Sobolev spaces and functions of Marcinkiewicz type with repeated averaging operations over spheres

Partial Differential Equations and Applications (2022) 3:66
https://doi.org/10.1007/s42985-022-00203-1

ORIGINAL PAPER

Sobolev spaces and functions of Marcinkiewicz type with repeated averaging operations over spheres

Shuichi Sato[1]

Received: 12 July 2022 / Accepted: 7 September 2022

Abstract
We consider the weighted Sobolev spaces with weights of the Muckenhoupt class and characterize the spaces by the square functions of Marcinkiewicz type defined by repeated averaging operations over spheres.

Keywords Littlewood–Paley functions · Marcinkiewicz integrals · Sobolev spaces

Mathematics Subject Classification Primary 46E35 · 42B25

1 Introduction

The function of Marcinkiewicz is defined by

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$
$$F(x) = \int_0^x f(y)\, dy.$$

Marcinkiewicz [7] in 1938 introduced an analogue of this square function in the setting of periodic functions on the torus. Results conjectured in [7] were proved by Zygmund [17] and the non-periodic version above was provided by Waterman [16]. Let $\mathscr{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing smooth functions on $\mathbb{R}^n$ and let $\mathscr{S}_0(\mathbb{R}^n)$ be the subspace of $\mathscr{S}(\mathbb{R}^n)$ consisting of functions f with $\widehat{f}$ vanishing in a neighborhood of the origin, where $\widehat{f}$ denotes the Fourier transform defined as

$$\widehat{f}(\xi) = \mathscr{F}(f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle}\, dx, \quad \langle x, \xi \rangle = \sum_{k=1}^n x_k \xi_k.$$

The manuscript belongs to Knowledge Transfer, editor by Eduardo Teixeira.

The author is partly supported by Grant-in-Aid for Scientific Research (C) No. 20K03651, Japan Society for the Promotion of Science.

✉ Shuichi Sato
shuichipm@gmail.com

[1] Department of Mathematics, Faculty of Education, Kanazawa University, Kanazawa 920-1192, Japan

Published online: 24 September 2022

Then, for $p \in (1, \infty)$, it is known that

$$\|\mu(f)\|_p \simeq \|f\|_p, \quad f \in \mathscr{S}_0(\mathbb{R}), \tag{1.1}$$

where $\|\cdot\|_p$ denotes the L^p norm and $\|\mu(f)\|_p \simeq \|f\|_p$ means that there exist positive constants C_1, C_2 independent of f such that

$$C_1\|f\|_p \le \|\mu(f)\|_p \le C_2\|f\|_p.$$

We can see that (1.1) is equivalent to

$$\|\nu(f)\|_p \simeq \|f'\|_p, \qquad f \in \mathscr{S}_0(\mathbb{R}), \tag{1.2}$$

where

$$\nu(f)(x) = \left(\int_0^\infty |f(x+t) + f(x-t) - 2f(x)|^2 \frac{dt}{t^3}\right)^{1/2}.$$

The relation (1.2) can be used to characterize the Sobolev space $W^{1,p}$.

We write

$$f(x+t) + f(x-t) - 2f(x) = 2\left(\int_{S^0} f(x - t\theta)\, d\sigma(\theta) - f(x)\right),$$

where $S^0 = \{-1, 1\}$ and σ is a measure on S^0 such that $\sigma(\{-1\}) = 1/2$, $\sigma(\{1\}) = 1/2$. By this observation we generalize ν to higher dimensions as follows. Let $n \ge 2$ and

$$\mathcal{A}(f)(x) = \left(\int_0^\infty \left|f(x) - \int_{S^{n-1}} f(x - t\theta)\, d\sigma(\theta)\right|^2 \frac{dt}{t^3}\right)^{1/2},$$

where $d\sigma$ is the Lebesgue surface measure on S^{n-1} normalized as $\int_{S^{n-1}} d\sigma = 1$. We also write

$$\Theta_t f(x) := \int_{S^{n-1}} f(x - ty)\, d\sigma(y) = \fint_{S(x,t)} f\, d\sigma_{x,t} = \sigma_{x,t}(S(x,t))^{-1} \int_{S(x,t)} f\, d\sigma_{x,t},$$

where $S(x,t) = \{y \in \mathbb{R}^n : |x - y| = t\}$ and $\sigma_{x,t}$ is the Lebesgue surface measure on $S(x,t)$. We note that if f is a locally integrable Borel measurable function on $\mathbb{R}^n$, then the integral $\Theta_t f(x)$ is defined for all $x \in \mathbb{R}^n$ and $t > 0$ and it is a Borel measurable function in $(x,t) \in \mathbb{R}^n \times (0,\infty)$ (see [3, pp. 74–75], [14, pp. 1285–1287]). If f is a locally integrable Lebesgue measurable function, then $\Theta_t f(x)$ is defined for a.e. $x \in \mathbb{R}^n$ and all $t > 0$ and measurable in (x,t) on $\mathbb{R}^n \times (0,\infty)$; also $\Theta_t f(x)$ is measurable on $\mathbb{R}^n$ for each fixed $t > 0$. When $n \ge 3$, this can be seen from [14, pp. 1285–1287]), where the condition $n \ge 3$ is assumed to apply the maximal inequality (8-12) there. When $n = 2$, we also have similar results for $\Theta_t f(x)$, since we have a maximal inequality analogous to (8-12) of [14] by [2].

Let $S(f) = \mathcal{A}(I_1 f)$:

$$S(f)(x) = \left(\int_0^\infty \left|I_1(f)(x) - \int_{S^{n-1}} I_1(f)(x - ty)\, d\sigma(y)\right|^2 \frac{dt}{t^3}\right)^{1/2},$$

where for $\beta \in \mathbb{R}$, I_β is the Riesz potential operator defined by

$$\mathscr{F}(I_\beta(f))(\xi) = (2\pi|\xi|)^{-\beta} \hat{f}(\xi) \tag{1.3}$$

for $f \in \mathscr{S}_0$.

The following is known ([5]).

Theorem A *Suppose that* $1 < p < \infty$, $n \geq 2$. *Let* $f \in \mathscr{S}_0(\mathbb{R}^n)$. *Then*

$$\|S(f)\|_p \simeq \|f\|_p.$$

This is used to characterize the Sobolev space $W^{1,p}(\mathbb{R}^n)$ in terms of $\mathcal{A}(f)$. Theorem A was motivated by results of Alabern et al. [1], where the operator

$$E(f)(x) = \left(\int_0^\infty \left|f(x) - \fint_{B(x,t)} f(y)\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2} \tag{1.4}$$

was considered and used to characterize $W^{1,p}$. Here

$$\fint_{B(x,t)} f(y)\,dy = |B(x,t)|^{-1} \int_{B(x,t)} f(y)\,dy,$$

where $B(x,t)$ is a ball in $\mathbb{R}^n$ with center x and radius t.

We generalize the operators $\mathcal{A}$ and S defined above. Let

$$\mathcal{A}_\alpha(f)(x) = \left(\int_0^\infty \left|f(x) - \int_{S^{n-1}} f(x-ty)\,d\sigma(y)\right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2}, \tag{1.5}$$

$$S_\alpha(f)(x) = \left(\int_0^\infty \left|I_\alpha(f)(x) - \int_{S^{n-1}} I_\alpha(f)(x-ty)\,d\sigma(y)\right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2}. \tag{1.6}$$

Then, we have an analogue of Theorem A for $1 < \alpha < 2$ (see [9]).

Theorem B *Let* S_α *be as in* (1.6) *and* $f \in \mathscr{S}_0(\mathbb{R}^n)$, $n \geq 2$. *Then if* $1 < \alpha < 2$, *we have*

$$\|S_\alpha(f)\|_p \simeq \|f\|_p$$

for $1 < p < \infty$.

This can be used to characterize the Sobolev spaces $W^{\alpha,p}$ for $1 < \alpha < 2$ by $\mathcal{A}_\alpha$ in (1.5).

In this note, to characterize $W^{\alpha,p}$ for $2 \leq \alpha < n$, we generalize S_α by considering iterated averaging operations. For $k \in \mathbb{Z}$ (the set of integers), $k \geq 1$, let

$$\mathcal{A}_\alpha^{(k)}(f)(x) = \left(\int_0^\infty \left|(I - \Theta_t)^k f(x)\right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2}, \tag{1.7}$$

$$S_\alpha^{(k)}(f)(x) = \left(\int_0^\infty \left|(I - \Theta_t)^k I_\alpha(f)(x)\right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2}, \tag{1.8}$$

where I is the identity operator and

$$(I - \Theta_t)^k = I + \sum_{j=1}^k (-1)^j \binom{k}{j} \Theta_t^j, \quad \binom{k}{j} = \frac{k!}{(k-j)!\,j!},$$

$$\Theta_t^j f(x) = f * \underbrace{\sigma_t * \cdots * \sigma_t}_{j}(x), \quad j \geq 2,$$

$$\Theta_t^1 f(x) = \Theta_t f(x) = \int_{S^{n-1}} f(x-ty)\,d\sigma(y) = f * \sigma_t(x), \quad \sigma_t = \sigma_{0,t}(S(0,t))^{-1}\sigma_{0,t}.$$

We note that $f * \sigma_t(x) = \fint_{S(x,t)} f\,d\sigma_{x,t}$.

