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# **Quantum characteristics of stimulated Cerenkov radiation in dielectric-lined waveguide operating at optical wavelengths**

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In this paper, a quantum mechanical model is proposed to describe the basic features of stimulated Cerenkov radiation in the small-signal gain regime. In this model, the electron is described by a wave packet with finite spreading length and the electron wave function is a solution of the Schrödinger equation. We show that the quantum effects are manifested when the spreading length of the electron wave is much longer than the electromagnetic (EM) wavelength such as in the optical wavelength range. The effect of electron relaxation due to Coulomb's collisions with neighboring electrons is introduced to characterize the damping of the vibration of the electron wave with time. When the relaxation effect is neglected, we prove that our essential results matches with other classical and quantum approaches based on different theoretical concepts.

Keywords: Cerenkov laser, free-electron laser, small-signal gain.

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## 1. Introduction

In the Cerenkov free-electron laser (CFEL), an electron can exchange energy with an electromagnetic (EM) wave in a homogeneous medium represented by a refractive index  $n(\omega)$ . To realize the Cerenkov laser, various interaction schemes have been proposed as with ultrarelativistic electron beams in a gaseous medium as well as with moderately relativistic beams in the dielectric waveguide [1-8]. In the CFEL with a dielectric-lined slab waveguide, the electron beam passes over the surface of a dielectric waveguide in the vacuum. When the electron beam velocity nearly equals the phase velocity of the guided wave of the waveguide, the electron beam energy is transferred to the EM wave. The usage of dielectric waveguide made of high refractive index material as a slow-wave structure enables the achievement of electron-EM wave synchronism by mildly to moderately relativistic electron beam.

A Cerenkov laser utilizes the leaked radiation that is evanescent in the direction normal to the electrons propagation direction for interaction with the electron beam. As the wavelength of the synchronized EM wave decreases, due to the evanescent nature of the leaked surface wave, the electron beam trajectory must be parallel to the waveguide surface in the region of close proximity to enable an efficient operation. By the help of high-brightness field emission electrons sources, the Cerenkov laser in an infrared [9-12] or even in an optical wavelength range [13-15] (micro-Cerenkov FEL) is quite possible.

The quantum analysis of the stimulated Cerenkov radiation was first considered by V. L. Ginzburg [16] basing on the energy and momentum conservations laws during the interaction of classical electrons and the EM wave in a medium. In previous quantum-based treatments [17,18], for a relativistic electron beam, the wave function of a quantum-mechanical electron is expanded in terms of infinite plane waves, and the motion of the electron is determined by the solution of the Klien-Gordon equation in the presence of the existing EM field. M. Yamada [19] has been analyzed the dynamics of each electron on the basis of the density matrix method derived from the Schrödinger equation, whereas an electron is expressed as a spatial spreading plane wave with finite spreading length.

In this paper, we use a highly idealized model, a monochromatic electron beam interacting with a monochromatic plane-wave field for a finite time. Since the solid state dielectric waveguide is used, the relativistic effect is of less importance. In the current analyses of the small-signal non-collective regime, an electron is described by a wave packet with finite spreading length  $\ell$ . The electron wave spreads over a finite length due to the Coulomb's repulsion forces of neighboring electrons. Experimentally, the spreading length of a single electron has been evaluated through measurements of emission spectrum and confirmed that this

length corresponds to the separating distance between an electron and neighboring electrons [14,15]. For an electron beam with current density  $\bar{N} = 1.5 \times 10^{13} \text{ m}^{-3}$ , the spreading length of an electron was estimated to be  $\ell \approx 40 \text{ }\mu\text{m}$  giving the approximate formula  $\bar{N} \approx 1/\ell^3$  whereas the volume of  $\ell^3$  is occupied by one electron [14,15]. In the sense of quantum mechanics, the spreading length of electron wavepacket corresponds to the uncertainty in the classical electron position. The quantum nature is manifested when the spreading length of an electron becomes comparable with the wavelength of the EM wave, thus our quantum model should be applied in the optical wavelength range. In our analytical model, the Schrödinger equation is used to describe the wave function of an electron. We take into account the effect of electrons collisions represented by the so-called the electron relaxation time. The relaxation time characterizes the damping phenomenon on the timely variation of the electron wave. Using semiclassical formulations, we will derive the power gain in the CFEL operation. In the absence of collisions, it will be shown that the dispersion function of the power gain is identical to that of the classical theory. In the quantum frame work, good agreement is shown between the current analysis and other quantum analysis based on different theoretical approaches. Moreover, we show the conditions at which the derived gain in our quantum model turns out to be the well-known classical gain in the longitudinal CFEL .

In the second section, we employ the Schrödinger equation as an equation for the wave function in the presence of a monochromatic plane-wave propagating in the direction of the electron beam. The small-signal gain for the CFEL operation characterized by the warm beam and the limited interaction length is calculated. In the third section, comparisons with other classical and quantum analyses are presented, and that the compatibility with these analyses is shown. The conclusion is summarized in section IV.

## 2. Semiclassical formulation

### 2.1 The small-signal gain

Schematic layout of Cerenkov-FEL interaction is shown in Fig. 1. If started from spontaneous emission, two mirrors are added at both ends and the device would be run as an oscillator.

In the electron beam, due to Coulomb's repulsion forces between electrons, an electron must be separated from other electrons and should be expressed by a wavepacket with a finite spreading length  $\ell$ . In the case of symmetric Coulomb's forces exerted on an electron, an electron wave can be represented by a boxlike wave packet with volume  $\ell^3$  corresponding to

isotropic separations with neighboring electrons. In the domain of Compton regime when the electron beam current density is low ( $\bar{J} \leq 1000 \text{ A/m}^2$ ), the spreading length of the wave packet should be longer than the EM wavelength such as in the optical wavelength range. Thus, we can assume an electron wave is subjected to a spatially varying electric field from the EM wave within the space of a single electron, and that the energy transfer to the radiation field is caused by coupling between the single electron wave and the EM wave, i.e. the wave-wave coupling.

