PALEY'S INEQUALITY FOR THE JACOBI EXPANSIONS

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Dedicated to Professor Kôzô Yabuta on his sixtieth birthday

ABSTRACT

Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in the unit disc satisfying

$$\sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})|\, d\theta < \infty.$$

Then $\left(\sum_{k=1}^{\infty} |a_{2^k}|^2\right)^{1/2} < \infty$, which is familiar as Paley's inequality. In this paper, an analogue of this inequality with respect to the Jacobi expansions is established.

1. *Introduction*

The classical Paley inequality [4] says that if $F(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to $H^1(\mathbb{D})$, then

$$\left(\sum_{k=1}^{\infty} |a_{2^k}|^2\right)^{1/2} \leqslant C\|F\|_{H^1},$$

where $H^1(\mathbb{D})$ is the Hardy space on the unit disc $\mathbb{D}$ which consists of the analytic functions $F(z)$ on $\mathbb{D}$ satisfying

$$\|F\|_{H^1} = \sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})|\, d\theta < \infty.$$

Let $\Re H^1$ be the real Hardy space consisting of the boundary functions $f(\theta) = \lim_{r\to 1} \Re F(re^{i\theta})$ of $F \in H^1(\mathbb{D})$ and $\|f\|_{\Re H^1} = \|F\|_{H^1}$ with real $F(0)$. Then, we restate the Paley inequality: If $f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$ is in $\Re H^1$, then

$$\left\{\sum_{k=1}^{\infty} \left(|c_{2^k}|^2 + |c_{-2^k}|^2\right)\right\}^{1/2} \leqslant C\|f\|_{\Re H^1}.$$

The aim of this paper is to establish an analogue of this inequality with respect to the Jacobi expansions.

Let $R_n^{(\alpha,\beta)}(\theta)$ be the Jacobi functions defined by

$$R_n^{(\alpha,\beta)}(\theta) = t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta) \left(\sin\frac{\theta}{2}\right)^{\alpha+1/2} \left(\cos\frac{\theta}{2}\right)^{\beta+1/2},$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree n and of order $\alpha, \beta > -1$, and

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$t_n^{(\alpha,\beta)}$ is the normalization coefficient; that is,

$$t_n^{(\alpha,\beta)} = \left(\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \right)^{1/2}.$$

The system $\{R_n^{(\alpha,\beta)}(\theta)\}_{n=0}^{\infty}$ is complete and orthonormal in $L^2(0,\pi)$ with respect to the ordinary Lebesgue measure $d\theta$. For a function $f(\theta)$ on $(0,\pi)$, we have the Jacobi expansion

$$f(\theta) \sim \sum_{n=0}^{\infty} c_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\theta), \qquad c_n^{(\alpha,\beta)} = \int_0^{\pi} f(\theta) R_n^{(\alpha,\beta)}(\theta)\, d\theta.$$

When $(\alpha,\beta) = (-1/2,-1/2)$ and $(\alpha,\beta) = (1/2,1/2)$, the Jacobi expansions are the cosine and sine expansions, respectively:

$$f(\theta) \sim \frac{c_0^{(-1/2,-1/2)}}{\sqrt{\pi}} + \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} c_n^{(-1/2,-1/2)} \cos n\theta,$$

$$c_n^{(-1/2,-1/2)} = \begin{cases} \frac{1}{\sqrt{\pi}} \int_0^{\pi} f(\theta)\, d\theta, & n = 0, \\ \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(\theta) \cos n\theta\, d\theta, & n = 1,2,\dots; \end{cases}$$

$$f(\theta) \sim \sum_{n=0}^{\infty} c_n^{(1/2,1/2)} \sin(n+1)\theta, \tag{1}$$

$$c_n^{(1/2,1/2)} = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(\theta) \sin(n+1)\theta\, d\theta, \quad n = 0,1,\dots.$$

The Jacobi polynomials are explicitly represented in the form

$$P_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \binom{n+\beta}{j} \left(\frac{x-1}{2}\right)^j \left(\frac{x+1}{2}\right)^{n-j},$$

where $\binom{a}{j} = a(a-1)\dots(a-j+1)/j!$. Also, the polynomials are given by Rodrigues' formula:

$$(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} \left(\frac{d}{dx}\right)^n \{(1-x)^{n+\alpha}(1+x)^{n+\beta}\}.$$

We refer to the work of Szegö [5] for the Jacobi polynomials.