If $k = 2$ in (1.7), we have

$$\mathcal{A}_\alpha^{(2)}(f)(x) = \left(\int_0^\infty \left| f(x) - 2\fint_{S(x,t)} f(y)\, d\sigma_{x,t}(y) + \fint_{S(x,t)} (f)_{S(y,t)}\, d\sigma_{x,t}(y) \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2},$$

where $(f)_{S(y,t)} = \fint_{S(y,t)} f$. We note that $\mathcal{A}_\alpha^{(1)} = \mathcal{A}_\alpha$, $\mathcal{A}_1 = \mathcal{A}$, $S_\alpha^{(1)} = S_\alpha$, $S_1 = S$.

Also, we consider discrete parameter versions of $\mathcal{A}_\alpha^{(k)}$ and $S_\alpha^{(k)}$:

$$\mathcal{B}_\alpha^{(k)}(f)(x) = \left(\sum_{\ell=-\infty}^{\infty} \left| (I - \Theta_{2^\ell})^k f(x) \right|^2 2^{-2\ell\alpha} \right)^{1/2}, \tag{1.9}$$

$$U_\alpha^{(k)}(f)(x) = \left(\sum_{\ell=-\infty}^{\infty} \left| (I - \Theta_{2^\ell})^k I_\alpha f(x) \right|^2 2^{-2\ell\alpha} \right)^{1/2}. \tag{1.10}$$

Let $1 < p < \infty$, $0 < \alpha < n$ and $w \in A_p$ (the weight class of Muckenhoupt). We recall that a weight function w belongs to A_p, if

$$\sup_B \left(|B|^{-1} \int_B w(x)\, dx \right) \left(|B|^{-1} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. Let L_w^p be the weighted Lebesgue space consisting of all functions f such that

$$\|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx \right)^{1/p} < \infty.$$

Define the weighted Sobolev space $W_w^{\alpha,p}$, $0 < \alpha < n$, by

$$W_w^{\alpha,p} = \{ f \in L_w^p : f = I_\alpha(g) \text{ for some } g \in L_w^p \}, \tag{1.11}$$

where $f = I_\alpha(g)$ signifies that

$$\int_{\mathbb{R}^n} f(x) h(x)\, dx = \int_{\mathbb{R}^n} g(x) I_\alpha(h)\, dx \quad \text{for all } h \in \mathscr{S}_0;$$

such function $g \in L_w^p$ is uniquely determined by f, since I_α is a bijection on $\mathscr{S}_0$, which is dense in $L^{p'}(w^{-p'/p})$, the dual space of $L^p(w)$, with $1/p + 1/p' = 1$. Define $g = I_{-\alpha}(f)$, and for $f \in W_w^{\alpha,p}$ let

$$\|f\|_{p,\alpha,w} = \|f\|_{p,w} + \|I_{-\alpha}(f)\|_{p,w}. \tag{1.12}$$

(See Remarks 1.5 and 1.6 below.) We simply write $W^{\alpha,p}$ when $w = 1$ (unweighted case). In this note, we mainly concentrate on the case $1 \le \alpha < n$.

We shall prove the following theorems.

Theorem 1.1 *Suppose that* $1 < \alpha < \min(2k, n)$, $1 < p < \infty$ *and* $w \in A_p$. *Let* $S_\alpha^{(k)}$ *be as in* (1.8). *Then we have*

$$\|S_\alpha^{(k)}(f)\|_{p,w} \simeq \|f\|_{p,w}, \quad f \in \mathscr{S}_0(\mathbb{R}^n).$$

Theorem 1.2 *Let* $1 < \alpha < \min(2k, n)$ *and let* $\mathcal{A}_\alpha^{(k)}$ *be as in* (1.7). *Let* $w \in A_p$ *with* $1 < p < \infty$. *Then,* f *is in the space* $W_w^{\alpha,p}$ *if and only if* $f \in L_w^p$ *and* $\mathcal{A}_\alpha^{(k)}(f) \in L_w^p$; *also, we have*

$$\|I_{-\alpha}(f)\|_{p,w} \simeq \|\mathcal{A}_\alpha^{(k)}(f)\|_{p,w},$$

where $I_{-\alpha}(f)$ *is as in* (1.12).

Theorem 1.3 *Suppose that* $1 < \alpha < \min(2k, n)$, $1 < p < \infty$ *and* $w \in A_p$. *Let* $U_\alpha^{(k)}$ *be as in* (1.10). *Then we have*

$$\|U_\alpha^{(k)}(f)\|_{p,w} \simeq \|f\|_{p,w}, \quad f \in \mathscr{S}_0(\mathbb{R}^n).$$

Theorem 1.4 *Let* $1 < \alpha < \min(2k, n)$ *and let* $\mathcal{B}_\alpha^{(k)}$ *be as in* (1.9). *Suppose that* $1 < p < \infty$ *and* $w \in A_p$. *Then,* f *belongs to* $W_w^{\alpha,p}$ *if and only if* $f \in L_w^p$ *and* $\mathcal{B}_\alpha^{(k)}(f) \in L_w^p$*; also, we have*

$$\|I_{-\alpha}(f)\|_{p,w} \simeq \|\mathcal{B}_\alpha^{(k)}(f)\|_{p,w}.$$

Analogues of Theorems 1.1 and 1.2 are obtained by Theorems 4.1 and 4.2 of [12], where averaging over spheres is replaced by averaging over balls.

We shall prove Theorems 1.1 and 1.3 by applying results for more general Littlewood-Paley operators. Let ψ be a function in $L^1(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} \psi(x)\, dx = 0. \tag{1.13}$$

The Littlewood-Paley function on $\mathbb{R}^n$ is defined by

$$g_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t}\right)^{1/2}, \tag{1.14}$$

where $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. Also a discrete parameter version of g_ψ is defined by

$$\Delta_\psi(f)(x) = \left(\sum_{\ell=-\infty}^{\infty} \left|f * \psi_{2^\ell}(x)\right|^2\right)^{1/2}. \tag{1.15}$$

The following results are known.

Theorem C *Suppose that*

(1) *there exists* $\epsilon > 0$ *such that* $B_\epsilon(\psi) < \infty$*, where* $B_\epsilon(\psi) = \int_{|x|>1} |\psi(x)|\,|x|^\epsilon\, dx$;
(2) *there exists* $u > 1$ *such that* $C_u(\psi) < \infty$*, where* $C_u(\psi) = \int_{|x|<1} |\psi(x)|^u\, dx$;
(3) H_ψ *belongs to* $L^1(\mathbb{R}^n)$*, where* $H_\psi(x) = \sup_{|y|\ge|x|} |\psi(y)|$.

Then

$$\|g_\psi(f)\|_{p,w} \le C_{p,w}\|f\|_{p,w}$$

for all $p \in (1, \infty)$ *and* $w \in A_p$. *If we further assume the non-degeneracy condition*: $\sup_{t>0} |\hat{\psi}(t\xi)| > 0$ *for all* $\xi \ne 0$*, then we also have the reverse inequality and hence* $\|f\|_{p,w} \simeq \|g_\psi(f)\|_{p,w}$, $f \in L_w^p$*, for* $p \in (1, \infty)$ *and* $w \in A_p$.

Theorem D *Let* $B_\epsilon(\psi)$, H_ψ *be as in Theorem C. Suppose that*

(1) *there exists* $\epsilon > 0$ *such that* $B_\epsilon(\psi) < \infty$;
(2) *there exists* $\delta > 0$ *such that* $|\hat{\psi}(\xi)| \le C|\xi|^{-\delta}$ *for all* $\xi \in \mathbb{R}^n \setminus \{0\}$;
(3) *the function* H_ψ *is in* $L^1(\mathbb{R}^n)$.

Then

$$\|\Delta_\psi(f)\|_{p,w} \le C_{p,w}\|f\|_{p,w}$$

for every $w \in A_p$ *and every* $p \in (1, \infty)$. *If we further have the non-degeneracy condition*: $\sup_{\ell\in\mathbb{Z}} |\hat{\psi}(2^\ell\xi)| > 0$ *for all* $\xi \ne 0$*, then the reverse inequality also holds and hence* $\|f\|_{p,w} \simeq \|\Delta_\psi(f)\|_{p,w}$, $f \in L_w^p$*, for* $p \in (1, \infty)$ *and* $w \in A_p$.

See [8, 11] for Theorems C and D.

Remark 1.5 The definition (1.11) of $W^{\alpha,p}_w$ is the same as that in [9, 11], where $W^{\alpha,p}_w$ is defined by using the Bessel potentials (see [13, Chap. V] for related results). This can be seen as follows. The space $W^{\alpha,p}_w$ with the definition (1.11) is characterized by a certain square function in [12, Theorem 1.5]. The same square function also characterizes the space $W^{\alpha,p}_w$ defined in terms of the Bessel potentials, which is shown in [11, Corollary 5.2]. Consequently, we see that the two definitions coincide.

Remark 1.6 Let $\mathscr{S}_{00}(\mathbb{R}^n)$ be the subspace of $\mathscr{S}_0(\mathbb{R}^n)$ consisting of functions f with $\widehat{f}$ vanishing outside a compact set not containing the origin. Then we can replace $\mathscr{S}_0(\mathbb{R}^n)$ by $\mathscr{S}_{00}(\mathbb{R}^n)$ in the definition of the weighted Sobolev spaces $W^{\alpha,p}_w$ without changing the definition of the spaces. This is because $\mathscr{S}_{00}(\mathbb{R}^n)$ is also dense in L^p_w for $w \in A_p$, $1 < p < \infty$.

