions which are defined with repeated uses of the averaging operation.

Finally, in Section 5 we shall illustrate by example how the Sobolev spaces $W_w^{\alpha,p}$ defined above can be characterized by distributional derivatives in some cases, by the arguments similar to the one in [28, Chap. V, proof of Theorem 3].

2. Invertibility of Fourier multipliers homogeneous with respect to δ_t^* and Littlewood–Paley operators. We consider a majorant of ψ defined by

$$H_\psi(x) = h(\rho(x)) = \sup_{\rho(y)\geq\rho(x)} |\psi(y)|$$

and two seminorms B_ϵ and D_u defined as

$$B_\epsilon(\psi) = \int_{|x|>1} |\psi(x)|\,|x|^\epsilon\,dx \qquad \text{for } \epsilon > 0,$$

$$D_u(\psi) = \Big(\int_{|x|<1} |\psi(x)|^u\,dx\Big)^{1/u} \qquad \text{for } u > 1.$$

In proving Theorem 1.4 we apply the following result.

THEOREM 2.1. *Suppose that $\psi \in L^1(\mathbb{R}^n)$ satisfies* (1.4). *Let $\epsilon > 0$, $u > 1$ and $C_j > 0$, $1 \leq j \leq 3$. Suppose that*

(1) $B_\epsilon(\psi) \leq C_1$;
(2) $D_u(\psi) \leq C_2$;
(3) $\|H_\psi\|_1 \leq C_3$.

Then g_ψ defined by (1.19) *is bounded on L^p_w:*

$$\|g_\psi(f)\|_{p,w} \leq C\|f\|_{p,w} \quad \textit{for all } p \in (1,\infty) \textit{ and } w \in A_p, \tag{2.1}$$

where the constant C depends only on p, w, ϵ, u and C_j, $1 \leq j \leq 3$, and does not otherwise depend on ψ. If we further assume the non-degeneracy condition

$$\sup_{t>0} |\hat\psi(\delta_t^*\xi)| > 0 \quad \textit{for } \xi \neq 0, \tag{2.2}$$

then we also have the reverse inequality of (2.1) *and hence*

$$\|g_\psi(f)\|_{p,w} \simeq \|f\|_{p,w} \quad \textit{for all } p \in (1,\infty) \textit{ and } w \in A_p.$$

By [25, Theorem 1.1], which generalizes a result of [17] to the case of non-isotropic dilations, we have the boundedness (2.1) under conditions (1)–(3) of Theorem 2.1, and the quantitative property of the constant C specified follows by checking the proof in [25]. The proof of [25, Theorem 1.1] is based on estimates for oscillatory integrals in [18].

REMARK 2.2. If there exist positive numbers σ_1, σ_2 such that

$$|\psi(x)| \leq C(1+\rho(x)^{-1})^{\gamma-\sigma_1}(1+\rho(x))^{-\gamma-\sigma_2} \quad \text{for all } x \in \mathbb{R}^n,$$

then conditions (1)–(3) of Theorem 2.1 are satisfied with some ϵ, u and C_j, $1 \leq j \leq 3$. To see this, the formula (1.9) is useful.

To prove the reverse inequality of (2.1), we apply a result on the invertibility on weighted L^p spaces of Fourier multipliers homogeneous with respect to δ_t^*. Let $m \in L^\infty(\mathbb{R}^n)$, $w \in A_p$, $1 < p < \infty$. The Fourier multiplier operator T_m is defined by

$$T_m(f)(x) = \int_{\mathbb{R}^n} m(\xi)\hat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}\,d\xi. \tag{2.3}$$

We say that m is a *Fourier multiplier for* L_w^p and write $m \in M_w^p$ (we also write $M^p(w)$ for M_w^p) if there exists a constant $C > 0$ such that

$$\|T_m(f)\|_{p,w} \le C\|f\|_{p,w} \quad \text{for all } f \in \mathcal{S}. \tag{2.4}$$

We define $\|m\|_{M^p(w)}$ to be the infimum of the constants C satisfying (2.4). Since $\mathcal{S}$ is dense in L_w^p, we have a unique extension of T_m to a bounded linear operator on L_w^p if $m \in M_w^p$. We observe that $M^p(w) = M^{p'}(\widetilde{w}^{-p'/p})$ by duality, where $\widetilde{w}(x) = w(-x)$. See [12] for relevant results.

We need the following result generalizing [22, Theorem 2.5] to the case of non-isotropic dilations.

THEOREM 2.3. *Let m be a bounded function on $\mathbb{R}^n$ which is continuous on $\mathbb{R}^n \setminus \{0\}$. Suppose that m is homogeneous of degree 0 with respect to δ_t^* and that $m(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Also, suppose that $m \in M_v^r$ for all $r \in (1,\infty)$ and all $v \in A_r$. Let $1 < p < \infty$, $w \in A_p$ and let $F(z)$ be holomorphic in $D = \mathbb{C} \setminus \{0\}$. Then $F(m(\xi)) \in M_w^p$.*

For $m \in M_w^p$, $1 < p < \infty$, $w \in A_p$, we consider the spectral radius operator

$$\rho_{p,w}(m) = \lim_{k\to\infty} \|m^k\|_{M^p(w)}^{1/k}.$$

To prove Theorem 2.3, we need the following.

PROPOSITION 2.4. *Suppose that $1 < p < \infty$, $w \in A_p$ and $m \in L^\infty(\mathbb{R}^n)$. Let m be homogeneous of degree 0 with respect to the dilations δ_t^* and continuous on S^{n-1}. Assume that $m \in M_v^r$ for all $r \in (1,\infty)$ and all $v \in A_r$. Then, for any $\epsilon > 0$, there exists $\ell \in M_w^p$ which is homogeneous of degree 0 with respect to δ_t^* and in $C^\infty(\mathbb{R}^n\setminus\{0\})$ such that $\|m-\ell\|_\infty < \epsilon$ and $\rho_{p,w}(m-\ell) < \epsilon$.*

To prove Proposition 2.4, we apply the following lemmas.

LEMMA 2.5. *Let $\eta \in C^\infty(\mathbb{R})$, $\operatorname{supp}\eta \subset [1,2]$, $\eta \ge 0$ and $\int_0^\infty |\eta(t)|^2\,dt/t = 1$. Define a real function ψ in $\mathcal{S}_0(\mathbb{R}^n)$ by $\hat{\psi}(\xi) = \eta(\rho^*(\xi))$. Then*

$$\|g_\psi(f)\|_{p,w} \simeq \|f\|_{p,w} \quad \textit{for all } p \in (1,\infty) \textit{ and } w \in A_p.$$

LEMMA 2.6. *Suppose that $m \in L^\infty(\mathbb{R}^n)$, $m \in C^\infty(\mathbb{R}^n\setminus\{0\})$ and that m is homogeneous of degree 0 with respect to δ_t^*. Then $m \in M_w^p$ for all $p \in (1,\infty)$ and $w \in A_p$ and*

$$\|m\|_{M^p(w)} \le C \sup_{1\le\rho^*(\xi)\le 2,\, |a|\le[\gamma]+1} |(\partial_\xi)^a m(\xi)|$$

with a constant C independent of m, where $(\partial_\xi)^a = (\partial/\partial\xi_1)^{a_1} \dots (\partial/\partial\xi_n)^{a_n}$ with $a = (a_1, \dots, a_n)$, $a_j \in \mathbb{Z}$, $a_j \geq 0$, $1 \leq j \leq n$.