Let $H^1(0,\pi)$ be the space defined by

$$H^1(0,\pi) = \left\{ h|_{(0,\pi)} : h \in \Re H^1, \text{ even} \right\}.$$

We endow the space $H^1(0,\pi)$ with the norm $\|f\|_{H^1(0,\pi)} = \|h\|_{\Re H^1}$, where $f = h|_{(0,\pi)}$.

Our theorem is as follows.

THEOREM 1. *Let $\{n_k\}_{k=1}^{\infty}$ be a Hadamard sequence; that is, $n_{k+1}/n_k \geqslant \rho > 1$, where $k = 1,2,\dots$. Let $\alpha, \beta \geqslant -1/2$. Then, the Jacobi coefficients $c_{n_k}^{(\alpha,\beta)}$ of a function $f \in H^1(0,\pi)$ satisfy*

$$\left(\sum_{k=1}^{\infty} |c_{n_k}^{(\alpha,\beta)}|^2 \right)^{1/2} \leqslant C \|f\|_{H^1(0,\pi)}. \tag{2}$$

The proof of Paley's inequality by the real method (see the book by Torchinsky [6, Chapter XV, 4.3]) inspired us to obtain an inequality of Paley type for the Jacobi expansions. In our proof, the (H^1, BMO)-duality identified by C. Fefferman will play an essential role. The theorem will be proved in the next section. We add here, two further remarks.

REMARK 1. The theorem with the space $L^1(0,\pi)$ instead of $H^1(0,\pi)$ does not hold; that is, there exists a function $f \in L^1(0,\pi)$ such that

$$\sum_{k=1}^{\infty} |c_{n_k}^{(\alpha,\beta)}|^2 = \infty.$$

We shall give its proof here only for the case $\alpha = \beta = 1/2$; that is, the sine series (1). Although a further argument is required, other cases can be proved similarly. Suppose that $\sum_{k=1}^{\infty} |c_{n_k}^{(1/2,1/2)}|^2 < \infty$ for all $f \in L^1(0,\pi)$. Then, by the closed graph theorem, we have $\sum_{k=1}^{\infty} |c_{n_k}^{(1/2,1/2)}|^2 \leqslant C\|f\|_{L^1}^2$. Let $f_j(\theta) = j\chi_{(\theta_0-1/(2j),\theta_0+1/(2j))}(\theta)$, for $j = 1, 2, \ldots$, where $\chi_I(\theta)$ is the characteristic function of an interval I. Then, $\|f_j\|_{L^1} = 1$ and the coefficients $c_{n,j}^{(1/2,1/2)}$ of f_j satisfy $c_{n,j}^{(1/2,1/2)} \to (2/\pi)^{1/2}\sin(n+1)\theta_0$ as $j \to \infty$. Therefore, we have

$$C \geqslant \liminf_{j\to\infty} \sum_{k=0}^{\infty} \left|c_{n_k,j}^{(1/2,1/2)}\right|^2 \geqslant \frac{2}{\pi}\sum_{k=0}^{\infty} \sin^2(n_k+1)\theta_0.$$

On the other hand, there exists $\theta_0 \in (0,\pi)$ such that the set of points $\{(n_k+1)\theta_0\}_k$ is dense in $(0,\pi)$ (see [**2**, Theorem 1.40]), which leads to a contradiction.

REMARK 2. We note that $H^1(0,\pi) = \{h \in \Re H^1 : \operatorname{supp} h \subset [0,\pi]\}$, which follows from the argument of [**1**, p. 608, the last line, to p. 609, line 9].