In Sect. 2, we shall prove Theorem 1.1 by applying Theorem C. Theorem 1.2 will be proved in Sect. 3 as an application of Theorem 1.1. We shall prove Theorem 1.3 in Sect. 4 by applying Theorem D. Also, in Sect. 4, Theorem 1.4 will be proved by using Theorem 1.3. In Sect. 5, analogues of Theorems 1.3 and 1.4 for $\alpha = 1$ will be presented. This will be accomplished by applying ideas of [5] in an essential way. Finally, in Sect. 6 we shall have some further remarks and results.

2 Proof of Theorem 1.1

We write

$$(I - \Theta_t)^k = I - N_t, \quad N_t = -\sum_{j=1}^{k} (-1)^j \binom{k}{j} \Theta_t^j. \tag{2.1}$$

We note that $N_t f = f * \mu_t$ with a measure μ_t, $t > 0$, satisfying

$$\begin{aligned} N_t f(x) &= \int f(x-y)\, d\mu_t(y) = \int f(x-ty)\, d\mu(y), \\ \widehat{\mu_t}(\xi) &= -\sum_{j=1}^{k} (-1)^j \binom{k}{j} \hat{\sigma}(t\xi)^j, \end{aligned} \tag{2.2}$$

and hence $\widehat{\mu_t}(0) = -\sum_{j=1}^{k} (-1)^j \binom{k}{j} = 1$, where $\mu = \mu_1$. Using $\int d\mu = 1$, for $f \in \mathscr{S}_0$, we see that

$$\begin{aligned} (I - \Theta_t)^k I_\alpha f(x) &= I_\alpha f(x) - I_\alpha f * \mu_t(x) \\ &= \int_{\mathbb{R}^n} (I_\alpha f(x) - I_\alpha f(x - ty))\, d\mu(y). \end{aligned}$$

Recall that if $L_\alpha(x) = \tau(\alpha)|x|^{\alpha-n}$, $0 < \alpha < n$, with

$$\tau(\alpha) = \frac{\Gamma(n/2 - \alpha/2)}{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)},$$

then $\widehat{L_\alpha}(\xi) = (2\pi|\xi|)^{-\alpha}$. Let

$$\zeta(x) = L_\alpha(x) - \int_{\mathbb{R}^n} L_\alpha(x-y)\, d\mu(y) = \int_{\mathbb{R}^n} (L_\alpha(x) - L_\alpha(x-y))\, d\mu(y). \tag{2.3}$$

The following results will be used in estimating ζ.

Lemma 2.1 *We have the following properties of μ.*

(1) *the measure μ is compactly supported*;
(2) *for any compact set K in $\mathbb{R}^n$ and α, $1 < \alpha < n$, we have*

$$\sup_{x \in K} \left| \int_{\mathbb{R}^n} L_\alpha(x - y)\, d\mu(y) \right| < \infty;$$

(3) $\int_{\mathbb{R}^n} y^\gamma \, d\mu(y) = 0$ *if* $1 \le |\gamma| \le 2k - 1$, *where* $\gamma = (\gamma_1, \ldots, \gamma_n)$ *is a multi-index*; $\gamma_j \in \mathbb{Z}$, $\gamma_j \ge 0$, $|\gamma| = \gamma_1 + \cdots + \gamma_n$, $y^\gamma = y_1^{\gamma_1} \ldots y_n^{\gamma_n}$.

Proof We can see the assertion of part (1) from

$$\mu = -\sum_{j=1}^{k} (-1)^j \binom{k}{j} \sigma^{(j)}, \quad \sigma^{(j)} = \underbrace{\sigma * \cdots * \sigma}_{j}, \quad j \ge 2, \quad \sigma^{(1)} = \sigma,$$

since σ is concentrated on S^{n-1}. To prove part (2), we first see that, by a direct computation,

$$\sup_{x \in K} \int L_\alpha(x - y)\, d\sigma(y) < \infty,$$

where we are assuming that $\alpha > 1$. By induction, this holds with σ replaced by $\sigma^{(j)}$ for any $2 \le j \le k$, which easily implies what is claimed.

Proof of part (3). Since $1 - \hat{\sigma}(\xi) = O(|\xi|^2)$ for $|\xi| \le 1$, we see that

$$1 - \hat{\mu}(\xi) = 1 + \sum_{j=1}^{k} (-1)^j \binom{k}{j} \hat{\sigma}(\xi)^j = (1 - \hat{\sigma}(\xi))^k = O(|\xi|^{2k})$$

for $|\xi| \le 1$. On the other hand, applying Taylor's formula for $\hat{\mu}(\xi)$ at $\xi = 0$, we have

$$\hat{\mu}(\xi) = 1 + \sum_{1 \le |\gamma| \le 2k-1} c_\gamma \partial^\gamma \hat{\mu}(0) \xi^\gamma + O(|\xi|^{2k})$$

for $|\xi| \le 1$, where $\partial^\gamma = \partial_1^{\gamma_1} \ldots \partial_n^{\gamma_n} = (\partial/\partial\xi_1)^{\gamma_1} \ldots (\partial/\partial\xi_n)^{\gamma_n}$. Thus we see that

$$\sum_{1 \le |\gamma| \le 2k-1} c_\gamma \partial^\gamma \hat{\mu}(0) \xi^\gamma = O(|\xi|^{2k})$$

for $|\xi| \le 1$. It follows that $\partial^\gamma \hat{\mu}(0) = 0$ if $1 \le |\gamma| \le 2k - 1$, which implies what we need. $\square$

We apply Taylor's formula in (2.3) for $L_\alpha(x - y)$ as a function of y at $y = 0$. Then, by (1) and (3) of Lemma 2.1, since $[\alpha] \le 2k - 1$, if $\alpha < 2k$, we have, if $|x|$ is sufficiently large,

$$|\zeta(x)| \le C|x|^{\alpha - n - [\alpha] - 1}. \tag{2.4}$$

Combining (2.4) with part (2) of Lemma 2.1, we see that

$$|\zeta(x)| \le \begin{cases} C|x|^{\alpha - n} & \text{if } |x| \le 1, \\ C|x|^{\alpha - n - [\alpha] - 1} & \text{if } |x| > 1. \end{cases} \tag{2.5}$$

It follows, in particular, that $\zeta \in L^1(\mathbb{R}^n)$. Also, by (2.3) and part (3) of Lemma 2.1 we have

$$\hat{\zeta}(\xi) = (2\pi|\xi|)^{-\alpha}(1 - \hat{\mu}(\xi)) = O(|\xi|^{-\alpha + 2k})$$

for $|\xi| \le 1$. Since $\alpha < 2k$, this implies that $\hat{\zeta}(0) = 0$, or $\int \zeta = 0$. Also, we see that $\sup_{t>0} |\hat{\zeta}(t\xi)| > 0$ for every $\xi \ne 0$, since $\hat{\mu}(\xi) \to 0$ as $|\xi| \to \infty$. By this and (2.5) we can apply Theorem C to conclude that $\|g_\zeta(f)\|_{p,w} \simeq \|f\|_{p,w}$ for $f \in L^p_w$, $p \in (1, \infty)$ and $w \in A_p$, which implies Theorem 1.1 since $S^{(k)}_\alpha(f) = g_\zeta(f)$.

Remark 2.2 It is known that

$$\sigma * \sigma(x) = \begin{cases} c|x|^{-n+2} \left[(2^2 - |x|^2)|x|^2\right]^{(n-3)/2}, & \text{if } 0 < |x| < 2; \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\sigma^{(j)}$ is a compactly supported radial function when $j \ge 2$, where $\sigma^{(j)}$ is as in the proof of Lemma 2.1.

3 Proof of Theorem 1.2

We can easily prove the following two lemmas (see [12] for the proofs).

Lemma 3.1 *Let $f \in L^p_w$, where $1 < p < \infty$ and $w \in A_p$. Let m be a positive integer and define $f_{(m)} = f\chi_{E_m}$ with*

$$E_m = \{x \in \mathbb{R}^n : |x| \le m, |f(x)| \le m\},$$

where χ_E denotes the characteristic function of a set E. Then we see that $f_{(m)} \to f$ almost everywhere and in L^p_w as $m \to \infty$.

Lemma 3.2 *Let $1 < p < \infty$ and $f \in L^p_w$ with $w \in A_p$. Choose an infinitely differentiable, non-negative, radial function ϕ on $\mathbb{R}^n$ such that $\phi(\xi) = 1$ for $|\xi| \le 1$, $\mathrm{supp}(\phi) \subset \{|\xi| \le 2\}$. Define $\eta^{(\epsilon)} \in \mathscr{S}_0$ for $\epsilon \in (0, 1/2)$ by*

$$\eta^{(\epsilon)}(\xi) = \phi(\epsilon\xi) - \phi(\epsilon^{-1}\xi).$$

Then $\eta^{(\epsilon)}(\xi) = \eta^{(\epsilon/2)}(\xi)\eta^{(\epsilon)}(\xi)$. Define $f^{(\epsilon)} = f * \mathscr{F}^{-1}(\eta^{(\epsilon)})$. Then $f^{(\epsilon)} \to f$ almost everywhere and in L^p_w as $\epsilon \to 0$.

Also, we need the following.