Proof of Lemma 2.5. By [25, Theorem 1.1] we see that $\|g_\psi(f)\|_{p,w} \leq C\|f\|_{p,w}$ for all $p \in (1,\infty)$ and $w \in A_p$. To prove the reverse inequality we note that $\|g_\psi(f)\|_2 = \|f\|_2$. Thus the polarization implies that for real-valued $f, h \in \mathcal{S}$,

$$\begin{aligned} 4 \int_{\mathbb{R}^n} f(x)h(x)\,dx &= \int_{\mathbb{R}^n} (f(x)+h(x))^2\,dx - \int_{\mathbb{R}^n} (f(x)-h(x))^2\,dx \\ &= \int_{\mathbb{R}^n} (g_\psi(f+h)(x))^2\,dx - \int_{\mathbb{R}^n} (g_\psi(f-h)(x))^2\,dx \\ &= 4 \int_{\mathbb{R}^n} \int_0^\infty f * \psi_t(x) h * \psi_t(x)\,\frac{dt}{t}\,dx. \end{aligned}$$

Therefore, by the inequalities of Schwarz and Hölder we have

$$\Big| \int_{\mathbb{R}^n} f(x)h(x)\,dx \Big| \leq \|g_\psi(f)\|_{p,w} \|g_\psi(h)\|_{p',w^{-p'/p}} \leq C\|g_\psi(f)\|_{p,w}\|h\|_{p',w^{-p'/p}}.$$

Taking the supremum over h with $\|h\|_{p',w^{-p'/p}} \leq 1$, we find that $\|f\|_{p,w} \leq C\|g_\psi(f)\|_{p,w}$, from which we can derive the desired estimates for complex-valued functions. ∎

Proof of Lemma 2.6. Let ψ be as in Lemma 2.5 and define ψ_m by $\mathcal{F}(\psi_m)(\xi) = \hat{\psi}(\xi)m(\xi)$. Then $g_\psi(T_m f) = g_{\psi_m}(f)$. So, by Lemma 2.5 for $w \in A_p$, $1 < p < \infty$, we have

$$\|T_m f\|_{p,w} \leq C\|g_\psi(T_m f)\|_{p,w} = C\|g_{\psi_m}(f)\|_{p,w}. \tag{2.5}$$

Since $\psi_m \in \mathcal{S}_0$, g_{ψ_m} is bounded on L^p_w. To specify the operator bounds, we apply the estimates (2.1). It is sufficient to observe the following estimates:

$$\begin{aligned} |\psi_m(x)| &= \Big| \int_{\mathbb{R}^n} \hat{\psi}(\xi)m(\xi)e^{2\pi i\langle x,\xi\rangle}\,d\xi \Big| \\ &\leq C(1+|x|)^{-[\gamma]-1} \sup_{1\leq\rho^*(\xi)\leq 2,\, |a|\leq[\gamma]+1} |(\partial_\xi)^a m(\xi)|, \end{aligned} \tag{2.6}$$

which follows by integration by parts, with the constant C independent of m. Combining (2.5), (2.6) and the estimates (2.1), we obtain the conclusion. ∎

Proof of Proposition 2.4. As in [11], [22], we take a sequence $\{\varphi_j\}_{j=1}^\infty$ of functions on the orthogonal group $O(n)$ with the following properties:

(1) each φ_j is infinitely differentiable, non-negative and $\int_{O(n)} \varphi_j(A)\,dA = 1$, where dA is the Haar measure on $O(n)$;
(2) for any neighborhood U of the identity of $O(n)$, there exists a positive integer N such that $\operatorname{supp}(\varphi_j) \subset U$ for $j \geq N$.

For $\xi \in S^{n-1}$, let

$$\widetilde{m}_j(\xi) = \int_{O(n)} m(A\xi)\varphi_j(A)\, dA.$$

Then $\widetilde{m}_j$ is C^∞ on S^{n-1} (see [11, pp. 123–124]). For $\xi \in \mathbb{R}^n \setminus \{0\}$, let

$$m_j(\xi) = \widetilde{m}_j(\delta^*_{\rho^*(\xi)^{-1}}\xi).$$

Then m_j is homogeneous of degree 0 with respect to δ_t^*, $m_j \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and $m_j = \widetilde{m}_j$ on S^{n-1}.

We prove

$$\rho_{r,v}(m_j) \le \|m\|_\infty, \qquad r \in (1,\infty),\ v \in A_r. \tag{2.7}$$

For this it suffices to show that

$$\|m_j^k\|_{M^r(v)} \le C_j k^{[\gamma]+1}\|m\|_\infty^k,$$

where C_j is independent of k. This follows by Lemma 2.6, since

$$\sup_{1\le\rho^*(\xi)\le 2,\, |a|\le[\gamma]+1} |(\partial_\xi)^a m_j(\xi)^k| \le C_j k^{[\gamma]+1}\|m\|_\infty^k.$$

To see this, it is helpful to refer to [11, pp. 123–124].

Since $m_j \to m$ as $j \to \infty$ uniformly on S^{n-1}, we can take $\ell = m_j$ for j large enough to get $\|m - \ell\|_\infty < \epsilon$. Let $p \in (1,\infty)$, $w \in A_p$. Confirming that a result analogous to [22, Proposition 2.2] holds true in the setting of non-isotropic dilations, we can find $r > 1$, $s > 1$ and $\theta \in (0,1)$ such that $w^s \in A_r$ and

$$\|(m-m_j)^k\|_{M^p(w)} \le \|(m-m_j)^k\|_\infty^{1-\theta}\|(m-m_j)^k\|_{M^r(w^s)}^\theta.$$

Thus

$$\rho_{p,w}(m-m_j) \le \|m-m_j\|_\infty^{1-\theta}\rho_{r,w^s}(m-m_j)^\theta.$$

Since

$$\rho_{r,w^s}(m-m_j) \le \rho_{r,w^s}(m) + \rho_{r,w^s}(m_j)$$

(see Riesz–Nagy [15, p. 426]), it follows that

$$\begin{aligned}\rho_{p,w}(m-m_j) &\le \|m-m_j\|_\infty^{1-\theta}(\rho_{r,w^s}(m) + \rho_{r,w^s}(m_j))^\theta \\ &\le \|m-m_j\|_\infty^{1-\theta}(\rho_{r,w^s}(m) + \|m\|_\infty)^\theta,\end{aligned}$$

where the last inequality follows from (2.7). Since $\|m-m_j\|_\infty \to 0$ as $j \to \infty$, for a given $\epsilon > 0$, taking $\ell = m_j$ with j large enough, we have $\rho_{p,w}(m-\ell) < \epsilon$ and $\|m - \ell\|_\infty < \epsilon$. ∎

Proof of Theorem 2.3. The proof is similar to that of [22, Theorem 2.5]. Let

$$\epsilon_0 = \frac{1}{4}\min_{\xi\in S^{n-1}} |m(\xi)|.$$

Applying Proposition 2.4, we can find $\ell \in M^p_w$ which is homogeneous of degree 0 with respect to δ^*_t and belongs to $C^\infty(\mathbb{R}^n \setminus \{0\})$ such that $\|m-\ell\|_\infty < \epsilon_0$ and $\rho_{p,w}(m-\ell) < \epsilon_0$. Let $C : \ell(\xi) + 2\epsilon_0 e^{i\theta}$, $0 \le \theta \le 2\pi$, be a circle in D. Apply Cauchy's formula to get

$$(2.8) \qquad F(m(\xi)) = \frac{1}{2\pi i}\int_C \frac{F(\zeta)}{\zeta - m(\xi)}\,d\zeta = \frac{\epsilon_0}{\pi}\int_0^{2\pi} \frac{F(\ell(\xi)+2\epsilon_0 e^{i\theta})}{2\epsilon_0 e^{i\theta} + \ell(\xi) - m(\xi)} e^{i\theta}\,d\theta$$

for $\xi \in \mathbb{R}^n \setminus \{0\}$. We expand the integrand in the last integral into a power series by using

$$(2.9) \qquad \frac{e^{i\theta}}{2\epsilon_0 e^{i\theta} + \ell(\xi) - m(\xi)} = \frac{1}{2\epsilon_0}\sum_{k=0}^\infty \left(\frac{m(\xi)-\ell(\xi)}{2\epsilon_0 e^{i\theta}}\right)^k,$$

where the series converges uniformly in $\theta \in [0, 2\pi]$ since

$$\left|\frac{m(\xi)-\ell(\xi)}{2\epsilon_0 e^{i\theta}}\right| \le \frac{1}{2}.$$