2. *Proof of the theorem*

We shall prove the theorem by using Lemmas 1 and 2, which will be stated and proved in the next section. Since $|c_0^{(\alpha,\beta)}| = |\int_0^\pi f(\theta)\,d\theta| \leqslant \|f\|_{H^1(0,\pi)}$, it is enough to show inequality (2) for a function f in $H^1(0,\pi)$ with Jacobi expansion $f(\theta) \sim \sum_{n=1}^{\infty} c_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\theta)$. Let $\{r_k\}_{k=1}^{\infty}$ be a sequence such that $\sum_{k=1}^{\infty} |r_k|^2 < \infty$. Let $g_N(\theta)$, where $N = 1, 2, \ldots$, be functions defined by

$$g_N(\theta) = \sum_{k=1}^{N} r_k R_{n_k}^{(\alpha,\beta)}(\theta).$$

We extend the functions $f(\theta)$ and $g_N(\theta)$, for $N = 1, 2, \ldots$, to the outside of $(0,\pi)$ as 2π-periodic even functions. We may denote them by the same notations without confusion. By the (H^1, BMO)-duality, we have

$$\left|\int_{-\pi}^{\pi} f(\theta) g_N(\theta)\,d\theta\right| \leqslant C\|g_N\|_* \|f\|_{\Re H^1},$$

where $\|h\|_* = \sup_I (1/|I|)\int_I |h(\theta) - h_I|\,d\theta$, the sup being taken over all intervals I of the real line $\mathbb{R}$, where $h_I = (1/|I|)\int_I h(\theta)\,d\theta$, and where $|I|$ is the length of I (see [**3**, Chapter X] and [**1**]). Since

$$\int_{-\pi}^{\pi} f(\theta) g_N(\theta)\,d\theta = 2\int_0^{\pi} f(\theta) g_N(\theta)\,d\theta = 2\sum_{k=1}^{N} r_k c_{n_k}^{(\alpha,\beta)},$$

if we show that

$$\|g_N\|_* \leqslant C \left(\sum_{k=1}^{\infty} |r_k|^2 \right)^{1/2} \tag{3}$$

with a constant C independent of N and a sequence $\{r_k\}_{k=1}^{\infty}$, then

$$\left| \sum_{k=1}^{N} r_k c_{n_k}^{(\alpha,\beta)} \right| \leqslant C \left(\sum_{k=1}^{\infty} |r_k|^2 \right)^{1/2} \|f\|_{\Re H^1},$$

which implies that

$$\left(\sum_{k=1}^{N} |c_{n_k}^{(\alpha,\beta)}|^2 \right)^{1/2} \leqslant C \|f\|_{\Re H^1}.$$

Thus, letting $N \to \infty$, we get the desired inequality (2).

We now prove the inequality (3). Our task is to show that for every finite interval I with $|I| \leqslant \pi$, there exists a constant c_I such that

$$\frac{1}{|I|} \int_I |g_N(\theta) - c_I| \, d\theta \leqslant C \left(\sum_{k=1}^{\infty} |r_k|^2 \right)^{1/2}, \tag{4}$$

where C is independent of I, N and a sequence $\{r_k\}_{k=1}^{\infty}$ (see [3, Chapter X, 1°]). Further, we may assume that $I \subset [0, \pi]$ since $g_N(\theta)$ is even and 2π-periodic. Indeed, let $I = I_1 \cup I_2 = [-a, 0] \cup [0, b]$ $(0 < a \leqslant b \leqslant \pi)$, for example. Then

$$\begin{aligned} \frac{1}{|I|} \int_I |g_N(\theta) - c_{I_2}| \, d\theta &= \frac{1}{|I|} \left(\int_0^a |g_N(\theta) - c_{I_2}| \, d\theta + \int_0^b |g_N(\theta) - c_{I_2}| \, d\theta \right) \\ &\leqslant \frac{2}{|I_2|} \int_{I_2} |g_N(\theta) - c_{I_2}| \, d\theta. \end{aligned}$$

The other cases are similar.