Lemma 3.3 *Suppose that $f \in L^p_w$, $w \in A_p$, $1 < p < \infty$. Let $f^{(\epsilon)}$ be as in Lemma 3.2. Let $d\mu$ be as in (2.2). Then we have the following.*

(1) *there exists a sequence $\{\epsilon_k\}$, $\epsilon_k \to 0$, such that*

$$\int_{S^{n-1}} f^{(\epsilon_k)}(x - ty)\, d\mu(y) \to \int_{S^{n-1}} f(x - ty)\, d\mu(y) \tag{3.1}$$

for a.e. $(x, t) \in \mathbb{R}^n \times (0, \infty)$;

(2) *we can find a sequence $\{\epsilon_k\}$, $\epsilon_k \to 0$, such that we have the convergence (3.1) for a.e. $x \in \mathbb{R}^n$ and for all $t = 2^\ell$ with $\ell \in \mathbb{Z}$.*

Proof Let $K_M = B(0, M) \times (0, M)$, $M = 1, 2, 3, \ldots$. By part (1) of Lemma 2.1, the measure μ is supported in $B(0, N)$ for some $N > 0$. We see that

$$
\begin{aligned}
I_{M,\epsilon} &:= \iint_{K_M} \left| \int f^{(\epsilon)}(x-ty)\,d\mu(y) - \int f(x-ty)\,d\mu(y) \right| dx\,dt \\
&\le \int_{B(0,N)} M \int_{B(0,M+MN)} \left| f^{(\epsilon)}(x) - f(x) \right| dx\,d|\mu|(y) \\
&\le CM \int_{B(0,M+MN)} \left(\left| f^{(\epsilon)}(x) - f(x) \right|^p w(x)\,dx \right)^{1/p} \left(\int_{B(0,M+MN)} w(x)^{-p'/p}\,dx \right)^{1/p'},
\end{aligned}
$$

where the last inequality follows by Hölder's inequality. By Lemma 3.2, it follows that $I_{M,\epsilon} \to 0$ as $\epsilon \to 0$. Therefore, there exists a sequence $\{\epsilon_k\}$ for which we have (3.1) for a.e. $(x,t) \in K_M$. Applying this arguments, we can find sequences $\{\epsilon_k^{(M)}\}$, $M = 1, 2, 3, \ldots$, such that $\{\epsilon_k^{(M+1)}\}$ is a subsequence of $\{\epsilon_k^{(M)}\}$ and we have

$$
\int_{S^{n-1}} f^{(\epsilon_k^{(M)})}(x-ty)\,d\mu(y) \to \int_{S^{n-1}} f(x-ty)\,d\mu(y)
$$

for a.e. $(x,t) \in K_M$. Thus we can get the conclusion of part (1) of the lemma by applying the diagonal process arguments.

Part (2) can be shown similarly, since we have the convergence (3.1) for a.e. $x \in \mathbb{R}^n$ and for each fixed $t = 2^\ell$ with some $\{\epsilon_k\}$ by the arguments of the proof of part (1). $\square$

For $\delta \in (0, 1/2)$, $\beta \in \mathbb{R}$ and $f \in L^p_w$, let $I_\beta^{(\delta)}(f) = \mathscr{F}^{-1}(\eta^{(\delta)}(\xi)(2\pi|\xi|)^{-\beta}) * f$, where $\eta^{(\delta)}$ is as in Lemma 3.2.

Lemma 3.4 *Let $f \in L^p_w$, $w \in A_p$, $1 < p < \infty$ and let $f^{(\epsilon)}$ be as in Lemma 3.2. Let $\mathcal{A}_\alpha^{(k)}$ be as in Theorem 1.2. Then*

$$
\|\mathcal{A}_\alpha^{(k)}(f^{(\epsilon)})\|_{p,w} \simeq \|I_{-\alpha}^{(\epsilon/2)} f^{(\epsilon)}\|_{p,w}, \quad 0 < \epsilon < 1/2.
$$

Proof For $f \in L^p_w$, $\epsilon \in (0, 1/2)$ and a positive integer m, define $f_{m,\epsilon} \in \mathscr{S}_0$ by $f_{m,\epsilon} = (f_{(m)})^{(\epsilon)}$, where $f_{(m)}$ is as in Lemma 3.1. By Theorem 1.1 we have

$$
\|\mathcal{A}_\alpha^{(k)}(f_{m,\epsilon})\|_{p,w} = \|S_\alpha^{(k)}(I_{-\alpha} f_{m,\epsilon})\|_{p,w} \simeq \|I_{-\alpha}^{(\epsilon/2)} f_{m,\epsilon}\|_{p,w}, \tag{3.2}
$$

where we have used the relation $I_{-\alpha} f_{m,\epsilon} = I_{-\alpha}^{(\epsilon/2)} f_{m,\epsilon}$.

Let K be a compact set in $\mathbb{R}^n$. Then we see that $f_{m,\epsilon}(x) \to f^{(\epsilon)}(x)$ uniformly for $x \in K$, since by Hölder's inequality, we have

$$
\begin{aligned}
\left| f_{m,\epsilon}(x) - f^{(\epsilon)}(x) \right| &= \left| \int (f_{(m)}(y) - f(y)) \mathscr{F}^{-1}(\eta^{(\epsilon)})(x-y)\,dy \right| \\
&\le \|f_{(m)} - f\|_{p,w} \left(\int \left| \mathscr{F}^{-1}(\eta^{(\epsilon)})(x-y) \right|^{p'} w(y)^{-p'/p}\,dy \right)^{1/p'}.
\end{aligned}
$$

Thus by Lemma 3.1 we can easily see the uniform convergence claimed. It follows that $f_{m,\epsilon}(x) - f_{m,\epsilon} * \mu_t(x) \to f^{(\epsilon)}(x) - f^{(\epsilon)} * \mu_t(x)$ as $m \to \infty$ for all $x \in \mathbb{R}^n$ and $t > 0$ (see (2.2)). Therefore, recalling the definition of $\mathcal{A}_\alpha^{(k)}$ and noting (2.1), by Fatou's lemma and (3.2), we see that,

$$
\begin{aligned}
\|\mathcal{A}_\alpha^{(k)}(f^{(\epsilon)})\|_{p,w} &\le \liminf_{m\to\infty} \|\mathcal{A}_\alpha^{(k)}(f_{m,\epsilon})\|_{p,w} \\
&\le C \liminf_{m\to\infty} \|I_{-\alpha}^{(\epsilon/2)} f_{m,\epsilon}\|_{p,w} = C\|I_{-\alpha}^{(\epsilon/2)} f^{(\epsilon)}\|_{p,w},
\end{aligned} \tag{3.3}
$$

where we have the last equality since $I_{-\alpha}^{(\epsilon/2)}$ is bounded on L^p_w. In particular, we see that $\mathcal{A}_\alpha^{(k)}(f^{(\epsilon)}) \in L^p_w$.

To complete the proof of Lemma 3.4, we first note that

$$\begin{aligned}\|\mathcal{A}_\alpha^{(k)}(f^{(\epsilon)}) - \mathcal{A}_\alpha^{(k)}(f_{m,\epsilon})\|_{p,w} &\le \|\mathcal{A}_\alpha^{(k)}(f^{(\epsilon)} - f_{m,\epsilon})\|_{p,w} \\ &= \|\mathcal{A}_\alpha^{(k)}((f - f_{(m)})^{(\epsilon)})\|_{p,w}. \end{aligned} \tag{3.4}$$

We can see that

$$(f_{(j)} - f_{(m)})^{(\epsilon)}(x) - (f_{(j)} - f_{(m)})^{(\epsilon)} * \mu_t(x) \to (f - f_{(m)})^{(\epsilon)}(x) - (f - f_{(m)})^{(\epsilon)} * \mu_t(x)$$

as $j \to \infty$ for all x and t, as we have shown above that $f_{m,\epsilon}(x) - f_{m,\epsilon} * \mu_t(x) \to f^{(\epsilon)}(x) - f^{(\epsilon)} * \mu_t(x)$. Thus by Fatou's lemma we see that

$$\|\mathcal{A}_\alpha^{(k)}((f - f_{(m)})^{(\epsilon)})\|_{p,w} \le \liminf_{j\to\infty} \|\mathcal{A}_\alpha^{(k)}((f_{(j)} - f_{(m)})^{(\epsilon)})\|_{p,w}. \tag{3.5}$$

Since $(f_{(j)} - f_{(m)})^{(\epsilon)} \in \mathscr{S}_0$, by Theorem 1.1 we have

$$\begin{aligned}&\|\mathcal{A}_\alpha^{(k)}((f_{(j)} - f_{(m)})^{(\epsilon)})\|_{p,w} \simeq \|I_{-\alpha}((f_{(j)} - f_{(m)})^{(\epsilon)})\|_{p,w} \\ &= \|I_{-\alpha}^{(\epsilon/2)}((f_{(j)} - f_{(m)})^{(\epsilon)})\|_{p,w}.\end{aligned}$$

Since $f_{(m)} \to f$ in L^p_w, from this it follows that

$$\lim_{j,m\to\infty} \|\mathcal{A}_\alpha^{(k)}((f_{(j)} - f_{(m)})^{(\epsilon)})\|_{p,w} = 0,$$

which combined with (3.4) and (3.5) implies that $\mathcal{A}_\alpha^{(k)}(f_{m,\epsilon}) \to \mathcal{A}_\alpha^{(k)}(f^{(\epsilon)})$ in L^p_w as $m \to \infty$. Thus, letting $m \to \infty$ in (3.2), we have the conclusion of Lemma 3.4. □

Furthermore, we need the following.