Substituting (2.9) in (2.8), we have

$$(2.10) \qquad F(m(\xi)) = \frac{1}{2\pi}\sum_{k=0}^\infty \left(\frac{m(\xi)-\ell(\xi)}{2\epsilon_0}\right)^k N_k(\xi),$$

where

$$N_k(\xi) = \int_0^{2\pi} F(\ell(\xi)+2\epsilon_0 e^{i\theta}) e^{-ik\theta}\,d\theta$$

and the series on the right hand side of (2.10) converges uniformly in $\xi \in \mathbb{R}^n \setminus \{0\}$, since

$$\left|\frac{m(\xi)-\ell(\xi)}{2\epsilon_0}\right| \le \frac{1}{2}, \qquad \epsilon_0 \le |\ell(\xi) + 2\epsilon_0 e^{i\theta}| \le \|m\|_\infty + 3\epsilon_0.$$

Also, $N_k(\xi)$ is homogeneous of degree 0 with respect to δ^*_t and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$ and

$$\sup_{1 \le \rho^*(\xi) \le 2,\, |a| \le [\gamma]+1} |(\partial_\xi)^a N_k(\xi)| \le C$$

with C independent of k. Therefore, by Lemma 2.6 we have $\|N_k\|_{M^p(w)} \le C$ with a constant C independent of k. Thus we see that

$$\sum_{k=0}^\infty (2\epsilon_0)^{-k}\|(m-\ell)^k\|_{M^p(w)}\|N_k\|_{M^p(w)} \le C\sum_{k=0}^\infty (2\epsilon_0)^{-k}\|(m-\ell)^k\|_{M^p(w)},$$

and the last series converges since $\|(m-\ell)^k\|_{M^p(w)} \le \epsilon_0^k$ if k is sufficiently large. From this and (2.10) we can infer that $F(m) \in M^p_w$. ■

By Theorem 2.3 in particular we have the following.

COROLLARY 2.7. *Let $1 < p < \infty$ and $w \in A_p$. Suppose that m is homogeneous of degree 0 with respect to δ_t^* and that $m \in M_v^r$ for all $r \in (1,\infty)$ and all $v \in A_r$. Assume further that m is continuous on S^{n-1} and does not vanish there. Then $m^{-1} \in M_w^p$.*

Proof. Take $F(z) = 1/z$ in Theorem 2.3. ∎

Applying Corollary 2.7 in the theory of Littlewood–Paley functions, we can prove the following.

THEOREM 2.8. *Let $\psi \in L^1(\mathbb{R}^n)$ satisfy (1.4). Suppose that $\|g_\psi(f)\|_{r,v} \le C_{r,v}\|f\|_{r,v}$, $f \in \mathcal{S}$, for all $r \in (1,\infty)$ and all $v \in A_r$ and that $m(\xi) = \int_0^\infty |\hat{\psi}(\delta_t^*\xi)|^2\, dt/t$ is continuous and strictly positive on S^{n-1}. Let $f \in \mathcal{S}$. Then*

$$\|f\|_{p,w} \le C_{p,w}\|g_\psi(f)\|_{p,w}$$

for all $p \in (1,\infty)$ and all $w \in A_p$.

To prove Theorem 2.8, we also need the following lemma.

LEMMA 2.9. *Suppose that $\|g_\psi(f)\|_{r,v} \le C_{r,v}\|f\|_{r,v}$, $f \in \mathcal{S}$, for all $r \in (1,\infty)$ and all $v \in A_r$. If $m(\xi)$ is defined as in Theorem 2.8 and $1 < p < \infty$, $w \in A_p$, then $m \in M_w^p$.*

Proof. For $\epsilon \in (0,1)$, let

$$\Psi^{(\epsilon)}(x) = \int_\epsilon^{\epsilon^{-1}} \int_{\mathbb{R}^n} \psi_t(x+y)\bar{\psi}_t(y)\, dy\, \frac{dt}{t},$$

where $\bar{\psi}_t$ denotes the complex conjugate. We note that

$$\mathcal{F}(\Psi^{(\epsilon)})(\xi) = \int_\epsilon^{\epsilon^{-1}} \hat{\psi}(\delta_t^* t\xi)\hat{\bar{\psi}}(-\delta_t^*\xi)\, \frac{dt}{t} = \int_\epsilon^{\epsilon^{-1}} |\hat{\psi}(\delta_t^*\xi)|^2\, \frac{dt}{t} =: m^{(\epsilon)}(\xi).$$

Therefore $\Psi^{(\epsilon)} * f = T_{m^{(\epsilon)}} f$. We observe that

$$\Psi^{(\epsilon)} * f(x) = \int_\epsilon^{\epsilon^{-1}} \int_{\mathbb{R}^n} \psi_t * f(y)\bar{\psi}_t(y-x)\, dy\, \frac{dt}{t};$$

$$\int_{\mathbb{R}^n} \Psi^{(\epsilon)} * f(x)h(x)\, dx = \int_\epsilon^{\epsilon^{-1}} \int_{\mathbb{R}^n} \psi_t * f(y)\bar{\psi}_t * h(y)\, dy\, \frac{dt}{t}$$

for $f, h \in \mathcal{S}$. Thus by the inequalities of Schwarz and Hölder we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} \Psi^{(\epsilon)} * f(x)h(x)\, dx\Big| &\le \int_{\mathbb{R}^n} g_\psi(f)(y)g_\psi(\bar{h})(y)\, dy \\ &\le \|g_\psi(f)\|_{p,w}\|g_\psi(\bar{h})\|_{p',w^{-p'/p}} \\ &\le C\|g_\psi(f)\|_{p,w}\|h\|_{p',w^{-p'/p}}.\end{aligned}$$

Taking the supremum over functions h with $\|h\|_{p',w^{-p'/p}} \le 1$, we have

$$\|T_{m^{(\epsilon)}} f\|_{p,w} \le C\|g_\psi(f)\|_{p,w}.$$

Letting $\epsilon \to 0$ and noting $m^{(\epsilon)} \to m$, we have

$$\|T_m f\|_{p,w} \le C\|g_\psi(f)\|_{p,w}. \tag{2.11}$$

Since $\|g_\psi(f)\|_{p,w} \le C\|f\|_{p,w}$, we see that $m \in M^p_w$. ∎

Proof of Theorem 2.8. Let m be as in Theorem 2.8. Then by Lemma 2.9, $m \in M^p_w$ for all $p \in (1,\infty)$ and $w \in A_p$. So we can apply Corollary 2.7 to m to conclude that $m^{-1} \in M^p_w$ if $1 < p < \infty$, $w \in A_p$ and hence by (2.11),

$$\|f\|_{p,w} = \|T_{m^{-1}} T_m f\|_{p,w} \le C\|T_m f\|_{p,w} \le C\|g_\psi(f)\|_{p,w}$$

for $f \in \mathcal{S}$, which implies the conclusion. ∎

Proof of Theorem 2.1. It remains to prove the reverse inequality of (2.1). If $m(\xi) = \int_0^\infty |\hat\psi(\delta_t^*\xi)|^2\,dt/t$, then by the non-degeneracy (2.2) we have $m(\xi) \ne 0$ for $\xi \ne 0$. Therefore, by Theorem 2.8 we only have to show that m is continuous on S^{n-1}. In [25], it has been shown that

$$\int_{2^k}^{2^{k+1}} |\hat\psi(\delta_t^*\xi)|^2 \frac{dt}{t} \le C \min(|\delta_{2^k}^*\xi|^\epsilon, |\delta_{2^k}^*\xi|^{-\epsilon})$$

for $\xi \in S^{n-1}$ and $k \in \mathbb{Z}$ with some $\epsilon > 0$ (see [25, Lemmas 3.1 and 3.3]). By analogues for δ_t^* of (a), (b) for δ_t in Section 1, it follows that

$$\int_{2^k}^{2^{k+1}} |\hat\psi(\delta_t^*\xi)|^2 \frac{dt}{t} \le C \min(2^{k\epsilon}, 2^{-k\epsilon}).$$