We put $I = [\theta_0, \theta_1] \subset [0, \pi]$. If $|I| > 1/n_1$, then

$$\begin{aligned} \frac{1}{|I|} \int_I |g_N(\theta)| \, d\theta &\leqslant \left(\frac{1}{|I|} \int_I |g_N(\theta)|^2 \, d\theta \right)^{1/2} \\ &\leqslant \left(\frac{1}{|I|} \int_0^{\pi} |g_N(\theta)|^2 \, d\theta \right)^{1/2} \leqslant n_1^{1/2} \left(\sum_{k=1}^{\infty} |r_k|^2 \right)^{1/2}. \end{aligned}$$

Thus, it is enough to treat the case where there exists a positive integer M, such that $1/n_{M+1} < |I| \leqslant 1/n_M$. We shall show inequality (4) with $c_I = g_M(\theta_0)$. We write

$$\begin{aligned} g_N(\theta) &= g_M(\theta) + \sum_{k=M+1}^{N} r_k R_{n_k}^{(\alpha,\beta)}(\theta) \\ &= g_M(\theta) + E_{M,N}(\theta), \quad \text{say.} \end{aligned}$$

It follows that

$$\frac{1}{|I|} \int_I |g_N(\theta) - g_M(\theta_0)| \, d\theta \leqslant \frac{1}{|I|} \int_I |g_M(\theta) - g_M(\theta_0)| \, d\theta + \frac{1}{|I|} \int_I |E_{M,N}(\theta)| \, d\theta. \tag{5}$$

We consider the first term on the right-hand side. By Schwarz's inequality and inequality (16) of Lemma 1 in Section 3, we have

$$|g_M(\theta)-g_M(\theta_0)|^2 \leqslant \sum_{k=1}^{M}|r_k|^2\sum_{k=1}^{M}\left|R_{n_k}^{(\alpha,\beta)}(\theta)-R_{n_k}^{(\alpha,\beta)}(\theta_0)\right|^2$$
$$\leqslant C\sum_{k=1}^{M}|r_k|^2\sum_{k=1}^{M}\left(n_k^{\delta}|\theta-\theta_0|^{\delta}\right)^2,$$

where $\delta=\min\{\alpha+1/2,\beta+1/2\}$ if $0<\alpha+1/2<1$ or $0<\beta+1/2<1$, and $\delta=1$ otherwise. The second sum on the right-hand side is bounded by $|I|^{2\delta}\sum_{k=1}^{M}n_k^{2\delta}$ for $\theta\in I$, which is dominated by $C|I|^{2\delta}n_M^{2\delta}$ since the sequence $\{n_k\}$ is a Hadamard sequence with $n_{k+1}/n_k\geqslant\rho>1$, where C depends only on ρ. It follows from $|I|n_M\leqslant 1$ that $|g_M(\theta)-g_M(\theta_0)|^2\leqslant C\sum_{k=1}^{M}|r_k|^2$. Thus, we have

$$\frac{1}{|I|}\int_I|g_M(\theta)-g_M(\theta_0)|\,d\theta \leqslant \left(\frac{1}{|I|}\int_I|g_M(\theta)-g_M(\theta_0)|^2\,d\theta\right)^{1/2}$$
$$\leqslant C\left(\sum_{k=1}^{M}|r_k|^2\right)^{1/2}. \tag{6}$$

We next evaluate the second term, $(1/|I|)\int_I|E_{M,N}(\theta)|\,d\theta$, on the right-hand side of inequality (5). We have

$$\left(\frac{1}{|I|}\int_I|E_{M,N}(\theta)|\,d\theta\right)^2\leqslant\frac{1}{|I|}\int_I|E_{M,N}(\theta)|^2\,d\theta\leqslant\sum_{k,j=M+1}^{N}|r_k||r_j|U_{k,j},$$

where

$$U_{k,j}=\frac{1}{|I|}\left|\int_I R_{n_k}^{(\alpha,\beta)}(\theta)R_{n_j}^{(\alpha,\beta)}(\theta)\,d\theta\right|.$$