Lemma 3.5 *Let $w \in A_p$, $1 < p < \infty$. Suppose that $f \in W^{\alpha,p}_w$ and $g = I_{-\alpha}(f)$ $(0 < \alpha < n)$. Then we have*

$$I_{-\alpha}^{(\epsilon/2)} f^{(\epsilon)} = g^{(\epsilon)}.$$

Proof For $h \in \mathscr{S}_0$ we see that

$$\begin{aligned}\int g^{(\epsilon)}(x) I_\alpha(h)(x)\,dx &= \lim_{m\to\infty} \int g_{m,\epsilon}(x) I_\alpha(h)(x)\,dx \\ &= \lim_{m\to\infty} \int I_\alpha^{(\epsilon/2)}(g_{m,\epsilon})(x)h(x)\,dx \\ &= \int I_\alpha^{(\epsilon/2)}(g^{(\epsilon)})(x)h(x)\,dx,\end{aligned} \tag{3.6}$$

where $g_{m,\epsilon}$ is as in the proof of Lemma 3.4. We rewrite the integral $\int g^{(\epsilon)} I_\alpha(h)\,dx$ as follows:

$$\begin{aligned}\int g^{(\epsilon)}(x) I_\alpha(h)(x)\,dx &= \lim_{m\to\infty} \int g_{m,\epsilon}(x) I_\alpha(h)(x)\,dx \\ &= \lim_{m\to\infty} \int g_{(m)}(x) I_\alpha(h^{(\epsilon)})(x)\,dx = \int g(x) I_\alpha(h^{(\epsilon)})(x)\,dx.\end{aligned} \tag{3.7}$$

We have $\int g I_\alpha(h^{(\epsilon)})\,dx = \int f h^{(\epsilon)}\,dx$ by the definition of $g = I_{-\alpha}(f)$. Using this in (3.7), we see that

$$\begin{aligned}\int g^{(\epsilon)}(x) I_\alpha(h)(x)\,dx &= \int f(x) h^{(\epsilon)}(x)\,dx = \lim_{m\to\infty} \int f_{(m)}(x) h^{(\epsilon)}(x)\,dx \\ &= \lim_{m\to\infty} \int f_{m,\epsilon}(x) h(x)\,dx = \int f^{(\epsilon)}(x) h(x)\,dx. \end{aligned} \tag{3.8}$$

By (3.6) and (3.8) for all $h \in \mathscr{S}_0$ we have

$$\int I_\alpha^{(\epsilon/2)}(g^{(\epsilon)})(x) h(x)\,dx = \int f^{(\epsilon)}(x) h(x)\,dx.$$

Thus we see that $I_\alpha^{(\epsilon/2)}(g^{(\epsilon)}) = f^{(\epsilon)}$. We note that $I_\alpha^{(\epsilon/2)}$ and $I_{-\alpha}^{(\epsilon/2)}$ are bounded on L^p_w and the mapping $f \to f^{(\epsilon)}$ is also bounded on L^p_w. Therefore, using Lemma 3.1 we have

$$I_{-\alpha}^{(\epsilon/2)}(f^{(\epsilon)}) = I_{-\alpha}^{(\epsilon/2)}(I_\alpha^{(\epsilon/2)}(g^{(\epsilon)})) = \lim_{m\to\infty} I_{-\alpha}^{(\epsilon/2)}(I_\alpha^{(\epsilon/2)}(g_{m,\epsilon})).$$

Since $g_{m,\epsilon} \in \mathscr{S}_0$ and $\eta^{(\epsilon/2)} = 1$ on the support of $\mathscr{F}(g_{m,\epsilon})$, we easily see that

$$\begin{aligned} I_{-\alpha}^{(\epsilon/2)}(I_\alpha^{(\epsilon/2)}(g_{m,\epsilon}))(x) &= \int (\eta^{(\epsilon/2)}(\xi))^2 \mathscr{F}(g_{m,\epsilon})(\xi) e^{2\pi i\langle x,\xi\rangle}\,d\xi \\ &= \int \mathscr{F}(g_{m,\epsilon})(\xi) e^{2\pi i\langle x,\xi\rangle}\,d\xi = g_{m,\epsilon}(x). \end{aligned}$$

Using this, we see that

$$I_{-\alpha}^{(\epsilon/2)}(f^{(\epsilon)}) = \lim_{m\to\infty} g_{m,\epsilon} = g^{(\epsilon)}.$$

This completes the proof of Lemma 3.5. □

Proof of Theorem 1.2 Let $f \in W_w^{\alpha,p}$ and $g = I_{-\alpha}(f)$. From Lemmas 3.4 and 3.5, it follows that

$$\|\mathcal{A}_\alpha^{(k)}(f^{(\epsilon)})\|_{p,w} \le C\|g^{(\epsilon)}\|_{p,w} \le C\|M(g)\|_{p,w} \le C\|g\|_{p,w},$$

where M denotes the Hardy-Littlewood maximal operator, which is bounded on L^p_w. From part (1) of Lemma 3.3 and Lemma 3.2, we can find a sequence $\{\epsilon_j\}$ such that $f^{(\epsilon_j)}(x) - f^{(\epsilon_j)} * \mu_t(x) \to f(x) - f * \mu_t(x)$ for a.e. $(x,t) \in \mathbb{R}^n \times (0,\infty)$ as $j \to \infty$. Therefore, by Fatou's lemma we have

$$\|\mathcal{A}_\alpha^{(k)}(f)\|_{p,w} \le \liminf_{j\to\infty} \|A_\alpha^{(k)}(f^{(\epsilon_j)})\|_{p,w} \le C\|I_{-\alpha} f\|_{p,w}. \tag{3.9}$$

Conversely, we assume that $f \in L^p_w$ and $\mathcal{A}_\alpha^{(k)}(f) \in L^p_w$. Then, Minkowski's inequality and the L^p_w boundedness of M imply that

$$\|\mathcal{A}_\alpha^{(k)}(f^{(\epsilon)})\|_{p,w} \le C\|M(\mathcal{A}_\alpha^{(k)}(f))\|_{p,w} \le C\|\mathcal{A}_\alpha^{(k)}(f)\|_{p,w}. \tag{3.10}$$

Using Lemma 3.4 and (3.10), we have

$$\sup_{\epsilon\in(0,1/2)} \|I_{-\alpha}^{(\epsilon/2)} f^{(\epsilon)}\|_{p,w} \le C \sup_{\epsilon\in(0,1/2)} \|\mathcal{A}_\alpha^{(k)}(f^{(\epsilon)})\|_{p,w} \le C\|\mathcal{A}_\alpha^{(k)}(f)\|_{p,w}.$$

By compactness, we can find a sequence $\{\epsilon_j\}$, $0 < \epsilon_j < 1/2$, and a function $g \in L^p_w$ such that $\epsilon_j \to 0$,

$$\|g\|_{p,w} \le C\|\mathcal{A}_\alpha^{(k)}(f)\|_{p,w} \tag{3.11}$$

and $I_{-\alpha}^{(\epsilon_j/2)} f^{(\epsilon_j)} \to g$ weakly in L^p_w as $j \to \infty$.

Now we prove that $f = I_\alpha g$. Let $h \in \mathscr{S}_0$. From Lemma 3.2, we see that $f^{(\epsilon_j)} \to f$ in L^p_w as $j \to \infty$. Using this, we have

$$\int_{\mathbb{R}^n} f(x)h(x)\,dx = \lim_{j\to\infty} \int_{\mathbb{R}^n} f^{(\epsilon_j)}(x)h(x)\,dx = \lim_{j\to\infty}\lim_{m\to\infty} \int_{\mathbb{R}^n} f_{m,\epsilon_j}(x)h(x)\,dx$$
$$= \lim_{j\to\infty}\lim_{m\to\infty} \int_{\mathbb{R}^n} I_{-\alpha}(f_{m,\epsilon_j})(x)I_\alpha(h)(x)\,dx.$$

Thus, noting that $I_{-\alpha}(f_{m,\epsilon_j}) = I_{-\alpha}^{(\epsilon_j/2)}(f_{m,\epsilon_j})$, we see that

$$\int_{\mathbb{R}^n} f(x)h(x)\,dx = \lim_{j\to\infty}\lim_{m\to\infty} \int_{\mathbb{R}^n} I_{-\alpha}^{(\epsilon_j/2)}(f_{m,\epsilon_j})(x)I_\alpha(h)(x)\,dx$$
$$= \lim_{j\to\infty} \int_{\mathbb{R}^n} I_{-\alpha}^{(\epsilon_j/2)}(f^{(\epsilon_j)})(x)I_\alpha(h)(x)\,dx = \int_{\mathbb{R}^n} g(x)I_\alpha(h)(x)\,dx.$$

It follows that $f = I_\alpha g$ by definition. Thus (3.11) can be restated as

$$\|I_{-\alpha} f\|_{p,w} = \|g\|_{p,w} \le C\|\mathcal{A}_\alpha^{(k)}(f)\|_{p,w}. \tag{3.12}$$

By combining (3.9) and (3.12), we conclude the proof of Theorem 1.2. □

4 Proofs of Theorems 1.3 and 1.4

Let ζ be as in (2.3). In the proof of Theorem 1.1 in Section 2, we have already seen that $\zeta \in L^1(\mathbb{R}^n)$ and $\int \zeta = 0$. We observe that $U_\alpha^{(k)}(f) = \Delta_\zeta(f)$. By (2.5), we can see that ζ satisfies the conditions (1) and (3) of Theorem D. To see the condition (2) of Theorem D, we recall that

$$\hat{\zeta}(\xi) = (2\pi|\xi|)^{-\alpha}(1 - \hat{\mu}(\xi)).$$

Obviously, this implies the condition (2) of Theorem D and also the non-degeneracy condition required in Theorem D. So we can apply Theorem D to get Theorem 1.3.