This implies that

$$\int_\epsilon^{\epsilon^{-1}} |\hat\psi(\delta_t^*\xi)|^2 \frac{dt}{t} \to \int_0^\infty |\hat\psi(\delta_t^*\xi)|^2 \frac{dt}{t} \quad \text{as } \epsilon \to 0$$

uniformly in $\xi \in S^{n-1}$. We note that $\int_\epsilon^{\epsilon^{-1}} |\hat\psi(\delta_t^*\xi)|^2\,dt/t$ is continuous on S^{n-1} for each fixed $\epsilon > 0$. Thus the continuity of m on S^{n-1} follows by uniform convergence. ∎

Remark 2.10. Let $\psi^{(j)} \in L^1(\mathbb{R}^n)$ for $j = 1, 2, \ldots, \ell$. Suppose that $\psi^{(j)}$ satisfies (1.4) and (1)–(3) of Theorem 2.1 for every j, $1 \le j \le \ell$. Let

$$\Psi(x) = (\psi^{(1)}(x), \ldots, \psi^{(\ell)}(x)),$$
$$\Psi_t(x) = (\psi_t^{(1)}(x), \ldots, \psi_t^{(\ell)}(x)), \quad \mathcal{F}(\Psi_t)(\xi) = (\mathcal{F}(\psi_t^{(1)})(\xi), \ldots, \mathcal{F}(\psi_t^{(\ell)})(\xi)).$$

We further assume that

$$\sup_{t>0}\left|\mathcal{F}\left(\Psi_t\right)(\xi)\right| = \sup_{t>0}\Big(\sum_{j=1}^{\ell}|\mathcal{F}(\psi^{(j)})(\delta_t^*\xi)|^2\Big)^{1/2} > 0, \quad \forall\xi\in\mathbb{R}^n\setminus\{0\}. \tag{2.12}$$

Let

$$f * \Psi_t(x) = (f * \psi_t^{(1)}(x), \ldots, f * \psi_t^{(\ell)}(x))$$

and

$$g_\Psi(f)(x) = \Big(\int_0^\infty |f * \Psi_t(x)|^2 \frac{dt}{t}\Big)^{1/2}, \quad |f * \Psi_t(x)| = \Big(\sum_{j=1}^{\ell}|f * \psi_t^{(j)}(x)|^2\Big)^{1/2}.$$

Then by Theorem 2.1 we have $\|g_\Psi(f)\|_{p,w} \le C\|f\|_{p,w}$. We can also prove the reverse inequality by adapting the arguments given above when $\ell = 1$ for the present situation, applying the non-degeneracy (2.12). Thus

$$\|g_\Psi(f)\|_{p,w} \simeq \|f\|_{p,w}. \tag{2.13}$$

Example. We give an example in the case of the Euclidean structures ($\rho(x) = |x|$, $\delta_t(x) = tx$) for which we can apply Remark 2.10 to get the norm equivalence in (2.13). Let $P_t(x)$ be the Poisson kernel on the upper half-space $\mathbb{R}^n \times (0,\infty)$ defined by

$$P_t(x) = c_n\frac{t}{(|x|^2+t^2)^{(n+1)/2}} = \int_{\mathbb{R}^n} e^{-2\pi t|\xi|}e^{2\pi i\langle x,\xi\rangle}\,d\xi.$$

Let $\psi^{(j)}(x) = (\partial/\partial x_j)P_1(x)$, $1 \le j \le n$. Then

$$\mathcal{F}(\psi^{(j)})(\xi) = 2\pi i\xi_j e^{-2\pi|\xi|}.$$

We can see that all the requirements in Remark 2.10 for $\psi^{(j)}$, $1 \le j \le n$, needed in the proof of (2.13) are fulfilled; in particular, (2.12) follows from

$$|\mathcal{F}(\Psi_t)(\xi)| = 2\pi t|\xi|e^{-2\pi t|\xi|}.$$

Thus we have (2.13) for $\Psi = ((\partial/\partial x_1)P_1, \ldots, (\partial/\partial x_n)P_1)$.

3. Proofs of Theorems 1.4 and 1.5. We apply the following estimates in proving Theorem 1.4.

Lemma 3.1. *Let F be a function in $C^\infty(\mathbb{R}^n\setminus\{0\})$ which is homogeneous of degree d with respect to δ_t. Then for $\rho(x)\ge 1$ we have*

$$|(\partial_x)^a F(x)| \le C_a\rho(x)^{d-|a|}$$

for all multi-indices a with a positive constant C_a independent of x.

Proof. We write $\delta_t = (\delta_{ij}(t))$, $1 \le i,j \le n$. We have $t^dF(x) = F(\delta_t x)$. Differentiating both sides by using the chain rule on the right hand side, we have

$$t^d(\partial_x)^a F(x) = \Big[\Big(\prod_{j=1}^{n}\Big(\sum_{i=1}^{n}\delta_{ij}(t)\partial/\partial x_i\Big)^{a_j}\Big)F\Big](\delta_t x).$$

Substituting $t = \rho(x)^{-1}$ in this equation, we have

$$|(\partial_x)^a F(x)| \le C\Big(\sup_{|b|=|a|,\rho(x)=1}|(\partial_x)^b F(x)|\Big)\Big(\sup_{1\le i\le n,1\le j\le n}\delta_{ij}(\rho(x)^{-1})\Big)^{|a|}\rho(x)^d.$$

This implies what we need, since $|\delta_{ij}(t)| \le Ct$ for $0 < t \le 1$ by (b) of Section 1. ∎

Proof of Theorem 1.4. Let $0 < \alpha < \gamma$ and $\mathcal{L}_\alpha = \mathcal{F}^{-1}(\rho^*(\xi)^{-\alpha})$. Then $\mathcal{L}_\alpha$ is homogeneous of degree $\alpha-\gamma$ with respect to δ_t and belongs to $C^\infty(\mathbb{R}^n\setminus\{0\})$ (see [4, pp. 162–165]). Let $\psi^{(\alpha)} = \mathcal{L}_\alpha - \mathcal{L}_\alpha * \Phi$. Then $H_\alpha(f) = g_{\psi^{(\alpha)}}(f)$.

We easily see that

$$|\psi^{(\alpha)}(x)| \le C\rho(x)^{\alpha-\gamma} \quad \text{for } \rho(x) \le 2. \tag{3.1}$$

Since

$$\psi^{(\alpha)}(x) = \int_{\mathbb{R}^n}(\mathcal{L}_\alpha(x) - \mathcal{L}_\alpha(x-y))\Phi(y)\,dy,$$

and

$$|(\partial_x)^a\mathcal{L}_\alpha(x)| \le C_a\rho(x)^{\alpha-\gamma-|a|} \quad \text{for } \rho(x) \ge 2$$

for all multi-indices a by Lemma 3.1, using Taylor's formula with (1.16) and noting that Φ is compactly supported, we see that

$$|\psi^{(\alpha)}(x)| \le C\rho(x)^{\alpha-\gamma-[\alpha]-1} \quad \text{for } \rho(x) \ge 2, \tag{3.2}$$

where $\alpha-\gamma-[\alpha]-1 < -\gamma$. By (3.1), (3.2) and (1.9) it follows that $\psi^{(\alpha)} \in L^1$ (see Remark 2.2). Also, we have

$$|\mathcal{F}(\psi^{(\alpha)})(\xi)| = |\rho^*(\xi)^{-\alpha}(1-\hat{\Phi}(\xi))| \le C\rho^*(\xi)^{-\alpha}|\xi|^{[\alpha]+1} \le C\rho^*(\xi)^{-\alpha+[\alpha]+1}$$

for $\rho^*(\xi) \le 1$ by the analogue for ρ^* of (4) for ρ of Section 1. So we have $\mathcal{F}(\psi^{(\alpha)})(0) = 0$, i.e., $\int\psi^{(\alpha)} = 0$; combining this with (3.1), (3.2) and (5) for ρ of Section 1 we see that conditions (1)–(3) of Theorem 2.1 are satisfied for $\psi^{(\alpha)}$. Further, it is easy to see that

$$\sup_{t>0}|\mathcal{F}(\psi^{(\alpha)})(\delta_t^*\xi)| > 0$$

for $\xi \ne 0$. Thus all the assumptions of Theorem 2.1 are fulfilled for $\psi^{(\alpha)}$ and the conclusion of Theorem 1.4 follows by applying Theorem 2.1 to $g_{\psi^{(\alpha)}}$. ∎