We shall show that there exist positive constants C and $0<\gamma<1$, such that

$$U_{k,j}\leqslant C\gamma^{|k-j|} \tag{7}$$

for $M+1\leqslant k\leqslant N$ and $M+1\leqslant j\leqslant N$. Once this is done, we shall obtain

$$\sum_{k,j}|r_k||r_j|U_{k,j}\leqslant C\sum_{k=M+1}^{N}|r_k|^2,$$

and thus

$$\frac{1}{|I|}\int_I|E_{M,N}(\theta)|\,d\theta\leqslant C\left(\sum_{k=M+1}^{N}|r_k|^2\right)^{1/2}. \tag{8}$$

We may assume, without loss of generality, that $n_j\leqslant n_k$. By inequality (20) of Lemma 2 in Section 3, we have

$$U_{k,j}\leqslant C\left\{\left(\frac{n_j}{n_k}\right)^{\delta}+\frac{\log^+(n_k|I|)}{n_k|I|}+\frac{1}{n_k|I|}\right\}.$$

The first term on the right-hand side is bounded by $(1/\rho^{\delta})^{k-j}$. For the second term, we fix an arbitrary positive number ν satisfying $0<\nu<1$. Then

$$\frac{\log^+ n_k|I|}{|I|n_k}\leqslant C_\nu\left(\frac{1}{|I|n_j}\frac{n_j}{n_k}\right)^{\nu}\leqslant C_\nu\left(\frac{1}{\rho^{\nu}}\right)^{k-j}$$

since $|I|n_j > 1$, for $j = M+1, M+2, \ldots$, where C_ν depends only on ν. The last term $1/(|I|n_k)$ is bounded by $(1/\rho)^{k-j}$. The inequality (7) is proved. Therefore, we get inequality (8), which with inequality (6) completes the proof of the theorem.

3. *Lemmas*

In this section, we shall state the lemmas used in the proof of the theorem, and give their proofs. We need some properties of the Jacobi polynomial:

$$t_n^{(\alpha,\beta)} = O(n^{1/2}), \tag{9}$$

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x) \tag{10}$$

for $\alpha, \beta > -1$, and

$$\left|P_n^{(\alpha,\beta)}(\cos\theta)\right| \leqslant O(n^\alpha), \tag{11}$$

$$\left|R_n^{(\alpha,\beta)}(\theta)\right| \leqslant C_{\alpha,\beta} \tag{12}$$

for $0 < \theta \leqslant \pi/2$, $\alpha \geqslant -1/2$. See [5, (7.32.6)]. The polynomial $P_n^{(\alpha,\beta)}(x)$ satisfies

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{13}$$

See [5, (4.21.7)]. We use the following estimate [5, (8.21.17), (8.21.18)]:

$$\left(\sin\frac{\theta}{2}\right)^\alpha \left(\cos\frac{\theta}{2}\right)^\beta P_n^{(\alpha,\beta)}(\theta) = N^{-\alpha}\frac{\Gamma(n+\alpha+1)}{n!}\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_\alpha(N\theta)$$
$$+\begin{cases} \theta^{1/2}\, O(n^{-3/2}) & \text{if } cn^{-1} \leqslant \theta \leqslant \pi - \epsilon, \\ \theta^{\alpha+2}\, O(n^\alpha) & \text{if } 0 < \theta \leqslant cn^{-1}, \end{cases} \tag{14}$$

$$\pi^{1/2} n^{1/2} P_n^{(\alpha,\beta)}(\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2} = \cos(N\theta+\gamma) + \frac{O(1)}{n\sin\theta}, \tag{15}$$

if $c/n \leqslant \theta \leqslant \pi - c/n$, $\alpha, \beta > -1$, where

$$N = n + (\alpha+\beta+1)/2, \qquad \gamma = -\alpha\pi/2 - \pi/4,$$

and c and ϵ are fixed positive numbers.

The first lemma gives the order of the Lipschitz continuity of $R_n^{\alpha,\beta}(\theta)$ with respect to n.