Next we prove Theorem 1.4.

Lemma 4.1 *Let $w \in A_p$, $1 < p < \infty$ and $f \in L^p_w$. Let $f^{(\epsilon)}$ be as in Lemma 3.2. Let $\mathcal{B}_\alpha^{(k)}$ be as in Theorem 1.4. Then*

$$\|\mathcal{B}_\alpha^{(k)}(f^{(\epsilon)})\|_{p,w} \simeq \|I_{-\alpha}^{(\epsilon/2)} f^{(\epsilon)}\|_{p,w}, \quad 0 < \epsilon < 1/2.$$

Proof For $f \in L^p_w$, let $f_{m,\epsilon} \in \mathscr{S}_0$ be as in the proof of Lemma 3.4. Applying Theorem 1.3, we have

$$\|\mathcal{B}_\alpha^{(k)}(f_{m,\epsilon})\|_{p,w} = \|U_\alpha^{(k)}(I_{-\alpha} f_{m,\epsilon})\|_{p,w} \simeq \|I_{-\alpha}^{(\epsilon/2)} f_{m,\epsilon}\|_{p,w}. \tag{4.1}$$

Using (4.1) and arguing as in the proof of Lemma 3.4, we can prove Lemma 4.1. □

Proof of Theorem 1.4 The proof is similar to that of Theorem 1.2, so it is brief.

Suppose that $f \in W_w^{\alpha,p}$ and let $g = I_{-\alpha}(f)$. By Lemmas 4.1 and 3.5, we have

$$\|\mathcal{B}_\alpha^{(k)}(f^{(\epsilon)})\|_{p,w} \le C\|g^{(\epsilon)}\|_{p,w} \le C\|M(g)\|_{p,w} \le C\|g\|_{p,w}.$$

Thus, as in the proof of (3.9), by part (2) of Lemma 3.3, Lemma 3.2 and Fatou's lemma we see that

$$\|\mathcal{B}_\alpha^{(k)}(f)\|_{p,w} \le C\|I_{-\alpha} f\|_{p,w}. \tag{4.2}$$

Next, we assume that $f \in L^p_w$ and $\mathcal{B}^{(k)}_\alpha(f) \in L^p_w$. Using Minkowski's inequality we see that

$$\|\mathcal{B}^{(k)}_\alpha(f^{(\epsilon)})\|_{p,w} \le C\|M(\mathcal{B}^{(k)}_\alpha(f))\|_{p,w} \le C\|\mathcal{B}^{(k)}_\alpha(f)\|_{p,w}. \tag{4.3}$$

By Lemma 4.1 and (4.3), we have

$$\sup_{\epsilon\in(0,1/2)} \|I^{(\epsilon/2)}_{-\alpha} f^{(\epsilon)}\|_{p,w} \le C \sup_{\epsilon\in(0,1/2)} \|\mathcal{B}^{(k)}_\alpha(f^{(\epsilon)})\|_{p,w} \le C\|\mathcal{B}^{(k)}_\alpha(f)\|_{p,w}.$$

So, we can find a sequence $\{\epsilon_j\}$ and a function $g \in L^p_w$ such that $0 < \epsilon_j < 1/2$, $\epsilon_j \to 0$,

$$\|g\|_{p,w} \le C\|\mathcal{B}^{(k)}_\alpha(f)\|_{p,w} \tag{4.4}$$

and $I^{(\epsilon_j/2)}_{-\alpha} f^{(\epsilon_j)} \to g$ weakly in L^p_w as $j \to \infty$.

We can prove that $f = I_\alpha g$ as in the proof of Theorem 1.2. So by (4.4) we have

$$\|I_{-\alpha} f\|_{p,w} = \|g\|_{p,w} \le C\|\mathcal{B}^{(k)}_\alpha(f)\|_{p,w}. \tag{4.5}$$

We conclude the proof of Theorem 1.4 by combining (4.2) and (4.5). □

5 Characterization of $W^{1,p}_w$ by discrete parameter square functions

In [5] $\mathcal{A}_1$ is used to characterize $W^{1,p}_w$ for $1 < p < \infty$, $w \in A_p$. Here we consider $\mathcal{B}_1$ and prove a similar characterization by $\mathcal{B}_1$, where $\mathcal{B}_1 = \mathcal{B}^{(1)}_1$ (see (1.9)).

Theorem 5.1 *Let $1 < p < \infty$, $w \in A_p$ and $f \in L^p_w$. Then, $f \in W^{1,p}_w$ if and only if $\mathcal{B}_1(f) \in L^p_w$; further*

$$\|I_{-1}(f)\|_{p,w} \simeq \|\mathcal{B}_1(f)\|_{p,w}.$$

Let R_j, $1 \le j \le n$, be the Riesz transform:

$$R_j(f)(x) = \mathrm{p.v.}\, C_n \int f(x-y)\frac{y_j}{|y|^{n+1}}\,dy,$$

where $C_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$. It is known that $\mathscr{F}(R_j f)(\xi) = (-i\xi_j/|\xi|)\hat{f}(\xi)$, $f \in \mathscr{S}$.

To prove Theorem 5.1 we need the following results.

Lemma 5.2 *Let $\phi^{(j)}(x) = c_n^{-1} x_j|x|^{-n}\chi_{B(0,1)}$, $j = 1, \ldots, n$, where c_n is the surface area of S^{n-1}. Then we have the following.*

(1) $R_k(\phi^{(j)}) \in L^1(\mathbb{R}^n)$ *for* $1 \le j, k \le n$.
(2) $\mathscr{F}(R_k(\phi^{(j)}))(\xi) = (-i\xi_k/|\xi|)\mathscr{F}(\phi^{(j)})(\xi)$ *and* $\int R_k(\phi^{(j)})(x)\,dx = 0$ *for* $1 \le j, k \le n$.
(3) $\left|\mathscr{F}(\phi^{(j)})(\xi)\right| \le C\min(|\xi|^\epsilon, |\xi|^{-\epsilon})$ *for* $\xi \in \mathbb{R}^n \setminus \{0\}$ *with some* $\epsilon > 0$.
(4) $\sup_{\ell\in\mathbb{Z}} \left|\sum_{j=1}^n \mathscr{F}(R_j(\phi^{(j)}))(2^\ell\xi)\right| > 0$ *for* $\xi \in \mathbb{R}^n \setminus \{0\}$.

Proof Proof of part (1). This is valid since $\phi^{(j)}$ is essentially an atom for $H^1(\mathbb{R}^n)$ (the Hardy space) (see [4, Chap. III]). Here we give a proof for completeness. For $|x| > 2$ we have $|R_k(\phi^{(j)})(x)| \le C|x|^{-n-1}$ as follows. Since $\int \phi^{(j)} = 0$, we see that

$$
\begin{aligned}
|R_k(\phi^{(j)})(x)| &= \left| C_n \int_{|y|\le 1} \left(\frac{x_k - y_k}{|x-y|^{n+1}} - \frac{x_k}{|x|^{n+1}} \right) \phi^{(j)}(y)\, dy \right| \\
&\le C \int |x|^{-n-1} |\phi^{(j)}(y)|\, dy \le C|x|^{-n-1} \|\phi^{(j)}\|_1.
\end{aligned}
$$

Also, we note that $\phi^{(j)} \in L^p(\mathbb{R}^n)$ for $p \in (1, n/(n-1))$. Thus by Hölder's inequality and the L^p- boundedness of R_k, for $p \in (1, n/(n-1))$ we see that

$$
\int_{|x|\le 2} |R_k(\phi^{(j)})(x)|\, dx \le |B(0,2)|^{1/p'} \|R_k(\phi^{(j)})\|_p \le C\|\phi^{(j)}\|_p.
$$

Collecting results, we have $R_k(\phi^{(j)}) \in L^1(\mathbb{R}^n)$.

Proof of part (2). Let $1 < p < n/(n-1)$. We take a sequence $\{f_\ell\}_{\ell=1}^\infty$ in $\mathscr{S}$ such that $f_\ell \to \phi^{(j)}$ in L^p as $\ell \to \infty$. Since $\phi^{(j)}$ is supported on $|x| \le 1$, we may assume that $\operatorname{supp}(f_\ell) \subset B(0,2)$ and hence we also have $f_\ell \to \phi^{(j)}$ in L^1. Thus $\hat{f}_\ell(\xi) \to \mathscr{F}(\phi^{(j)})(\xi)$ for every ξ. By the L^p boundedness of R_k, it follows that $R_k(f_\ell) \to R_k(\phi^{(j)})$ in L^p. Applying the inequality of Hausdorff-Young, we see that $\mathscr{F}(R_k(f_\ell)) \to \mathscr{F}(R_k(\phi^{(j)}))$ in $L^{p'}$; also we may assume that $\mathscr{F}(R_k(f_\ell)) \to \mathscr{F}(R_k(\phi^{(j)}))$ a.e. by taking a subsequence, if necessary. Thus, for almost every ξ we have

$$
\mathscr{F}(R_k(\phi^{(j)}))(\xi) = \lim_{\ell\to\infty} \mathscr{F}(R_k(f_\ell))(\xi) = \lim_{\ell\to\infty} \frac{-i\xi_k}{|\xi|} \hat{f}_\ell(\xi) = \frac{-i\xi_k}{|\xi|} \mathscr{F}(\phi^{(j)})(\xi). \quad (5.1)
$$

Since $R_k(\phi^{(j)}) \in L^1$ by part (1), $\mathscr{F}(R_k(\phi^{(j)}))$ is continuous on $\mathbb{R}^n$. Also, $\mathscr{F}(\phi^{(j)})$ is continuous on $\mathbb{R}^n$. Thus by (5.1) $\mathscr{F}(R_k(\phi^{(j)}))(\xi) = (-i\xi_k/|\xi|)\mathscr{F}(\phi^{(j)})(\xi)$ holds for every $\xi \neq 0$.