When the initial velocities of electrons are perturbed, the separating distance between an electron and its neighbors will not be isotropic anymore, or the spreading length of the electron wave will be varied with time. In this case, an electron will be influenced by different Coulomb's forces from the neighboring electrons causing the so-called electron scattering. Since these asymmetric Coulomb's forces will try to relax the electron to its initial state, this process is termed as electron relaxation. In our quantum electron model, the electron relaxation process is viewed as a phase distortion in the electron wave. In the classical treatment as discussed in [20,21], the relaxation effect can be understood as the cause of the damping phenomena on modulations of the electron velocity and the electron density by the EM field in the so-called bunching mechanism. In other words, the relaxation effect works to relax the modulated electron velocity and density toward the average electron velocity and average density, respectively.

In this section, we consider the above described quantum electron interacting with a classical EM field during a finite time  $\Delta T = L/\bar{v}$ , where  $\bar{v}$  is the electron velocity and  $L$  is the interaction length. In the lined waveguide, the forward electric field of the transverse modes is assumed to be pointed along the direction of the electron beam (z direction).

Before application of an electric field at time  $t < 0$ , the electron wave function is given as

$$\psi_n(r, t) = |n, t\rangle = \frac{1}{\sqrt{\ell^3}} e^{jk_n z - j(\omega_n + 1/\tau_n)t}, \quad (1)$$

where  $k_n$  and  $\omega_n$  are the wave-number and frequency of the  $n$ -th energy state, respectively. For simplicity, we also define  $|n\rangle = (1/\sqrt{\ell^3})e^{jk_n z}$  and  $|t\rangle = e^{-j(\omega_n + 1/\tau_n)t}$  as the spatially and timely-dependent parts of the wave function, respectively. In Eq. (1),  $\tau_n$  is the electron relaxation time that characterizes the relaxation of the electron motion at the state  $n$ . The electron wave function described by Eq. (1) satisfies the orthonormality condition and the time-dependent Schrödinger equation having the Hamiltonian,

$$\hat{H}'_0 = \hat{H}_0 - j\hbar\hat{\Gamma}, \quad (2)$$

whereas  $\hat{H}_0$  is the principle Hamiltonian that results in an energy eigenvalue  $W_n$  for the state  $n$ ,

$$\hat{H}_0 \psi_n(r, t) = W_n \psi_n(r, t), \quad (3)$$

and  $\hat{\Gamma}$  represents the electron relaxation time, being defined as

$$\langle \psi_m | \hat{\Gamma} | \psi_n \rangle = \frac{1}{\tau} \delta_{mn}. \quad (4)$$

During application of the electric field after time  $t = 0$ , the state of an electron evolves according to

$$\Psi(r, t) = \sum_n a_n(t) \psi_n(r, t) = \sum_n a_n(t) |n\rangle e^{-(j\omega_n + 1/\tau_n)t}, \quad (5)$$

where  $a_n(t)$  is a time-dependent coefficient that weights the contributions from each eigenfunction. During the time  $t = 0$  to  $t = \Delta T$ , the time-dependent change in electron potential caused by the copropagating electric field  $E_z$  results in a transition between electron eigenstates. In this time interval, the electron state can be termed as transition “or mixed” state.

During the interaction between EM wave and electrons, the electron wave function  $\Psi(r, t)$  evolves in time according to the Schrödinger equation:

$$j\hbar \frac{\partial}{\partial t} \Psi(r, t) = (\hat{H}'_0 + \hat{H}_{\text{int}}) \Psi(r, t), \quad (6)$$

whereas a new Hamiltonian  $\hat{H}_{\text{int}}$  is created in the form of

$$\hat{H}_{\text{int}} = j \frac{e}{2m_0\omega} [\hat{p}(E_z - c.c) + (E_z - c.c)\hat{p}], \quad (7)$$

$\hat{p} = -i\hbar \frac{\partial}{\partial z}$  is the momentum operator. The longitudinal electric field is classically expressed in the form of

$$E_z = F(z) T_z(x, y) e^{j(\omega t - \beta z)}, \quad (8)$$

where  $F(z)$  is the field amplitude of the propagating wave and  $T_z(x, y)$  is a function of the transverse electric field distribution.

Firstly, we will embark on getting the coefficients  $a_n(t)$ . By substituting Eq. (5) into Eq. (6), one may rewrite Eq. (6) in the form of

$$j\hbar \sum_n \left[ \left( \frac{da_n(t)}{dt} \right) |n\rangle e^{-(j\omega_n + 1/\tau_n)t} + a_n(t) \left( \frac{\partial}{\partial t} |n\rangle e^{-(j\omega_n + 1/\tau_n)t} \right) \right] = (\hat{H}'_0 + \hat{H}_{\text{int}}) \sum_n a_n(t) |n\rangle e^{-(j\omega_n + 1/\tau_n)t}. \quad (9)$$

By using the Schrödinger equation in which the unperturbed Hamiltonian  $\hat{H}'_0$  acting on the wave function  $\psi_n(r, t)$ , Eq. (9) can be rewritten as

$$i\hbar \sum_n |n\rangle e^{-(j\omega_n + 1/\tau_n)t} \left( \frac{da_n(t)}{dt} \right) = \sum_n a_n(t) \hat{H}_{\text{int}} |n\rangle e^{-(j\omega_n + 1/\tau_n)t}. \quad (10)$$