LEMMA 1. *Let $\alpha, \beta \geqslant -1/2$. Then there exists a constant C such that*

$$|R_n^{(\alpha,\beta)}(\theta_1) - R_n^{(\alpha,\beta)}(\theta_2)| \leqslant Cn^\delta |\theta_1 - \theta_2|^\delta \tag{16}$$

for $0 \leqslant \theta_1 < \theta_2 \leqslant \pi$, where $\delta = \min\{\alpha+1/2, \beta+1/2\}$ if $0 < \alpha+1/2 < 1$ or $0 < \beta+1/2 < 1$, and $\delta = 1$ otherwise, and C is independent of θ_1, θ_2 and n.

Proof. Since $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$, it follows that

$$R_n^{(\alpha,\beta)}(\pi - \theta) = (-1)^n R_n^{(\beta,\alpha)}(\theta).$$

Using this, we have

$$R_n^{(\alpha,\beta)}(\theta_1) - R_n^{(\alpha,\beta)}(\theta_2) = (-1)^n \left(R_n^{(\beta,\alpha)}(\pi-\theta_1) - R_n^{(\beta,\alpha)}(\pi-\theta_2)\right) \tag{17}$$

where $\pi/2 \leqslant \theta_1 < \theta_2 \leqslant \pi$), and

$$\begin{aligned} &R_n^{(\alpha,\beta)}(\theta_1) - R_n^{(\alpha,\beta)}(\theta_2) \\ &\quad = \left(R_n^{(\alpha,\beta)}(\theta_1) - R_n^{(\alpha,\beta)}(\pi/2)\right) + (-1)^n \left(R_n^{(\beta,\alpha)}(\pi/2) - R_n^{(\beta,\alpha)}(\pi - \theta_2)\right) \end{aligned} \tag{18}$$

where $0 \leqslant \theta_1 \leqslant \pi/2 < \theta_2 \leqslant \pi$.

Thus it is enough to show inequality (16) for $0 \leqslant \theta_1 < \theta_2 \leqslant \pi/2$.

We first treat the case $\alpha + 1/2 \geqslant 1$ or $\alpha + 1/2 = 0$. It is enough to show that $|(d/d\theta)R_n^{(\beta,\alpha)}(\theta)| \leqslant Cn$ for $0 \leqslant \theta \leqslant \pi/2$, where C is independent of θ and n. We write $(d/d\theta)R_n^{(\alpha,\beta)}(\theta) = V_1 + V_2 + V_3$, where

$$V_1 = -(n+\alpha+\beta+1)t_n^{(\alpha,\beta)}P_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+3/2}\left(\cos\frac{\theta}{2}\right)^{\beta+3/2},$$

$$V_2 = \frac{1}{2}\left(\alpha+\frac{1}{2}\right)t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+3/2},$$

$$V_3 = -\frac{1}{2}\left(\beta+\frac{1}{2}\right)t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+3/2}\left(\cos\frac{\theta}{2}\right)^{\beta-1/2}.$$

It follows from inequality (12) and $0 \leqslant \theta \leqslant \pi/2$, that $|V_3| \leqslant C$ with C independent of n and θ. For V_1, we have

$$V_1 = -(n+\alpha+\beta+1)\frac{t_n^{(\alpha,\beta)}}{t_{n-1}^{(\alpha+1,\beta+1)}}R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta),$$

and $(n+\alpha+\beta+1)t_n^{(\alpha,\beta)}/t_{n-1}^{(\alpha+1,\beta+1)} = O(n)$. Thus, by inequality (12), we have $|V_1| \leqslant Cn$. If $\alpha + 1/2 = 0$, then the term V_2 does not appear. Thus we have inequality (16) in this case. Let $\alpha + 1/2 \geqslant 1$. For V_2, we have $|V_2| \leqslant C\theta^{-1}|R_n^{(\alpha,\beta)}(\theta)|$. Since $J_\alpha(z) \sim z^\alpha$ (for $z \to 0$) and $J_\alpha(z) = O(z^{-1/2})$ (for $z \to \infty$) (see [5, (1.71.10), (1.71.11)]), it follows from inequality (14) that

$$V_2 = \begin{cases} O(n) + O(n^{-2}) & \text{if } n^{-1} \leqslant \theta \leqslant \pi/2, \\ O(n^{\alpha+1/2}\theta^{\alpha-1/2}) + \theta^{\alpha+3/2}O(n^{\alpha+1/2}) & \text{if } 0 < \theta \leqslant n^{-1}. \end{cases} \tag{19}$$