Next, we observe that

$$
\left|\mathscr{F}(\phi^{(j)})(\xi)\right| = \left| \int \phi^{(j)}(x)(e^{-2\pi i\langle x,\xi\rangle} - 1) \right| \le C|\xi| \int |\phi^{(j)}(x)||x|\, dx \le C|\xi|. \quad (5.2)
$$

Thus

$$
\left|\mathscr{F}(R_k(\phi^{(j)}))(\xi)\right| = \left| i\frac{\xi_k}{|\xi|} \mathscr{F}(\phi^{(j)})(\xi) \right| \le C|\xi|,
$$

which implies $\mathscr{F}(R_k(\phi^{(j)}))(0) = 0$, in other words, $\int R_k(\phi^{(j)})(x)\, dx = 0$.

Proof of part (3). We write $\mathscr{F}(\phi^{(j)})$ as follows.

$$
\begin{aligned}
\mathscr{F}(\phi^{(j)})(\xi) &= \int \phi^{(j)}(x) e^{-2\pi i\langle x,\xi\rangle}\, dx = \int_0^1 \int_{S^{n-1}} \theta_j e^{-2\pi i r\langle \theta,\xi\rangle}\, d\sigma(\theta)\, dr \\
&= \int_{S^{n-1}} \theta_j \frac{(1 - e^{-2\pi i\langle\theta,\xi\rangle})}{2\pi i\langle\theta,\xi\rangle}\, d\sigma(\theta). \quad (5.3)
\end{aligned}
$$

Thus

$$
\left|\mathscr{F}(\phi^{(j)})(\xi)\right| \le \int_{S^{n-1}} |\theta_j| \pi^{-\epsilon} |\langle\theta,\xi\rangle|^{-\epsilon}\, d\sigma(\theta) = C|\xi|^{-\epsilon},
$$

where $0 < \epsilon < 1$ and C is a positive constant independent of ξ. From this and (5.2) we can deduce the inequality claimed.

Proof of part (4). We recall that

$$
\int_{S^{n-1}} e^{-2\pi i\langle\theta,\xi\rangle}\, d\sigma(\theta) = \frac{2\pi}{c_n} \frac{J_{(n-2)/2}(2\pi|\xi|)}{|\xi|^{(n-2)/2}} =: V(|\xi|),
$$

where J_β denotes the Bessel function of the first kind of order β (see [15, p.154]). It follows that

$$\int_{S^{n-1}} \theta_j e^{-2\pi i\langle\theta,\xi\rangle}\, d\sigma(\theta) = \frac{1}{-2\pi i}(\partial/\partial\xi_j)V(|\xi|) = \frac{1}{-2\pi i}\frac{\xi_j}{|\xi|}V'(|\xi|).$$

Using this in (5.3), we have

$$\mathscr{F}(\phi^{(j)})(\xi) = \frac{i}{2\pi}\frac{\xi_j}{|\xi|}\int_0^1 V'(r|\xi|)\, dr = \frac{\xi_j}{|\xi|}W(|\xi|),$$

where W is an analytic function defined by

$$W(u) = \frac{i}{2\pi}\int_0^1 V'(ru)\, dr.$$

So, using part (2), we see that

$$\left|\sum_{j=1}^n \mathscr{F}(R_j(\phi^{(j)}))(\xi)\right| = \left|\sum_{j=1}^n \frac{\xi_j^2}{|\xi|^2}W(|\xi|)\right| = |W(|\xi|)| \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}.$$

Thus, we need to show that

$$\sup_{\ell\in\mathbb{Z}} |W(|2^\ell\xi|)| > 0 \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{5.4}$$

We give a proof by contradiction. We first note that $W(0) = 0$. If there is $\xi \in \mathbb{R}^n \setminus \{0\}$ such that $W(2^{-\ell}|\xi|) = 0$ for all $\ell \in \mathbb{Z}$, then we have a sequence $\{2^{-\ell}|\xi|\}_{\ell=1}^\infty$ of distinct points such that $2^{-\ell}|\xi| \to 0$ and $W(2^{-\ell}|\xi|) = 0$ for all $\ell = 1, 2, \ldots$. This implies that the function W is identically 0 by the uniqueness of analytic continuation. Thus we have reached a contradiction, and hence we have (5.4). □

To prove Theorem 5.1, we apply the following result.

Theorem 5.3 *Suppose that* $1 < p < \infty$ *and* $w \in A_p$. *Let* $U_1 = U_1^{(1)}$, *where* $U_1^{(1)}$ *is as in* (1.10). *Then we have*

$$\|U_1(f)\|_{p,w} \simeq \|f\|_{p,w}, \quad f \in \mathscr{S}_0(\mathbb{R}^n).$$

Proof Let $f \in \mathscr{S}_0$. Then by [5, Lemma 2.1] we see that

$$I_1(f)(x) - \int_{S^{n-1}} I_1(f)(x-ty)\, d\sigma(y) = \frac{1}{c_n}\int_{B(x,t)} \left\langle \nabla I_1 f(y), \frac{x-y}{|x-y|^n}\right\rangle dy, \tag{5.5}$$

where c_n is the surface area of S^{n-1} as above and $\nabla g = (\partial_1 g, \ldots, \partial_n g)$. Let

$$\psi^{(j)}(x) = -R_j(\phi^{(j)})(x), \quad \psi(x) = \sum_{j=1}^n \psi^{(j)}(x).$$

Then by (5.5), we have

$$I_1(f)(x) - \int_{S^{n-1}} I_1(f)(x-ty)\, d\sigma(y) = t(f * \psi_t(x)).$$

Thus

$$U_1(f) = \Delta_\psi(f). \tag{5.6}$$

We note that $\Delta_{\psi^{(j)}}(f) = \Delta_{\phi^{(j)}}(R_j f)$. Using part (3) of Lemma 5.2, we can easily see that Theorem D is applicable to $\Delta_{\phi^{(j)}}$ to get its L^p_w boundedness. Thus, by Theorem D and the L^p_w boundedness of R_j, we see that

$$\begin{aligned}\|\Delta_\psi(f)\|_{p,w} &\le \sum_{j=1}^{n} \|\Delta_{\psi^{(j)}}(f)\|_{p,w} \\ &= \sum_{j=1}^{n} \|\Delta_{\phi^{(j)}}(R_j f)\|_{p,w} \le C\sum_{j=1}^{n} \|R_j f\|_{p,w} \le C\|f\|_{p,w}. \end{aligned} \tag{5.7}$$

□

To prove the reverse inequality, we apply the following result, which is essentially [11, Theorem 3.6].

Lemma 5.4 *Let $\psi \in L^1(\mathbb{R}^n)$ satisfy (1.13) and let Δ_ψ be as in (1.15). Suppose that*

$$\|\Delta_\psi(f)\|_{p,w} \le C\|f\|_{p,w}, \quad f \in \mathscr{S}_0,$$

for all $w \in A_p$ and all $p \in (1,\infty)$. Further, suppose that the function $m(\xi) = \sum_{\ell=-\infty}^{\infty} |\hat{\psi}(2^\ell\xi)|^2$ is continuous and strictly positive on $B_0 = \{1 \le |\xi| \le 2\}$. Then the reverse inequality

$$\|f\|_{p,w} \le C\|\Delta_\psi(f)\|_{p,w}, \quad f \in \mathscr{S}_0,$$

also holds for all $w \in A_p$ and all $p \in (1,\infty)$. Thus $\|f\|_{p,w} \simeq \|\Delta_\psi(f)\|_{p,w}$, $f \in \mathscr{S}_0$, for $p \in (1,\infty)$ and $w \in A_p$.

Let

$$b(\xi) = \sum_{\ell=-\infty}^{\infty} |\mathscr{F}(\psi)(2^\ell\xi)|^2 = \sum_{\ell=-\infty}^{\infty} \left| \sum_{j=1}^{n} \mathscr{F}\left(\psi^{(j)}\right)(2^\ell\xi) \right|^2.$$

Let N be a positive integer and

$$b_N(\xi) = \sum_{\ell=-N}^{N} |\mathscr{F}(\psi)(2^\ell\xi)|^2.$$

Then b_N is continuous on B_0, where B_0 is as in Lemma 5.4. By (2) and (3) of Lemma 5.2 b_N converges to b uniformly on B_0 and hence b is continuous on B_0. Also, Lemma 5.2 (4) implies that b is strictly positive on B_0. Thus, taking into account (5.7), we can apply Lemma 5.4 to Δ_ψ to get $\|f\|_{p,w} \le C\|\Delta_\psi(f)\|_{p,w}$. Recalling (5.6), we conclude the proof of Theorem 5.3. □

Proof of Theorem 5.1 The proof is similar to those of Theorems 1.2 and 1.4. We need the following.