By multiplying both sides of Eq. (10) by  $\langle m|e^{(j\omega_m-1/\tau_m)t}$ , performing the integration over the volume space of an electron, and using the orthonormal relationship, Eq. (10) becomes

$$j\hbar\left(\frac{da_m(t)}{dt}\right) = \sum_n a_n(t)\langle m|\hat{H}_{\text{int}}|n\rangle e^{-[j(\omega_n-\omega_m)+(1/\tau_n+1/\tau_m)]t}. \quad (11)$$

In the above equation, we define

$$\omega_{nm} = \omega_n - \omega_m, \quad (12)$$

where  $\omega_{nm}$  corresponds to the energy difference between levels  $n$  and  $m$ .

and,

$$1/\tau = 1/\tau_n + 1/\tau_m, \quad (13)$$

$\tau$  is the electron relaxation time that represents the effect of relaxation on the time variation of the phase of the beating electron wave at the transition state. By the help of Eqs. (12) and (13), Eq. (11) is rewritten as

$$\frac{da_m(t)}{dt} = \frac{1}{j\hbar} \sum_n a_n(t)\langle m|\hat{H}_{\text{int}}|n\rangle e^{-(j\omega_{nm}+1/\tau)t}. \quad (14)$$

By recalling Eq. (7), we can determine the term  $\langle m|\hat{H}_{\text{int}}|n\rangle e^{-(j\omega_{nm}+1/\tau)t}$  of Eq. (14),

$$\langle m|\hat{H}_{\text{int}}|n\rangle e^{-(j\omega_{nm}+1/\tau)t} = \frac{e\hbar}{2m_0\omega} \left\{ \langle m| -j\beta[\mathbf{E}_z + c.c.] |n\rangle + \langle m| 2jk_n[\mathbf{E}_z - c.c.] |n\rangle \right\} e^{-(j\omega_{nm}+1/\tau)t}, \quad (15)$$

Since  $\beta \ll k_n$ , the first term in  $\{ \}$  on the right-hand side of Eq. (15) can be ignored in comparison with the second term. Using Eq. (8) and considering that the amplitude of the electric field  $F(z)$  is changed negligibly over the space of a single electron, Eq. (15) becomes

$$\langle m|\hat{H}_{\text{int}}|n\rangle e^{-(j\omega_{nm}+1/\tau)t} = j \frac{e\hbar}{m_0\omega} k_n \left\{ \langle m| [F(z)T_z(x', y')e^{-j\beta z} - c.c.] |n\rangle \right\} e^{[j(\omega-\omega_{nm})-1/\tau]t}, \quad (16)$$

In Eq. (16),  $\langle m| [F(z)T_z(x', y')e^{-j\beta z}] |n\rangle$  can be estimated as

$$\langle m| [F(z)T_z(x', y')e^{-j\beta z}] |n\rangle = \left[ \frac{1}{\ell^2} \int_{-\ell/2}^0 \int_{\ell/2}^{\ell/2} T_z(x', y') dx' dy' \times \text{Sinc} \left[ (k_n - k_m - \beta) \frac{\ell}{2} \right] \times F(z) \right]. \quad (17)$$

Thus,

$$\begin{aligned} \langle m|\hat{H}_{\text{int}}|n\rangle e^{-(j\omega_{nm}+1/\tau)t} &= j \frac{e\hbar}{m_0\omega} k_n \times \\ &\left\{ \left[ \frac{1}{\ell^2} \int_{-\ell/2}^0 \int_{\ell/2}^{\ell/2} T_z(x', y') dx' dy' \times \text{Sinc} \left[ (k_n - k_m - \beta) \frac{\ell}{2} \right] \times F(z) \right] - c.c. \right\} e^{[j(\omega-\omega_{nm})-1/\tau]t}, \end{aligned} \quad (18)$$

By taking into account the momentum conservation rule  $k_n - k_m = \beta$  when the coherent

length of the electron wave  $\ell$  is sufficiently large, Eq. (18) can be written in the simplified form

$$\langle m | \hat{H}_{\text{int}} | n \rangle e^{-(j\omega_{nm}+1/\tau)t} \approx \Delta H_{nm} e^{[j(\omega-\omega_{nm})-1/\tau]t}, \quad (19)$$

where  $\Delta H_{nm}$  is the matrix element of the interaction Hamiltonian defined by

$$\Delta H_{nm} = j \frac{e\hbar}{m_0\omega} k_n \left\{ \left[ \frac{1}{\ell^2} \int_{-\ell/2}^0 \int_{\ell/2}^0 T_z(x', y') dx' dy' \times F(z) \right] - c.c \right\}. \quad (20)$$

Note that the integrals over the variables  $x'$  and  $y'$  are taken within the range of a single electron. By substituting Eq. (19) into Eq. (14), performing the integration on both sides from  $t = 0$  to  $t = t$ , and considering that  $a_n(t)$  is slowly varying with time, we can write

$$a_m(t) = \frac{1}{j\hbar} \sum_n a_n(t) \Delta H_{nm} \frac{e^{[j(\omega-\omega_{nm})-1/\tau]t} - 1}{[j(\omega-\omega_{nm})-1/\tau]}. \quad (21)$$

By writing the expectation value of the rate of energy gained by the EM field  $\left\langle \frac{d\Delta\Sigma}{dt} \right\rangle$  in the form of

$$\begin{aligned} \left\langle \frac{d\Delta\Sigma}{dt} \right\rangle &= \langle \Psi(r, t) | \left[ e \frac{\hat{p}}{m_0} \cdot (\mathbf{E}_z - \mathbf{E}_z^*) \right] | \Psi(r, t) \rangle \\ &= \frac{e\hbar}{m_0} \langle \Psi(r, t) | -j\beta(\mathbf{E}_z + \mathbf{E}_z^*) | \Psi(r, t) \rangle + \frac{e}{m_0} \langle \Psi(r, t) | (\mathbf{E}_z - \mathbf{E}_z^*) \hat{p} | \Psi(r, t) \rangle. \end{aligned} \quad (22)$$