Remark 3.2. If $\psi^{(\alpha)}$ is as in (1.6), in the case of the Euclidean norm and the ordinary dilation, to prove $\|f\|_{p,w} \le C\|g_{\psi^{(\alpha)}}(f)\|_{p,w}$, $0 < \alpha < 2$, $1 < p < \infty$, $w \in A_p$, we can also apply the polarization technique as in the proof of Lemma 2.5 (see also [19]) instead of using Theorem 2.1 with the non-degeneracy condition (2.2), which is applicable in a more general situation of Theorem 1.4. This is the case because $\mathcal{F}(\psi^{(\alpha)})$ is a radial function.

To prove Theorem 1.5 we prepare the following lemmas.

LEMMA 3.3. *Let $1 < p < \infty$, $w \in A_p$ and $f \in L^p_w$. For a positive integer m, let $f_{(m)} = f\chi_{E_m}$, where*

$$E_m = \{x \in \mathbb{R}^n : |x| \le m,\ |f(x)| \le m\}.$$

Then $f_{(m)} \to f$ almost everywhere and in L^p_w as $m \to \infty$.

LEMMA 3.4. *Let p, w and f be as in Lemma 3.3. Let φ be an infinitely differentiable, non-negative function on $\mathbb{R}^n$ such that $\varphi(\xi) = 1$ for $\rho^*(\xi) \le 1$, $\operatorname{supp}(\varphi) \subset \{\rho^*(\xi) \le 2\}$ and $\varphi(\xi) = \varphi_0(\rho^*(\xi))$ for some φ_0 on $\mathbb{R}$. Define $\zeta^{(\epsilon)} \in \mathcal{S}_0$ by*

$$\zeta^{(\epsilon)}(\xi) = \varphi(\delta^*_\epsilon \xi) - \varphi(\delta^*_{\epsilon^{-1}}\xi), \qquad \epsilon \in (0, 1/2).$$

Note that $\zeta^{(\epsilon)}(\xi) = \zeta^{(\epsilon/2)}(\xi)\zeta^{(\epsilon)}(\xi)$. Let $f^{(\epsilon)} = f * \mathcal{F}^{-1}(\zeta^{(\epsilon)})$. Then $f^{(\epsilon)} \to f$ almost everywhere and in L^p_w as $\epsilon \to 0$.

Proof of Lemma 3.3. The pointwise convergence is obvious and the norm convergence follows from Lebesgue's dominated convergence theorem since $|f_{(m)}| \le |f|$. ∎

Proof of Lemma 3.4. If $f \in \mathcal{S}$, we easily see that $f^{(\epsilon)} \to f$ pointwise as $\epsilon \to 0$. Therefore, for $f \in L^p_w$, we have

$$\begin{aligned}\Big\| \limsup_{\epsilon\to 0} |f^{(\epsilon)} - f| \Big\|_{p,w} &\le \Big\| \limsup_{\epsilon\to 0} |(f-h)^{(\epsilon)} - (f-h)| \Big\|_{p,w} \\ &\le C\|M(f-h)\|_{p,w} \le C\|f-h\|_{p,w}\end{aligned}$$

for any $h \in \mathcal{S}$. As $\mathcal{S}$ is dense in L^p_w, it follows that $\limsup_{\epsilon\to 0} |f^{(\epsilon)}(x) - f(x)| = 0$ a.e., which implies pointwise convergence. Norm convergence follows from pointwise convergence and the dominated convergence theorem since $|f^{(\epsilon)}| \le CM(f) \in L^p_w$. ∎

Proof of Theorem 1.5. Define $f_{m,\epsilon} = (f_{(m)})^{(\epsilon)}$ for $f \in L^p_w$. Then $f_{m,\epsilon} \in \mathcal{S}_0$. By Theorem 1.4, we see that

$$\|G_\alpha(f_{m,\epsilon})\|_{p,w} = \|H_\alpha(\mathcal{I}_{-\alpha} f_{m,\epsilon})\|_{p,w} \simeq \|\mathcal{I}^{(\epsilon/2)}_{-\alpha} f_{m,\epsilon}\|_{p,w}, \tag{3.3}$$

where $\mathcal{I}^{(\epsilon/2)}_\beta(f) = \mathcal{F}^{-1}(\zeta^{(\epsilon/2)}(\rho^*)^{-\beta}) * f$, $\beta \in \mathbb{R}$, for $f \in L^p_w$ and we have used the equality $\mathcal{I}_{-\alpha} f_{m,\epsilon} = \mathcal{I}^{(\epsilon/2)}_{-\alpha} f_{m,\epsilon}$. Using Lemma 3.3, we see that $f_{m,\epsilon} \to f^{(\epsilon)}$ in L^p_w, since

$$\|f_{m,\epsilon} - f^{(\epsilon)}\|_{p,w} \le C\|M(f_{(m)} - f)\|_{p,w} \le C\|f_{(m)} - f\|_{p,w}$$

and also $f_{m,\epsilon} \to f^{(\epsilon)}$ pointwise, since

$$\begin{aligned} |f_{m,\epsilon}(x) - f^{(\epsilon)}(x)| &= \Big| \int (f_{(m)}(y) - f(y)) \mathcal{F}^{-1}(\zeta^{(\epsilon)})(x-y)\, dy \Big| \\ &\le \|f_{(m)} - f\|_{p,w} \Big(\int |\mathcal{F}^{-1}(\zeta^{(\epsilon)})(x-y)|^{p'} w(y)^{-p'/p}\, dy \Big). \end{aligned}$$

Thus $f_{m,\epsilon} - \Phi_t * f_{m,\epsilon} \to f^{(\epsilon)} - \Phi_t * f^{(\epsilon)}$ a.e. as $m \to \infty$ and by (3.3) we have, via Fatou's lemma,

$$\begin{aligned} \|G_\alpha(f^{(\epsilon)})\|_{p,w} &\le \liminf_{m\to\infty} \|G_\alpha(f_{m,\epsilon})\|_{p,w} \\ &\le C \liminf_{m\to\infty} \|\mathcal{I}_{-\alpha}^{(\epsilon/2)} f_{m,\epsilon}\|_{p,w} = C \|\mathcal{I}_{-\alpha}^{(\epsilon/2)} f^{(\epsilon)}\|_{p,w}, \end{aligned}$$

where the last equality follows since $\mathcal{I}_{-\alpha}^{(\epsilon/2)}$ is bounded on L^p_w. Thus we see that $G_\alpha(f^{(\epsilon)}) \in L^p_w$. In fact, we also have the reverse inequality. To see this we first note that

$$\begin{aligned} \|G_\alpha(f^{(\epsilon)}) - G_\alpha(f_{m,\epsilon})\|_{p,w} &\le \|G_\alpha(f^{(\epsilon)} - f_{m,\epsilon})\|_{p,w} \\ &= \|G_\alpha((f - f_{(m)})^{(\epsilon)})\|_{p,w}. \end{aligned} \tag{3.4}$$