The case $n^{-1} \leqslant \theta \leqslant \pi/2$ is a precise estimate. For the case $0 < \theta \leqslant n^{-1}$, we have $O(n^{\alpha+1/2}\theta^{\alpha-1/2}) = O(n(n\theta)^{\alpha-1/2}) = O(n)$ since $\alpha - 1/2 \geqslant 0$. The error term satisfies $\theta^{\alpha+3/2}O(n^{\alpha+1/2}) = n^{-1}O((n\theta)^{\alpha+1/2}) = O(n^{-1})$. We get the desired estimate (16) for the case $\alpha + 1/2 \geqslant 1$.

Let $0 < \alpha + 1/2 < 1$. We here put

$$\begin{aligned} R_n^{(\alpha,\beta)}(\theta) &= t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2} \\ &= tP(\theta)s(\theta)c(\theta) = R(\theta) \end{aligned}$$

for simplicity. Let $0 \leqslant \theta_1 < \theta_2 \leqslant \pi/2$, and write

$$\begin{aligned} R(\theta_1) - R(\theta_2) &= t(P(\theta_1) - P(\theta_2))s(\theta_1)c(\theta_1) \\ &\quad + tP(\theta_2)(s(\theta_1) - s(\theta_2))c(\theta_1) + tP(\theta_2)s(\theta_2)(c(\theta_1) - c(\theta_2)) \\ &= W_1 + W_2 + W_3, \qquad \text{say.} \end{aligned}$$

For W_1 and W_3, the mean-value theorem allows us to follow the same proofs as for V_1 and V_3, respectively. We deal with the term W_2. By inequality (14) and the fact that $(\sin\theta/2)^{\alpha+1/2} \in \mathrm{Lip}_{\alpha+1/2}$, the space of continuous functions with order $\alpha+1/2$ of Lipschitz continuity, we have

$$W_2 = \begin{cases} \{O\left(n^{1/2}\theta_2^{-\alpha}(n\theta_2)^{-1/2}\right) + O\left(n^{1/2}\theta_2^{-\alpha}\theta_2^{1/2}n^{-3/2}\right)\}|\theta_1-\theta_2|^{\alpha+1/2} \\ \{O\left(n^{1/2}\theta_2^{-\alpha}(n\theta_2)^{\alpha}\right) + O\left(n^{1/2}\theta_2^{-\alpha}\theta_2^{\alpha+2}n^{\alpha}\right)\}|\theta_1-\theta_2|^{\alpha+1/2} \end{cases}$$

$$= \begin{cases} \{O\left(\theta_2^{-\alpha-1/2}\right) + O\left((n\theta_2)^{-1}\theta_2^{-\alpha-1/2}\theta_2^2\right)\}|\theta_1-\theta_2|^{\alpha+1/2} & \text{if } n^{-1} \leqslant \theta_2 \leqslant \pi/2, \\ \{O\left(n^{\alpha+1/2}\right) + O\left(n^{\alpha+1/2}\theta_2^2\right)\}|\theta_1-\theta_2|^{\alpha+1/2} & \text{if } 0 < \theta_2 \leqslant n^{-1}, \end{cases}$$

$$= O\left(n^{\alpha+1/2}\right)|\theta_1-\theta_2|^{\alpha+1/2},$$

which completes the proof of Lemma 1.

LEMMA 2. *Let $\alpha,\beta \geqslant -1/2$. Let $I=[\theta_0,\theta_1]$ be an interval with $0\leqslant\theta_0<\theta_1\leqslant\pi$. Then there exists a constant C independent of k, j and I, such that*

$$\left|\int_{\theta_0}^{\theta_1} R_k^{(\alpha,\beta)}(\theta)R_j^{(\alpha,\beta)}(\theta)\,d\theta\right| \leqslant C\left\{(\theta_1-\theta_0)\left(\frac{j}{k}\right)^{\delta} + \frac{\log^+ k(\theta_1-\theta_0)}{k} + \frac{1}{k}\right\} \tag{20}$$

for $j\leqslant k$, where $\delta=\min\{\alpha+1/2,\beta+1/2\}$ if $0<\alpha+1/2<1$ or $0<\beta+1/2<1$, and $\delta=1$ otherwise. The notation '$\log^+ u$' means that $\log^+ u=\log u$ for $1\leqslant u$ and $\log^+ u = 0$ for $u<1$.