Due to the relation  $\beta \ll k_n$ , we can safely neglect the first term on the right-hand side of Eq. (22) that contains  $(\mathbf{E}_z + \mathbf{E}_z^*)$ , and the expectation value of the energy lost by an electron  $\langle \Delta\Sigma \rangle$  during the electron transit will be

$$\langle \Delta\Sigma \rangle = \frac{e}{m_0} \int_0^t \sum_m \sum_n |a_m^* a_n| \langle m | (\mathbf{E}_z - \mathbf{E}_z^*) \hat{p} | n \rangle e^{-j(\omega_{nm}+1/\tau)t} dt. \quad (23)$$

Thus,

$$\langle \Delta\Sigma \rangle = \frac{e}{m_0} \int_0^t \sum_m \sum_n |a_m^* a_n| \left\{ \langle m | [F(z) T_z(x', y') e^{-j\beta z} - c.c] | n \rangle \right\} \times e^{[j(\omega-\omega_{nm})-1/\tau]t} dt. \quad (24)$$

By the help of Eq. (21) to get the coefficient  $a_m^*(t)$  and taking the summation over all states  $n \neq m$ , Eq. (24) becomes

$$\langle \Delta\Sigma \rangle = \frac{e}{\hbar} \int_0^t \sum_n |a_n(t)|^2 \left( \frac{\hbar k_n}{m_0} \right) \times \text{Re} \left\{ \sum_{n \neq m} \Delta H_{nm} \frac{e^{[j(\omega-\omega_{nm})-1/\tau]t} - 1}{[j(\omega-\omega_{nm})-1/\tau]} \times \left\{ \langle m | [F(z) T_z(x', y') e^{-j\beta z} - c.c] | n \rangle \right\} e^{[j(\omega-\omega_{nm})-1/\tau]t} dt \right\}. \quad (25)$$

In the small-signal gain regime by considering the first-order time-dependent perturbation



whereas  $|a_n(t)|^2 \approx 1$  when  $m \neq n$ , assuming the momentum conservation rule at which the Sinc-function is approximated to be 1 in Eq. (17), and recalling Eq. (20), the expectation value of  $\langle \Delta \Sigma \rangle$  is written as

$$\langle \Delta \Sigma \rangle = -\frac{e^2 \bar{v}^2 |F(z)|^2}{\hbar \omega} \left| \frac{1}{\ell^2} \int_{\ell-\ell/2}^0 \int_{-\ell/2}^{\ell/2} T_z(x', y') dx' dy' \right|^2 \text{Im} \left\{ \int_0^t \left[ \frac{e^{2[j(\omega-\omega_{nm})-1/\tau]t} - e^{[j(\omega-\omega_{nm})-1/\tau]t}}{[j(\omega-\omega_{nm})-1/\tau]} \right] dt \right\}. \quad (26)$$

By introducing the effective frequency between the electron and the EM wave or the wave frequency as seen by the electron as

$$\Omega = \omega - \omega_{nm} = \beta \bar{v} - \omega. \quad (27)$$

we can get,

$$\langle \Delta \Sigma \rangle = -\frac{1}{2} \frac{e^2 \bar{v}^2 |F(z)|^2}{\hbar \omega} \left| \frac{1}{\ell^2} \int_{\ell-\ell/2}^0 \int_{-\ell/2}^{\ell/2} T_z(x', y') dx' dy' \right|^2 \text{Im} \{X(\Omega, t)\}, \quad (28)$$

where  $X(\Omega, t)$  is the dispersion function defined as,

$$X(\Omega, t) = \left\{ \tau^2 \frac{1 - \left\{ 1 - (j\Omega\tau - 1) \frac{t}{\tau} \right\} e^{(j\Omega\tau - 1)t}}{(j\Omega\tau - 1)^2} \right\}. \quad (29)$$

Here, it is important to point out that the interaction mechanism is initiated at the spatial and time synchronizations between the electron wave function  $\psi_m^* \psi_n$ , during the transition from the initial state  $\psi_n(k_n, \omega_n)$  to the final state  $\psi_m(k_m, \omega_m)$ , and the EM wave  $(\beta, \omega)$ . The frequency of the electron wave function at the transition state, i.e. the beating electron wave  $\psi_m^* \psi_n(k_n - k_m, \omega_n - \omega_m)$ , corresponds to the difference between the frequency of electron wave function at initial and final states. The wave-wave synchronization is exhibited at the momentum and energy conservation rules, or equivalently when  $k_n - k_m \approx \beta$  and  $\omega_n - \omega_m \approx \omega$ , i.e.,  $\Omega \approx 0$ . In other words, the quantum characteristics in our analysis do not correspond to the de-Broglie wavelength of an electron ( $\approx 30$  pm) which is much smaller than any physical scale of the system, but it correspond to the wavelength of the beating electron wave  $(1/[(1/\lambda_n) - (1/\lambda_m)])$  which is much larger than the deBroglie wavelength and should be in order of the optical wavelength  $\lambda$ .