Since

$$(f_{(k)} - f_{(m)})^{(\epsilon)} - \Phi_t * (f_{(k)} - f_{(m)})^{(\epsilon)} \to (f - f_{(m)})^{(\epsilon)} - \Phi_t * (f - f_{(m)})^{(\epsilon)} \quad \text{a.e.}$$

as $k \to \infty$, by Fatou's lemma we have

$$\|G_\alpha((f - f_{(m)})^{(\epsilon)})\|_{p,w} \le \liminf_{k\to\infty} \|G_\alpha((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w}. \tag{3.5}$$

Since $(f_{(k)} - f_{(m)})^{(\epsilon)} \in \mathcal{S}_0$, by Theorem 1.4 we have

$$\begin{aligned} \|G_\alpha((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w} &\simeq \|\mathcal{I}_{-\alpha}((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w} \\ &= \|\mathcal{I}_{-\alpha}^{(\epsilon/2)}((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w}. \end{aligned}$$

Since $f_{(m)} \to f$ in L^p_w, this implies that

$$\lim_{k,m\to\infty} \|G_\alpha((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w} = 0.$$

Thus by (3.4) and (3.5), it follows that $G_\alpha(f_{m,\epsilon}) \to G_\alpha(f^{(\epsilon)})$ in L^p_w as $m \to \infty$. Therefore, letting $m \to \infty$ in (3.3), we have

$$\|G_\alpha(f^{(\epsilon)})\|_{p,w} \simeq \|\mathcal{I}_{-\alpha}^{(\epsilon/2)} f^{(\epsilon)}\|_{p,w}. \tag{3.6}$$

Suppose that $f \in W^{\alpha,p}_w$ and let $g = \mathcal{I}_{-\alpha}(f)$. We show that

$$\mathcal{I}_{-\alpha}^{(\epsilon/2)} f^{(\epsilon)} = g^{(\epsilon)} \tag{3.7}$$

as follows. For $h \in \mathcal{S}_0$ we have

$$\begin{aligned} \int g^{(\epsilon)} \mathcal{I}_\alpha(h)\, dx &= \lim_{m\to\infty} \int g_{m,\epsilon} \mathcal{I}_\alpha(h)\, dx = \lim_{m\to\infty} \int \mathcal{I}_\alpha^{(\epsilon/2)}(g_{m,\epsilon}) h\, dx \\ &= \int \mathcal{I}_\alpha^{(\epsilon/2)}(g^{(\epsilon)}) h\, dx. \end{aligned}$$

Also,

$$\int g^{(\epsilon)} \mathcal{I}_\alpha(h)\, dx = \lim_{m\to\infty} \int g_{m,\epsilon} \mathcal{I}_\alpha(h)\, dx$$
$$= \lim_{m\to\infty} \int g_{(m)} \mathcal{I}_\alpha(h^{(\epsilon)})\, dx = \int g \mathcal{I}_\alpha(h^{(\epsilon)})\, dx.$$

By the definition of $g = \mathcal{I}_{-\alpha}(f)$, $\int g\mathcal{I}_\alpha(h^{(\epsilon)})\, dx = \int fh^{(\epsilon)}\, dx$. Thus

$$\int g^{(\epsilon)} \mathcal{I}_\alpha(h)\, dx = \int fh^{(\epsilon)}\, dx = \lim_{m\to\infty} \int f_{(m)} h^{(\epsilon)}\, dx$$
$$= \lim_{m\to\infty} \int f_{m,\epsilon} h\, dx = \int f^{(\epsilon)} h\, dx.$$

Therefore

$$\int \mathcal{I}_\alpha^{(\epsilon/2)}(g^{(\epsilon)}) h\, dx = \int f^{(\epsilon)} h\, dx \quad \text{for all } h \in \mathcal{S}_0,$$

which implies that $\mathcal{I}_\alpha^{(\epsilon/2)}(g^{(\epsilon)}) = f^{(\epsilon)}$. Since $\mathcal{I}_\alpha^{(\epsilon/2)}$ and $\mathcal{I}_{-\alpha}^{(\epsilon/2)}$ are bounded on L^p_w and the mapping $f \mapsto f^{(\epsilon)}$ is also bounded on L^p_w, by Lemma 3.3 we see that

$$\mathcal{I}_{-\alpha}^{(\epsilon/2)}(f^{(\epsilon)}) = \mathcal{I}_{-\alpha}^{(\epsilon/2)} \mathcal{I}_\alpha^{(\epsilon/2)}(g^{(\epsilon)}) = \lim_{m\to\infty} \mathcal{I}_{-\alpha}^{(\epsilon/2)} \mathcal{I}_\alpha^{(\epsilon/2)}(g_{m,\epsilon})$$
$$= \lim_{m\to\infty} g_{m,\epsilon} = g^{(\epsilon)},$$

which proves (3.7).

By (3.6) and (3.7), we have

$$\|G_\alpha(f^{(\epsilon)})\|_{p,w} \le C\|g^{(\epsilon)}\|_{p,w} \le C\|M(g)\|_{p,w} \le C\|g\|_{p,w}.$$

Letting $\epsilon \to 0$ and applying Lemma 3.4 and Fatou's lemma, we have

$$\|G_\alpha(f)\|_{p,w} \le C\|\mathcal{I}_{-\alpha}(f)\|_{p,w}. \tag{3.8}$$

Conversely, let us assume that $f \in L^p_w$ and $G_\alpha(f) \in L^p_w$. By Minkowski's inequality we see that

$$\|G_\alpha(f^{(\epsilon)})\|_{p,w} \le C\|M(G_\alpha(f))\|_{p,w} \le C\|G_\alpha(f)\|_{p,w}. \tag{3.9}$$

Applying (3.6) and (3.9), we find that

$$\sup_{\epsilon\in(0,1/2)} \|\mathcal{I}_{-\alpha}^{(\epsilon/2)} f^{(\epsilon)}\|_{p,w} \le C \sup_{\epsilon\in(0,1/2)} \|G_\alpha(f^{(\epsilon)})\|_{p,w} \le C\|G_\alpha(f)\|_{p,w}.$$

Therefore, there exist a sequence $\{\epsilon_k\}$, $0 < \epsilon_k < 1/2$, and a function $g \in L^p_w$ such that $\epsilon_k \to 0$ and $\mathcal{I}_{-\alpha}^{(\epsilon_k/2)} f^{(\epsilon_k)} \to g$ weakly in L^p_w as $k \to \infty$ and

$$\|g\|_{p,w} \le C\|G_\alpha(f)\|_{p,w}. \tag{3.10}$$

We now show that $f = \mathcal{I}_\alpha g$. By Lemma 3.4, $f^{(\epsilon_k)} \to f$ in L^p_w. So, for $h \in \mathcal{S}_0$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} fh\,dx &= \lim_{k\to\infty} \int_{\mathbb{R}^n} f^{(\epsilon_k)} h\,dx = \lim_{k\to\infty}\lim_{m\to\infty} \int_{\mathbb{R}^n} f_{m,\epsilon_k} h\,dx \\ &= \lim_{k\to\infty}\lim_{m\to\infty} \int_{\mathbb{R}^n} \mathcal{I}_{-\alpha}(f_{m,\epsilon_k})\mathcal{I}_\alpha(h)\,dx \\ &= \lim_{k\to\infty}\lim_{m\to\infty} \int_{\mathbb{R}^n} \mathcal{I}_{-\alpha}^{(\epsilon_k/2)}(f_{m,\epsilon_k})\mathcal{I}_\alpha(h)\,dx \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^n} \mathcal{I}_{-\alpha}^{(\epsilon_k/2)}(f^{(\epsilon_k)})\mathcal{I}_\alpha(h)\,dx = \int_{\mathbb{R}^n} g\mathcal{I}_\alpha(h)\,dx.\end{aligned}$$