Proof. We see that it is enough to show inequality (20) for the case $I\subset[0,\pi/2]$. Let M be the greatest integer satisfying $2\pi M/K\leqslant\theta_1-\theta_0$, where $K=k+(\alpha+\beta+1)/2$, and let $\xi_m=\theta_0+2\pi m/K$, for $m=0,1,2,\ldots,M$, and $\xi_{M+1}=\theta_1$. We have

$$\begin{aligned}\int_{\theta_0}^{\theta_1} R_k^{(\alpha,\beta)}(\theta)R_j^{(\alpha,\beta)}(\theta)\,d\theta &= \sum_{m=0}^{M}\left\{\int_{\xi_m}^{\xi_{m+1}}\left(R_j^{(\alpha,\beta)}(\theta)-R_j^{(\alpha,\beta)}(\xi_m)\right)R_k^{(\alpha,\beta)}(\theta)\,d\theta\right.\\ &\qquad\left. + \; R_j^{(\alpha,\beta)}(\xi_m)\int_{\xi_m}^{\xi_{m+1}} R_k^{(\alpha,\beta)}(\theta)\,d\theta\right\}\\ &= \sum_{m=0}^{M}\{X_m^{(1)}+X_m^{(2)}\}, \quad \text{say.}\end{aligned}$$

For $X_m^{(1)}$, applying Lemma 1 and inequality (12), we have

$$|X_m^{(1)}| \leqslant Cj^{\delta}\int_{\xi_m}^{\xi_{m+1}}|\theta-\xi_m|^{\delta}\,d\theta \leqslant C\left(\frac{2\pi j}{K}\right)^{\delta}(\xi_{m+1}-\xi_m),$$

which leads to

$$\sum_{m=0}^{M}|X_m^{(1)}| \leqslant C\left(\frac{2\pi j}{K}\right)^{\delta}(\theta_1-\theta_0). \tag{21}$$

For $X_0^{(2)}$ and $X_M^{(2)}$, we apply inequality (12). It follows that

$$|X_p^{(2)}| \leqslant C\int_{\xi_m}^{\xi_{m+1}} d\theta \leqslant C\frac{2\pi}{K}, \quad \text{for } p=0,M. \tag{22}$$

For $X_1^{(2)},\ldots,X_{M-1}^{(2)}$, we use inequalities (12) and (15), together with the fact that

$\int_{\xi_m}^{\xi_{m+1}} \cos(K\theta + \gamma)\, d\theta = 0$. We have

$$\begin{aligned} |X_m^{(2)}| &\leqslant C \left| \int_{\xi_m}^{\xi_{m+1}} \left(\cos(K\theta+\gamma) + \frac{O(1)}{k\theta} \right) d\theta \right| \\ &\leqslant Ck^{-1} \int_{\xi_m}^{\xi_{m+1}} \frac{1}{\theta}\, d\theta = Ck^{-1} \left(\log \xi_{m+1} - \log \xi_m\right), \end{aligned}$$

which implies that

$$\begin{aligned} \sum_{m=1}^{M-1} |X_m^{(2)}| &\leqslant Ck^{-1}(\log \xi_M - \log \xi_1) \leqslant Ck^{-1} \log M \\ &\leqslant Ck^{-1} \log^+ \frac{K}{2\pi}(\theta_1 - \theta_0) \leqslant Ck^{-1}\left(\log^+ k(\theta_1 - \theta_0) + 1\right). \end{aligned} \tag{23}$$

Therefore, by inequalities (21), (22) and (23), we have inequality (20), which completes the proof of the lemma.

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