The instantaneous power lost by the electron beam can be given by

$$\Delta P = -\langle \Delta \Sigma \rangle \frac{I}{|e|}, \quad (30)$$

where  $I = \oint_S \bar{J} dx dy$  is the current of the electron beam defined as the flux of the current density  $\bar{J}$  through the cross-sectional area of the electron beam  $S$ . By substituting Eqs. (28) and (29) into Eq. (30), and recalling the relation of  $|F(z)|^2 \approx 2\sqrt{(\mu_0/\varepsilon_0)}(1/n_{\text{eff}})P(z)$  [14,15,20] whereas  $P(z)$  is the propagation power of the EM wave and  $n_{\text{eff}}$  is the effective refractive index defined with the propagation constant as  $\beta(\lambda) = \omega n_{\text{eff}}(\lambda)/c$ , we can get

$$\Delta P = \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{e\bar{J}\bar{v}^2}{\hbar\omega n_{\text{eff}}} \xi \text{Im}\{X(\Omega, t)\} \times P(z), \quad (31)$$

In Eq. (31),  $\xi$  is the spatial coupling coefficient between the optical field and the electron wave, and is given in term of the optical field distribution function as

$$\xi = \oint_S \left| \frac{1}{\ell^2} \int_{-\ell/2}^0 \int_{-\ell/2}^{\ell/2} T_z(x', y') dx' dy' \right|^2 dx dy. \quad (33)$$

From Eq. (31), the instantaneous power gain  $G(\Omega, t) = \Delta P/P$  will be given by

$$G(\Omega, t) = \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{e\bar{J}\bar{v}^2}{\hbar\omega n_{\text{eff}}} \xi \text{Im}\{X(\Omega, t)\}. \quad (32)$$

Now, we can get the temporally averaged gain by

$$\begin{aligned} \overline{G(\Omega, \Delta T)} &= \frac{1}{\Delta T} \int_0^{\Delta T} G(\Omega, t) dt \\ &= \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{e\bar{J}\bar{v}^2}{\hbar\omega n_{\text{eff}}} \xi \text{Im}\{\overline{X(\Omega, \Delta T)}\}, \end{aligned} \quad (34)$$

where

$$\begin{aligned} \overline{X(\Omega, \Delta T)} &= \frac{1}{\Delta T} \int_0^{\Delta T} X(\Omega, t) dt \\ &= \frac{2[1 - e^{(j\Omega - 1/\tau)\Delta T}] + \{(j\Omega - 1/\tau)\Delta T[1 + e^{(j\Omega - 1/\tau)\Delta T}]\}}{(j\Omega - 1/\tau)^3 \Delta T}. \end{aligned} \quad (35)$$

For convenience, we rewrite the averaged gain in the form

$$\overline{G(\Omega, \Delta T)} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{e\bar{J}\bar{v}^2(\Delta T)^2}{\hbar\omega n_{\text{eff}}} \xi \overline{Y(\Omega, \Delta T)}, \quad (36)$$

with an alternative dispersion function  $Y(\Omega, \Delta T)$

$$\overline{Y(\Omega, \Delta T)} = \frac{\text{Im}\{\overline{X(\Omega, \Delta T)}\}}{\Delta T^2}. \quad (37)$$

The dispersion function  $\overline{Y(\Omega, \Delta T)}$  gives the frequency dependence of the temporally averaged gain. It is very interesting to note that when the interaction time is much shorter than the

relaxation time  $\Delta T \ll \tau$ , i.e., when the relaxation effect can be omitted, the dispersion function  $\overline{Y(\Omega, \Delta T)}$  will be reduced to

$$\begin{aligned} Y(\Omega, \Delta T) \big|_{\Delta T \ll \tau} &= \text{Im} \left\{ \frac{2(1 - e^{j\Omega\Delta T}) + [j\Omega\Delta T(1 + e^{j\Omega\Delta T})]}{(j\Omega)^3 (\Delta T)^3} \right\} \\ &= \frac{1}{\Omega\Delta T} \left\{ \frac{\sin^2(\Omega\Delta T / 2)}{(\Omega\Delta T / 2)^2} - \frac{\sin^2(\Omega\Delta T)}{(\Omega\Delta T)} \right\} \end{aligned} \quad (38)$$

which is the famous dispersion function in old classical theories of free-electron lasers whose maximum value is 0.135 and occurs at  $\Omega\Delta T = 2.6$ . In this regime, the temporally averaged gain is independent of the relaxation time  $\tau$ , and is almost determined by the terms  $\Omega$  and  $\Delta T$ . We call this regime as the transition state in which the gain increases with increasing the interaction time  $\Delta T$ . On the other hand, when the interaction time is much longer than the relaxation time  $\Delta T \gg \tau$ , the dispersion function becomes

$$Y(\Omega, \Delta T) \big|_{\Delta T \gg \tau} = \frac{1}{(j\Omega\tau - 1)^2} \times \left( \frac{\tau}{\Delta T} \right)^2. \quad (39)$$

By substituting Eq. (39) into Eq. (36), one note that the temporally averaged gain is independent of the interaction time  $\Delta T$ , and is mostly characterized by the terms  $\Omega$  and  $\tau$ . Thus we call this regime as the steady state whereas the gain saturates at constant values with the time variation basing on the relaxation time  $\tau$ .

## 2.2 Numerical examples and discussions

A numerical example shows the variation in the peak value of the temporally averaged gain  $\overline{G(\Omega, \Delta T)}$  as a function of  $\Delta T / \tau$  is depicted in Fig. 2.  $\overline{G(\Omega, \Delta T)}$  is calculated using Eqs. (36) and (37). In this example, we assume that  $\lambda = 1.5 \mu\text{m}$ ,  $n_{\text{eff}} = 3.0$ ,  $\xi = 0.1$ ,  $J = 10^3 \text{ A/m}^2$ , and  $\tau = 10^{-9} \text{ sec}$ . These numerical values are identical to those obtained in experiments shown in Ref. [15] whereas the relaxation time was estimated to be  $\tau = 10^{-10} - 10^{-9} \text{ sec}$ . In Fig. 2, the peak values of the gain increases with the interaction time  $\Delta T$  until the interaction time reaches to approximately twice the relaxation time  $\tau$ , and saturates at certain value with further interaction time increase. As we mentioned, we call the former range the transition state and the latter range the steady state. In the quantum mechanics framework, the transition state should be understood as the time period in which the synchronization between the mixed electron wave (the beating electron wave) and the EM wave is building up as time increases. Since this synchronization is relaxed or damped by the relaxation effect, any further enhancement in the

synchronization is canceled by the relaxation effect and reveals the steady state operation.