This implies that $f = \mathcal{I}_\alpha g$ by definition. By (3.10) we have

$$\|\mathcal{I}_{-\alpha} f\|_{p,w} = \|g\|_{p,w} \le C\|G_\alpha(f)\|_{p,w},$$

which combined with (3.8), completes the proof of Theorem 1.5. ∎

4. Characterization of $W^{\alpha,p}_w$ by square functions defined by repeated averaging. Let $\Phi \in \mathcal{M}^1$. Define $\Lambda^j_t f(x)$, $j \ge 1$, by $\Lambda^j_t f(x) = f * \Phi^{(j)}_t(x)$, where

$$\Phi^{(1)}(x) = \Phi(x), \quad \Phi^{(j)}(x) = \overbrace{\Phi * \cdots * \Phi}^{j}(x), \quad j \ge 2.$$

We also write $\Lambda_t f(x)$ for $\Lambda^1_t f(x)$. Let I be the identity operator and k a positive integer. We consider

$$\begin{aligned}(I - \Lambda_t)^k f(x) &= f(x) + \sum_{j=1}^k (-1)^j \binom{k}{j} \Lambda^j_t f(x) \\ &= f(x) - K^{(k)}_t * f(x) = \int_{\mathbb{R}^n} (f(x) - f(x-y))K^{(k)}_t(y)\,dy\end{aligned} \tag{4.1}$$

for appropriate functions f, where

$$K^{(k)}(x) = -\sum_{j=1}^k (-1)^j \binom{k}{j} \Phi^{(j)}(x), \tag{4.2}$$

and we have used the equation

$$\int_{\mathbb{R}^n} K^{(k)}(x)\,dx = -\sum_{j=1}^k (-1)^j \binom{k}{j} = 1. \tag{4.3}$$

Define

$$E^{(k)}_\alpha(f)(x) = \left(\int_0^\infty |(I - \Lambda_t)^k f(x)|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2}, \quad \alpha > 0. \tag{4.4}$$

If $\Phi = \chi_0 = |B(0,1)|^{-1}\chi_{B(0,1)}$ and $k = 2$ in (4.4), we have

$$E_\alpha^{(2)}(f)(x) = \left(\int_0^\infty \left| f(x) - 2 \fint_{B(x,t)} f(y)\,dy + \fint_{B(x,t)} (f)_{B(y,t)}\,dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2},$$

where $(f)_{B(y,t)} = \fint_{B(y,t)} f$. Also, let

$$U_\alpha^{(k)}(f)(x) = \left(\int_0^\infty |(I - \Lambda_t)^k \mathcal{I}_\alpha(f)(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \tag{4.5}$$

where $0 < \alpha < \gamma$, $f \in \mathcal{S}_0$. Using (4.1), we can rewrite $E_\alpha^{(k)}(f)$ in (4.4) and $U_\alpha^{(k)}(f)$ in (4.5) as follows:

$$E_\alpha^{(k)}(f)(x) = \left(\int_0^\infty |f(x) - K_t^{(k)} * f(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \tag{4.6}$$

$$U_\alpha^{(k)}(f)(x) = \left(\int_0^\infty |\mathcal{I}_\alpha(f)(x) - K_t^{(k)} * \mathcal{I}_\alpha(f)(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \tag{4.7}$$

where $K^{(k)}$ is as in (4.2).

As applications of Theorems 1.4 and 1.5 we have the following theorems.

THEOREM 4.1. *Let* $0 < \alpha < \min(2k, \gamma)$, $1 < p < \infty$, $w \in A_p$ *and let* $U_\alpha^{(k)}$ *be as in* (4.5). *Then*

$$\|U_\alpha^{(k)}(f)\|_{p,w} \simeq \|f\|_{p,w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n).$$

THEOREM 4.2. *Let* $1 < p < \infty$, $w \in A_p$ *and* $0 < \alpha < \min(2k, \gamma)$. *Let* $E_\alpha^{(k)}$ *be as in* (4.4). *Then* $f \in W_w^{\alpha,p}$ *if and only if* $f \in L_w^p$ *and* $E_\alpha^{(k)}(f) \in L_w^p$; *moreover,*

$$\|\mathcal{I}_{-\alpha}(f)\|_{p,w} \simeq \|E_\alpha^{(k)}(f)\|_{p,w}.$$

Proofs of Theorems 4.1 and 4.2. Using the expressions of $E_\alpha^{(k)}(f)$ and $U_\alpha^{(k)}(f)$ in (4.6) and (4.7), we can derive Theorems 4.1 and 4.2 from Theorems 1.4 and 1.5, respectively, if $K^{(k)} \in \mathcal{M}^{2k-1}$, since then $K^{(k)} \in \mathcal{M}^\alpha$ for $\alpha \in (0, \min(2k, \gamma))$.

To show that $K^{(k)} \in \mathcal{M}^{2k-1}$, first we easily see that $K^{(k)}$ is bounded and compactly supported. Since we have already noted (4.3), it remains to show that

$$\int_{\mathbb{R}^n} y^a K^{(k)}(y)\,dy = 0 \quad \text{if } 1 \le |a| < 2k. \tag{4.8}$$

This can be shown as follows. Since $\Phi \in \mathcal{M}^1$, we have $\int y^a \Phi(y)\,dy = 0$ for $|a| = 1$, which implies that $\partial_\xi^a \hat{\Phi}(0) = 0$ for $|a| = 1$. Thus near $\xi = 0$, we have

$$(4.9)\quad 1-\mathcal{F}(K^{(k)})(\xi)=1+\sum_{j=1}^{k}(-1)^j\binom{k}{j}\hat{\Phi}(\xi)^j=(1-\hat{\Phi}(\xi))^k=O(|\xi|^{2k}).$$

Also, by Taylor's formula we see that

$$(4.10)\qquad \mathcal{F}(K^{(k)})(\xi)=1+\sum_{1\le|a|<2k}C_a\xi^a\partial_\xi^a\mathcal{F}(K^{(k)})(0)+O(|\xi|^{2k}).$$

From (4.9) and (4.10) it follows that

$$\sum_{1\le|a|<2k}C_a\xi^a\partial_\xi^a\mathcal{F}(K^{(k)})(0)=O(|\xi|^{2k}).$$

This implies that $\partial_\xi^a\mathcal{F}(K^{(k)})(0)=0$ for $1\le|a|<2k$, and hence we have (4.8). ■

Remark 4.3. In the definitions of $E_\alpha^{(k)}$ and $U_\alpha^{(k)}$ in (4.4) and (4.5), if we assume only that Φ belongs to $\mathcal{M}^0$, then we have analogues of Theorems 4.1 and 4.2 for the range $(0,\min(k,\gamma))$ of α.

5. The Sobolev spaces $W_w^{\alpha,p}$ and distributional derivatives. In $\mathbb{R}^2$, we consider $P=\operatorname{diag}(1,2)$, $\delta_t=\operatorname{diag}(t,t^2)$. Then $\gamma=3$ and

$$\rho(x_1,x_2)=\frac{1}{\sqrt{2}}\sqrt{x_1^2+\sqrt{x_1^4+4x_2^2}},$$

$\rho^*=\rho$, $\delta_t^*=\delta_t$. Under this setting, let $W_w^{\alpha,p}$ be the weighted Sobolev space on $\mathbb{R}^2$ defined in Section 1 with $0<\alpha<3$, $1<p<\infty$, $w\in A_p$. Then $W_w^{2,p}$ can be characterized by using distributional derivatives as follows.