Lemma 5.5 *Suppose that $w \in A_p$, $1 < p < \infty$ and $f \in L^p_w$. Let $f^{(\epsilon)}$ be as in Lemma 3.2. Let $\mathcal{B}_1$ be as in Theorem 5.1. Then*

$$\|\mathcal{B}_1(f^{(\epsilon)})\|_{p,w} \simeq \|I_{-1}^{(\epsilon/2)} f^{(\epsilon)}\|_{p,w}, \quad 0 < \epsilon < 1/2.$$

Proof Using Theorem 5.3, we have

$$\|\mathcal{B}_1(f_{m,\epsilon})\|_{p,w} = \|U_1(I_{-1}f_{m,\epsilon})\|_{p,w} \simeq \|I_{-1}^{(\epsilon/2)}f_{m,\epsilon}\|_{p,w}, \tag{5.8}$$

where $f \in L^p_w$ and $f_{m,\epsilon} \in \mathscr{S}_0$ is as in the proof of Lemma 3.4. We can prove Lemma 5.5 by applying (5.8) and by arguing similarly to the proofs of Lemmas 3.4 and 4.1. □

Applying Lemma 5.5, we can prove Theorem 5.1 in the same way as we have proved Theorem 1.4 by applying Lemma 4.1. □

6 Some further remarks and results

6.1 On availability of polarization techniques

In proving Theorem 1.1, if we have the inequality $\|S_\alpha^{(k)}(f)\|_{p,w} \le C\|f\|_{p,w}$, then the reverse inequality can be shown by the polarization techniques as in [5, 9] by using the identity $\|S_\alpha^{(k)}(f)\|_2 = c\|f\|_2$ (see [4, Chap. V, p. 507, 5.6 (b)]). In proving Theorems 1.3 and 5.3 we have difficulties in applying similar arguments due to absence of the corresponding L^2 equalities. So we need to apply different arguments using non-degeneracy.

6.2 Comments on S_α

In theorems of this note, we have considered square functions involving averaging over spheres $S(x,t) = \{y \in \mathbb{R}^n : |x-y| = t\}$. To define analogues in metric measure spaces of those square functions involving averaging over $S(x,t)$, we have difficulties in defining suitable measures on the spheres (boundaries of balls) in general spaces. This is not the case for square functions involving averaging over balls like the one in (1.4).

On the other hand, in relation to harmonic analysis on the Euclidean spaces, the square function $S_\alpha(f)$ in (1.6) has an interesting pointwise relation with the square functions arising from the Bochner-Riesz operators. Let

$$S_R^\beta(f)(x) = \int_{|\xi|<R} \widehat{f}(\xi)(1 - R^{-2}|\xi|^2)^\beta \, e^{2\pi i\langle x,\xi\rangle}\, d\xi$$

be the Bochner-Riesz mean of order β and let σ_β be a Littlewood-Paley operator defined as

$$\begin{aligned}\sigma_\beta(f)(x) &= \left(\int_0^\infty \left|R(\partial/\partial_R)S_R^\beta(f)(x)\right|^2 dR/R\right)^{1/2}\\ &= \left(\int_0^\infty \left|-2\beta\left(S_R^\beta(f)(x) - S_R^{\beta-1}(f)(x)\right)\right|^2 dR/R\right)^{1/2}.\end{aligned}$$

Then the following result is known (see [6, 10]).

Theorem E *Suppose that* $0 < \alpha < 2$ *and* $\beta = \alpha + \frac{n}{2}$. *Then we have*

$$\sigma_\beta(f)(x) \simeq S_\alpha(f)(x),$$

for $f \in \mathscr{S}_0(\mathbb{R}^n)$.

6.3 Discrete parameter square functions defined with repeated uses of averaging operations over balls

We can also consider a discrete parameter version of the square function in (1.4) as follows:

$$\left(\sum_{\ell=-\infty}^{\infty}\left|f(x)-\fint_{B(x,2^\ell)} f(y)\,dy\right|^2 2^{-2\ell}\right)^{1/2}.$$

Furthermore, we can consider analogues of $\mathcal{B}_\alpha^{(k)}(f)$ and $U_\alpha^{(k)}(f)$ in (1.9) and (1.10), respectively, where the averaging operation $f*\sigma_t$ is replaced by $f*\Phi_t$ with $\Phi=|B(0,1)|^{-1}\chi_{B(0,1)}$, and we can prove analogues of Theorems 1.3 and 1.4, as follows.

We define $\Lambda_t^j f(x)$, $j\ge 1$, by $\Lambda_t^j f(x)=f*\Phi_t^{(j)}(x)$, where

$$\Phi^{(1)}(x)=\Phi(x), \qquad \Phi^{(j)}(x)=\underbrace{\Phi*\cdots*\Phi}_{j}(x), \quad j\ge 2.$$

We also write $\Lambda_t f$ for $\Lambda_t^1 f$. Let I be the identity operator and for a positive integer k we consider

$$(I-\Lambda_t)^k f(x)=f(x)+\sum_{j=1}^{k}(-1)^j\binom{k}{j}\Lambda_t^j f(x).$$

For $0<\alpha<n$, let

$$\mathcal{G}_\alpha^{(k)}(f)(x)=\left(\sum_{\ell=-\infty}^{\infty}\left|(I-\Lambda_{2^\ell})^k f(x)\right|^2 2^{-2\alpha\ell}\right)^{1/2},$$

and

$$R_\alpha^{(k)}(f)(x)=\left(\sum_{\ell=-\infty}^{\infty}\left|(I-\Lambda_{2^\ell})^k I_\alpha(f)(x)\right|^2 2^{-2\alpha\ell}\right)^{1/2}.$$

We state the following results without proofs.

Theorem 6.1 *Let* $0<\alpha<\min(2k,n)$, $1<p<\infty$ *and* $w\in A_p$. *Then*

$$\|R_\alpha^{(k)}(f)\|_{p,w}\simeq\|f\|_{p,w}, \quad f\in\mathscr{S}_0(\mathbb{R}^n).$$

Theorem 6.2 *Suppose that* $1<p<\infty$, $w\in A_p$ *and* $0<\alpha<\min(2k,n)$. *Then* $f\in W_w^{\alpha,p}$ *if and only if* $f\in L_w^p$ *and* $\mathcal{G}_\alpha^{(k)}(f)\in L_w^p$; *further,*

$$\|I_{-\alpha}(f)\|_{p,w}\simeq\|\mathcal{G}_\alpha^{(k)}(f)\|_{p,w}.$$

Theorems 6.1 and 6.2 can be shown arguing similarly to the proofs of Theorems 1.3 and 1.4, respectively. Analogues of Theorems 6.1 and 6.2 for continuous parameter square functions are obtained in Theorems 4.1 and 4.2 of [12], respectively, where more general settings are considered.

Data Availability There are no data because this study is purely mathematical.

Declarations

Conflict of interest The author declares no conflicts of interest associated with this paper.

References

1. Alabern, R., Mateu, J., Verdera, J.: A new characterization of Sobolev spaces on $\mathbb{R}^n$. Math. Ann. **354**, 589–626 (2012)
2. Bourgain, J.: Averages in the plane over convex curves and maximal operators. J. Anal. Math. **47**, 69–85 (1986)
3. Folland, G.B.: Real Analysis Modern Techniques and Their Applications. A Wiley-Interscience Publication (1984)
4. Garcia-Cuerva, J., Rubio de Francia, J.L.: Weighted Norm Inequalities and Related Topics. North-Holland, Amsterdam (1985)
5. Hajłasz, P., Liu, Z.: A Marcinkiewicz integral type characterization of the Sobolev space. Publ. Mat. **61**, 83–104 (2017)
6. Kaneko, M., Sunouchi, G.: On the Littlewood-Paley and Marcinkiewicz functions in higher dimensions. Tôhoku Math. J. **37**, 343–365 (1985)
7. Marcinkiewicz, J.: Sur quelques intégrales du type de Dini. Ann. Soc. Polonaise **17**, 42–50 (1938)
8. Sato, S.: Remarks on square functions in the Littlewood-Paley theory. Bull. Aust. Math. Soc. **58**, 199–211 (1998)
9. Sato, S.: Littlewood-Paley operators and Sobolev spaces. Illinois J. Math. **58**, 1025–1039 (2014)
10. Sato, S.: Spherical square functions of Marcinkiewicz type with Riesz potentials. Arch. Math. **108**, 415–426 (2017)
11. Sato, S.: Littlewood-Paley equivalence and homogeneous Fourier multipliers. Integr. Equ. Oper. Theory **87**, 15–44 (2017)
12. Sato, S.: *Sobolev spaces with non-isotropic dilations and square functions of Marcinkiewicz type*, arXiv:2106.12216 [math.FA] (to appear in Studia Math.)
13. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton Univ. Press, Princeton (1970)
14. Stein, E.M., Wainger, S.: Problems in harmonic analysis related to curvature. Bull. Am. Math. Soc. **84**, 1239–1295 (1978)
15. Stein, E.M., Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces. Princeton Univ, Press, Princeton (1971)
16. Waterman, D.: On an integral of Marcinkiewicz. Trans. Am. Math. Soc. **91**, 129–138 (1959)
17. Zygmund, A.: On certain integrals. Trans. Am. Math. Soc. **58**, 170–204 (1944)