By using the same parameters values assumed to draw Fig. 2, a numerical example of the gain dispersion versus  $\Omega\tau$  for different values of  $\Delta T/\tau$  is shown in Fig. 3a,b. As shown in Fig. 3a, the profile of the distribution is periodical with  $\Omega\tau$  when  $\Delta T/\tau$  is small ( $\Delta T/\tau \ll 2$ ) and is almost determined by the dispersion function  $Y(\Omega, \Delta T)|_{\Delta T \ll \tau}$ . In Fig. 3b, the gain profile has a smooth curve when  $\Delta T/\tau$  is large ( $\Delta T/\tau \gg 2$ ) and is almost determined by the dispersion function  $Y(\Omega, \Delta T)|_{\Delta T \gg \tau}$ . The gain profile becomes sharper with increasing  $\Delta T/\tau$ .

### 3. Comparisons with other classical and quantum treatments

In Ref. [19], a quantum mechanical analysis of the interaction between the optical beam and the electron beam has been presented basing on the density matrix formalism. By keeping the same symbols used in this paper, if the electron scattering is small enough or the interaction time  $\Delta T$  is much shorter than the relaxation time, the gain coefficient  $g$  defined as the spatial growth rate of the power is given by [19]

$$g = \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{e\bar{J}L}{\hbar\omega n_{\text{eff}}} \xi \times D, \quad (40)$$

where  $D$  is the dispersion function given as

$$D = \text{Sinc}^2[(k_b - k_a - \beta)\ell/2] - \text{Sinc}^2[(k_c - k_b - \beta)\ell/2]. \quad (41)$$

The first and second terms on the right-hand side of Eq. (41) represent the electron transition from an initial energy level  $b$  to a lower level  $a$  and to a higher level  $c$  in the stimulated emission and induced absorption processes, respectively. The dispersion function determines the difference between the optical emission ( $D > 0$ ) and the optical absorption ( $D < 0$ ).  $D$  increases with  $\ell$  and the maximum value of  $D$  is 1.

In this paper, when the relaxation time is not taken into account, the power gain parameter  $G$  given by Eqs (36) and (38) can be simplified as

$$\overline{G(\Delta T)} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{e\bar{J}L^2}{n_{\text{eff}}\hbar\omega} \xi \times Y(\Omega, \Delta T)|_{\Delta T \ll \tau}. \quad (42)$$

If we replace the dispersion function  $Y(\Omega, \Delta T)|_{\Delta T \ll \tau}$  by  $D$  in Eq. (42), we can confirm that our results in Eq. (42) coincide with the result in Eq. (40) published in Ref. [19] whereas the power gain in our analysis  $G$  equal to  $g \times L$ . It is worth to point out that, the dispersion function  $D$  is determined by the spreading length of the electron wave  $\ell$  corresponding to

the uncertainty in the electron momentum, while the dispersion function  $Y(\Omega, \Delta T)|_{\Delta T \ll \tau}$  is determined by the interaction time  $\Delta T$  corresponding to the uncertainty in the electron energy. Thus the dispersion function  $D$  should be applicable when the interaction time tends to infinity, while the dispersion function  $Y(\Omega, \Delta T)|_{\Delta T \ll \tau}$  should be applicable when the electron beam interacts with an EM field for a finite time and the coulomb collisions are weak.

In the pioneering articles based on classical treatments [4,5,22,23], using the symbols of this paper and neglecting the relativistic effects whereas  $\gamma = [1 - (\bar{v}/c)^2]^{-1/2} \approx 1$  in our present problem, the power lost by the electron beam in low gain limit is given as [23]

$$\Delta P = \frac{e\omega I}{2m_0} (\Delta T)^3 \times |F(z)T_z(x, y)|^2 Y(\Omega, \Delta T)|_{\Delta T \ll \tau}. \quad (43)$$

Again, by the help of the relations of  $I = \iint_S \bar{J} dx dy$  and  $|F(z)|^2 \approx 2\sqrt{(\mu_0/\epsilon_0)}(1/n_{\text{eff}})P(z)$ , we

can rewrite Eq. (43) as

$$\Delta P = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{e\bar{J}\omega(\Delta T)^3}{n_{\text{eff}} m_0} \xi_{\text{cl}} Y(\Omega, \Delta T)|_{\Delta T \ll \tau} \times P(z), \quad (44)$$

where  $\xi_{\text{cl}} = \iint_S |T_z(x, y)|^2 dx dy$  is the coupling coefficient in the classical limit. So that the

power gain parameter  $G_{\text{cl}}$  in these well-known classical treatments can be given by

$$G_{\text{cl}} = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{e\bar{J}\omega(\Delta T)^3}{n_{\text{eff}} m_0} \xi_{\text{cl}} Y(\Omega, \Delta T)|_{\Delta T \ll \tau}. \quad (45)$$