Theorem 5.1. *Let $f\in L_w^p$ with $1<p<\infty$, $w\in A_p$. Let $(\partial/\partial x_1)^2f$, $\partial/\partial x_2 f$ be the distributional derivatives in $\mathcal{S}'$ (the space of tempered distributions). Then $f\in W_w^{2,p}$ if and only if $(\partial/\partial x_1)^2f\in L_w^p$ and $\partial/\partial x_2f\in L_w^p$; further,*

$$\|\mathcal{I}_{-\alpha}(f)\|_{2,w}\simeq\|(\partial/\partial x_1)^2f\|_{p,w}+\|\partial/\partial x_2f\|_{p,w}.$$

Proof. Suppose that $f\in W_w^{2,p}$. Let $g=\mathcal{I}_{-2}(f)\in L_w^p$. Then

$$(5.1)\qquad \int fh\,dx=\int g\mathcal{I}_2(h)\,dx\quad\text{for all } h\in\mathcal{S}_0.$$

Let $k(\xi)=-4\pi^2\xi_1^2$. Let $g_{m,\epsilon}=g_{(m)}*\mathcal{F}^{-1}(\zeta^{(\epsilon)})$ be as in Section 3. Then by (5.1) we see that for $h\in\mathcal{S}_0$,

$$\begin{aligned}(5.2)\qquad \int f(\partial/\partial x_1)^2h\,dx&=\int g\mathcal{I}_2((\partial/\partial x_1)^2h)\,dx=\int g\mathcal{I}_2(T_kh)\,dx\\&=\lim_{\epsilon\to0}\lim_{m\to\infty}\int g_{m,\epsilon}\mathcal{I}_2(T_kh)\,dx=\lim_{\epsilon\to0}\lim_{m\to\infty}\int T_{k(\rho^*)^{-2}}(g_{m,\epsilon})h\,dx.\end{aligned}$$

Since $k(\rho^*)^{-2}$ is homogeneous of degree 0 with respect to δ_t^* and infinitely differentiable in $\mathbb{R}^2\setminus\{0\}$, by Lemma 2.6 the multiplier operator $T_{k(\rho^*)^{-2}}$ is bounded on L^p_w. Thus $T_{k(\rho^*)^{-2}}(g_{m,\epsilon}) \to T_{k(\rho^*)^{-2}}(g)$ in L^p_w as $m \to \infty$ and $\epsilon \to 0$ since $g_{m,\epsilon} \to g$ in L^p_w as $m \to \infty$ and $\epsilon \to 0$. Therefore, by (5.2) we have

$$\int f(\partial/\partial x_1)^2 h\, dx = \int T_{k(\rho^*)^{-2}}(g) h\, dx \qquad h \in \mathcal{S}_0, \tag{5.3}$$

which implies that

$$\int f(\partial/\partial x_1)^2 \psi\, dx = \int T_{k(\rho^*)^{-2}}(g) \psi\, dx \quad \text{for all } \psi \in \mathcal{S}. \tag{5.4}$$

It follows that

$$(\partial/\partial x_1)^2 f = T_{k(\rho^*)^{-2}}(g) \quad \text{in } \mathcal{S}'. \tag{5.5}$$

To see (5.4), substitute $\psi - \mathcal{F}^{-1}(\varphi(\delta_\epsilon^{-1}\xi)\hat{\psi}(\xi))$ for h in (5.3), where φ is as in Lemma 3.4, and let $\epsilon \to 0$.

Let $\ell(\xi) = 2\pi i \xi_2$. Then, arguing similarly to the above and noting that $\ell(\rho^*)^{-2}$ is homogeneous of degree 0 with respect to δ_t^* and infinitely differentiable in $\mathbb{R}^2 \setminus \{0\}$, we see that $T_{\ell(\rho^*)^{-2}}(g) \in L^p_w$ and

$$-\int f\partial/\partial x_2 \psi\, dx = \int T_{\ell(\rho^*)^{-2}}(g)\psi\, dx \quad \text{for all } \psi \in \mathcal{S},$$

which implies that

$$\partial/\partial x_2 f = T_{\ell(\rho^*)^{-2}}(g) \quad \text{in } \mathcal{S}'. \tag{5.6}$$

Combining (5.5) and (5.6), we have

$$\|(\partial/\partial x_1)^2 f\|_{p,w} + \|\partial/\partial x_2 f\|_{p,w} \le C\|g\|_{p,w} = C\|\mathcal{I}_{-2}(f)\|_{p,w}. \tag{5.7}$$

Conversely, suppose that $(\partial/\partial x_1)^2 f =: \Theta \in L^p_w$ and $\partial/\partial x_2 f =: \Xi \in L^p_w$. Then, for $h \in \mathcal{S}_0$ we have

$$\int f(\partial/\partial x_1)^2 h\, dx = \int \Theta h\, dx, \qquad -\int f\partial/\partial x_2 h\, dx = \int \Xi h\, dx,$$

and hence

$$\int f(T_k h - T_\ell h)\, dx = \int f((\partial/\partial x_1)^2 h - \partial/\partial x_2 h)\, dx = \int (\Theta + \Xi) h\, dx, \tag{5.8}$$

where $k(\xi)$ and $\ell(\xi)$ are as above. Let

$$N(\xi) = \frac{k(\xi) - \ell(\xi)}{\rho^*(\xi)^2} = \frac{-4\pi^2\xi_1^2 - 2\pi i \xi_2}{\rho^*(\xi)^2}.$$

Then, substituting $\mathcal{I}_2(h)$ for h in (5.8), we have

$$\int f T_N h\, dx = \int (\Theta + \Xi)\mathcal{I}_2(h)\, dx. \tag{5.9}$$

We note that the functions N and $\widetilde{N}^{-1}$ are homogeneous of degree 0 with respect to δ_t^* and infinitely differentiable in $\mathbb{R}^2 \setminus \{0\}$, where $\widetilde{N}(\xi) = N(-\xi)$. So, $T_{\widetilde{N}^{-1}}$ is bounded on L^p_w by Lemma 2.6. Substituting $T_{N^{-1}}h$ for h in (5.9), we have

$$\int fh\, dx = \int (\Theta + \Xi) T_{N^{-1}}(\mathcal{I}_2(h))\, dx = \int T_{\widetilde{N}^{-1}}(\Theta + \Xi)\mathcal{I}_2(h)\, dx, \tag{5.10}$$

where the last equality follows as (5.3), since $T_{\widetilde{N}^{-1}}$ is bounded on L^p_w. By (5.10) we see that $f \in W^{2,p}_w$ and

$$\mathcal{I}_{-2}(f) = T_{\widetilde{N}^{-1}}(\Theta + \Xi)$$

and

$$\|\mathcal{I}_{-2}(f)\|_{p,w} \le C\|\Theta\|_{p,w} + C\|\Xi\|_{p,w} = C\|(\partial/\partial x_1)^2 f\|_{p,w} + C\|\partial/\partial x_2 f\|_{p,w},$$

which combined with (5.7) completes the proof of the theorem. ∎

We conclude this note with two remarks.

Remark 5.2. To characterize the (unweighted) Sobolev spaces $W^{\alpha,p}$ we can also apply the square functions of Luzin area integral type instead of Littlewood–Paley function type (see [26]). In [23], certain (H^1) Sobolev spaces were characterized by using square functions of Luzin area integral type. The characterization of those Sobolev spaces by square functions of Littlewood–Paley type analogous to Theorem 1.5 is yet to be proved.

Remark 5.3. Let us consider another square function of Marcinkiewicz type:

$$D_\alpha(f)(x) = \Big(\int_{\mathbb{R}^n} |I_\alpha(f)(x+y) - I_\alpha(f)(x)|^2 |y|^{-n-2\alpha}\,dy\Big)^{1/2},$$

where I_α is as in (1.2). Let $0 < \alpha < 1$ and $p_0 = 2n/(n+2\alpha) > 1$. Then it is known that the operator D_α is bounded on $L^p(\mathbb{R}^n)$ if $p_0 < p < \infty$ ([27]) and that D_α is of weak type (p_0, p_0) ([8]). In [24] analogues of these results were established in the case of dilations $\delta_t = t^P$ when P is diagonal.

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