On the other hand, if we denote the average velocity  $\bar{v}$  in Eq. (36) with a subscript  $b$  refereeing to the average velocity at the initial state, and by using the relations of  $\bar{v}_b \approx \omega/\beta$  and  $\hbar\beta = m_0\bar{v}_b - m_0\bar{v}_a = m_0\bar{v}_b(1 - \bar{v}_a/\bar{v}_b)$ , the gain in our analysis when the relaxation effect is neglected is given by

$$\overline{G(\Delta T)} = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{e\bar{J}\omega(\Delta T)^3}{n_{\text{eff}} m_0 (1 - \bar{v}_a/\bar{v}_b) \beta L} \xi \times Y(\Omega, \Delta T)|_{\Delta T \ll \tau}. \quad (46)$$

The gain in Eq. (46) derived in our analysis well coincide with the gain of old classical theory given by Eq. (44) at the condition of

$$(1 - \bar{v}_a/\bar{v}_b) \beta L \rightarrow 1. \quad (47)$$

By using the energy conservation rule  $E_b - E_a = \hbar\omega$  and the relation of  $\hbar\omega \ll E_b$ , Eq. (47) is reduced to

$$\left[1 - \left(1 - \frac{\hbar\omega}{(1/2)m_0\bar{v}_b^2}\right)^{1/2}\right]\beta L \approx \left(\frac{\hbar\omega}{m_0\bar{v}_b^2}\right)\beta L \rightarrow 1 \quad (48)$$

Since  $\hbar\omega \ll E_b$ , the condition described by Eq. (48) is mathematically met as the wavelengths decreases or when  $\beta L \gg 1$  verified at the optical and shorter wavelengths. For example, if the interaction length is 1 cm and the refractive index of the waveguide is 3, Eq. (18) can be approximated to be  $5 \times 10^{-12} (1/\lambda^2)$  and almost become 1 in the range of optical wavelengths.

Note that, as the wavelength increases, the distribution of the EM field can be considered as constant over the spreading length of single electron wave and  $T_z(x', y')$  may be taken outside the internal integration sign in Eq. (33). So that the coupling coefficient in our quantum treatment  $\xi$  defined in Eq. (33) approaches to the classical coupling coefficient  $\xi_{cl} = \iint_S |T_z(x, y)|^2 dx dy$ . This case can be verified in the microwave region when the EM wavelength becomes longer than the electron wave packet length and the electron can be considered as a localized point particle (classical particle).

#### 4. Conclusions

We propose a quantum analytical model to describe the stimulated Cerenkov radiation in the small-signal gain limit. In this model, the interacting electron is expressed as a wave packet with a finite spatial extend. The mechanism of the electron relaxation caused by the Coulomb's repulsion forces between electrons is taken into account. By introducing the so-called electron relaxation time, the interaction regimes induced by the stimulated emission is classified into the transition state and steady state. In the transition state when the interaction time is shorter than the relaxation time, the power gain increases simultaneously with the interaction time and the relaxation time is of less importance. In the steady state when the interaction time is much longer than the relaxation time, the gain becomes almost constant, being independent of the interaction time. If the relaxation effect is neglected as in the transition state, and by imposing special conditions, we show the compatibility between our results and those of other classical and quantum analyses.

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## Figures Captions

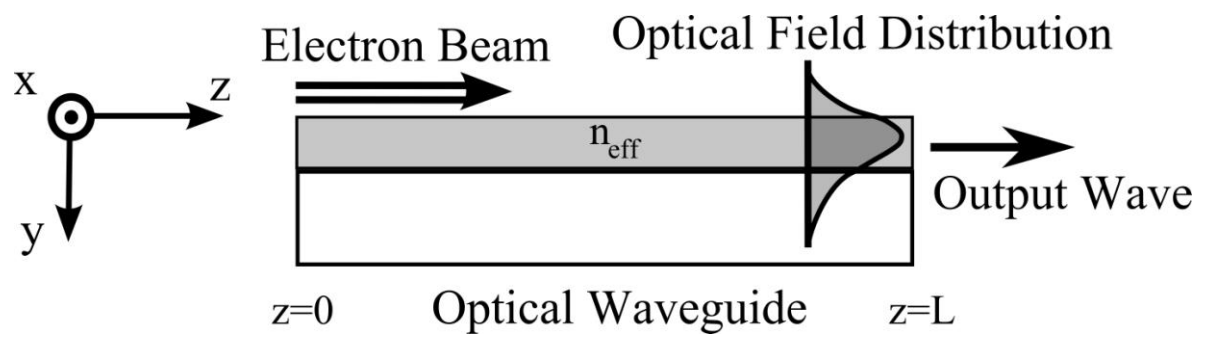
Fig. 1. Schematic diagram of the electron beam-dielectric guide interaction in a Cerenkov-FEL.

Fig. 2. The variation of the maximum values of the averaged gain  $\overline{G(\Omega, \Delta T)}$  with  $\Delta T / \tau$ . In the first region which is called the transition state, the gain increases with increasing  $\Delta T / \tau$  due to the enhancement of spatial and time synchronizations between the beating electron wave and the EM wave. Any further improvement in the synchronization mechanism will be canceled under the effect of electron relaxation inducing the second region called the steady state in which the gain saturates at constant value.

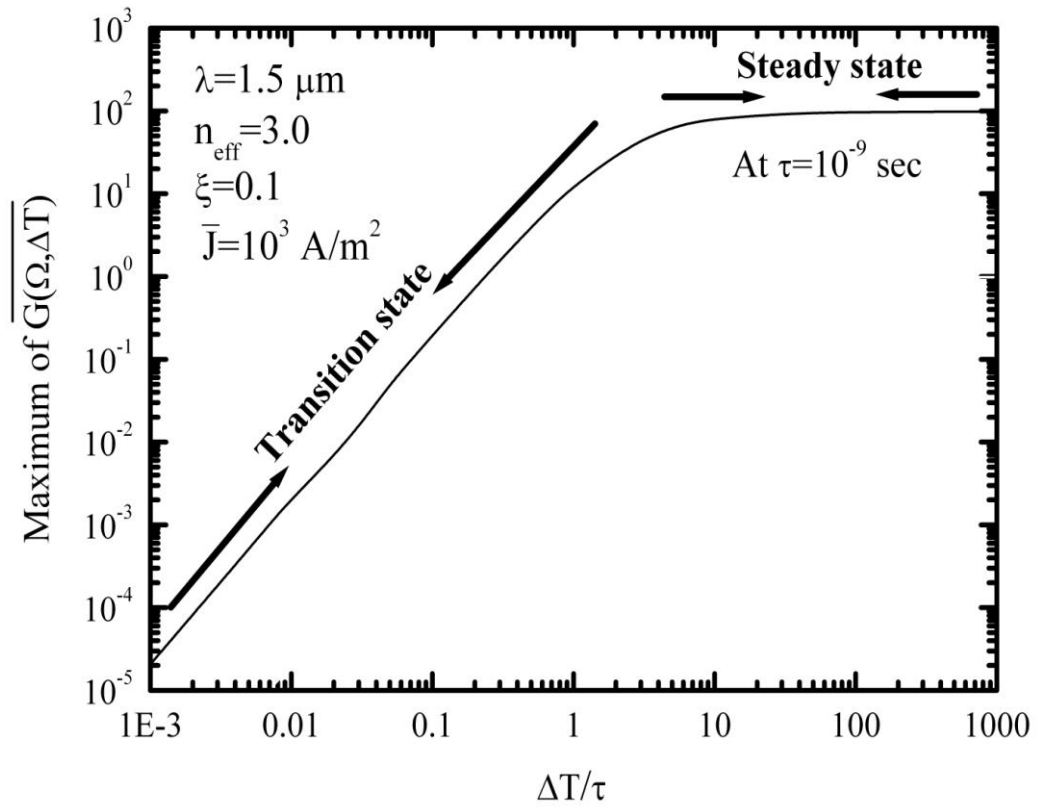
Fig. 3. The averaged gain  $\overline{G(\Omega, \Delta T)}$  versus  $\Omega\tau$  for small and large values of  $\Delta T / \tau$ . (a) When  $\Delta T / \tau \ll 2$ , the gain profile is represented by the function  $Y(\Omega, \Delta T)|_{\Delta T \ll \tau}$  characterized by periodic variation. (b) When  $\Delta T / \tau \gg 2$ , the gain profile is smoothly varied with  $\Omega\tau$  and is represented by the dispersion function  $Y(\Omega, \Delta T)|_{\Delta T \gg \tau}$ . In Fig. 3a,b, the positive gain is observed when  $\Omega\tau > 0$  (stimulated emission), the negative values are observed when  $\Omega\tau < 0$  (stimulated absorption), and the gain become zero at the synchronism condition  $\Omega = 0$ . The profile of the gain becomes sharper with increasing  $\Delta T / \tau$ .



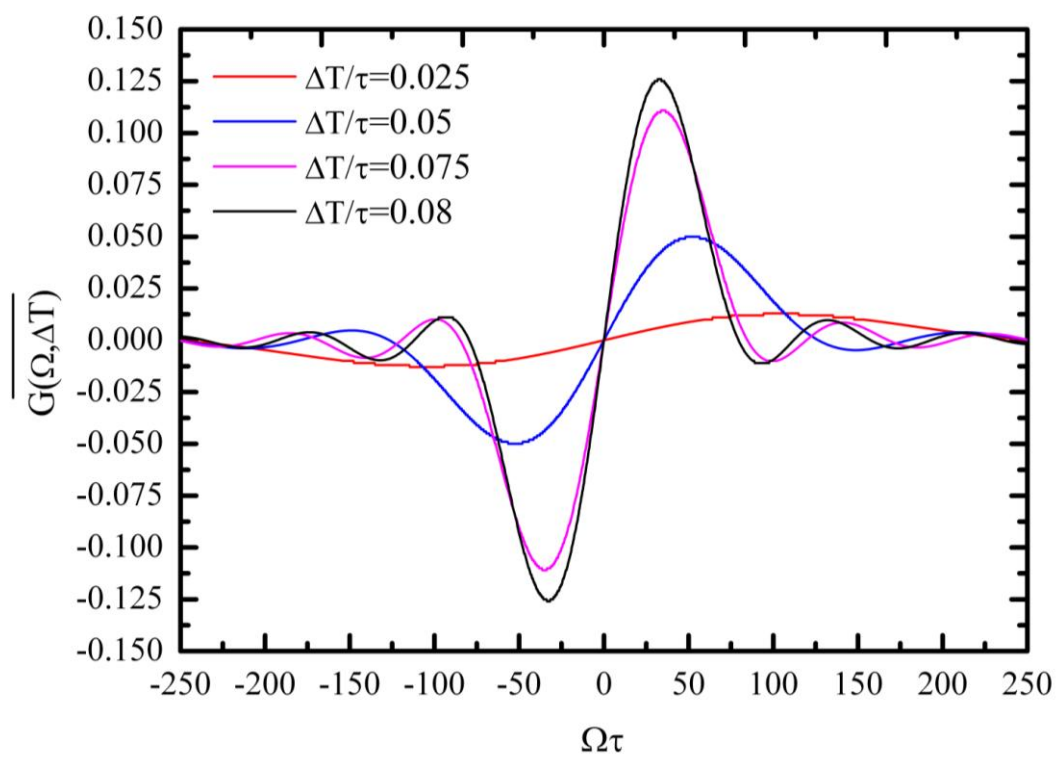
**Figure 1**



**Figure 2**



**Figure 3(a)**



**Figure 3(b)**

