Robotic Mapping and Localization Considering Unknown Noise Statistics*

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Abstract

This paper presents an analysis of H_∞ Filter(HF) for Robotics Mapping and Localization with unknown noise statistics. HF which is also known as the minimax filter is proposed in this paper to estimate the robot and landmarks location while robot moves through an unknown environment. Some of the conditions are proposed to ensure that the state covariance in HF is converging to a steady state value. Furthermore, the analysis of HF convergence for a robot observing landmarks are presented to examine its behavior through the observations. From the experimental results, HF gives a sufficient estimation about the environment. Subsequently, such a result can provide other available estimation methods with the capability to ensure and improved estimation in robotic mapping and localization problem.

Key words : Robotic Mapping, Localization, Unknown Noise Statistics

1. Introduction

The robotic mapping and localization problem has gained researcher's attention over some past decades. This problem illustrates an application of a mobile robot whose observing an environment and collecting information efficiently while it is moving through the environment. From its observations, the robot then attempts to build a map. In 1980's, Smith and Cheeseman et.al[1] initiated the robotic mapping problem based on the relationship between landmarks and a robot. Since then, a sequential of interesting developments dramatically has evolved its name to Simultaneous Localization and Mapping problem(SLAM)[2]. See Fig.1 for the illustration of SLAM problem which consists of a mobile robot observing some landmarks.

SLAM has been applied in a wide range of applications, indoors or outdoors such as satellite, mining, space exploration, rescue and military. The development of SLAM continues in 2D[3] or 3D applications[4][5] and amazingly expanded to home-based robot applications.
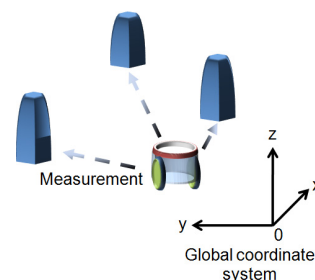

Fig. 1   SLAM problem

SLAM has enhanced from *Topological* and *Metric* approaches to *Behavioral* approach, *Mathematical-based model* approach and *Probabilistic* approach[2]. However, between these three techniques, the probabilistic approach has made a significant achievement. In spite of probabilistic approach remarkable achievement, such a technique suffers from some shortcomings such as computational complexity. Nevertheless, with modern development of software and research, a considerable support and solution to this problem may exist, thus inspire the development of probabilistic SLAM.

Recently, many approaches using the probabilistic whether parametric or non-parametric methods have been proposed to solve the SLAM problems e.g Kalman Filter, UKF, Particle Filter, etc. Currently, a non-parametric method called Fast-SLAM approach[2], efficiently constructs the unknown map by utilizing an amount of particle whose behaves as the uncertainty to the whole system. If more particles are used, then the estimation result will be better. Unfortunately, this method require higher computational cost. Due to such deficiencies, such a wonderful technique does not deter some classical methods. The readers are encouraged to look at the review about SLAM development explained by S.Thrun et.al[6], which is purposely discussed regarding the SLAM problem from various aspects.

It is a wise decision to model a system that can take into account for the worst case of noises or when the noise statistics are unknown but bounded. Hence, HF can be a complement estimator to tolerate such a robust system. The development of HF for SLAM is proposed herein and its performance has been briefly compared to Kalman Filter[7][8],[12]−[19] approach in some conditions. HF[9] is one of the set-membership approaches, which assumed that the noise is known in a bounded energy[10],[11]. The energy gain from the noise inputs to the estimation errors is guaranteed to be less than a certain level.

Throughout this paper, we examine the HF about its convergence properties. To guarantee the convergence of state covariance, some conditions are provided. If these conditions are not satisfied, then the estimation have erroneous results. Moreover, we investigate the convergence properties of a robot observing landmarks in an unknown environment. We study the results using a constant motion and sensors uncertainties with a perfect data association considering two cases of gaussian and unknown noise statistics but bounded. To this extent, H_∞ Filter is still new in the robotic mapping problem solution schemes, although it has a desirable property and is competitive compared to Kalman Filter. West et.al[12] showed that HF was competent with other well-known approaches such as Kalman Filter and Particle Filter for SLAM problem. However, no theoretical explanation or contribution about HF properties was discussed. In this paper, HF convergence properties are analyzed to understand its behavior in SLAM.

This paper is organized as follows. In *Section 2*, HF-Based SLAM is presented with a brief comparison to the Kalman Filter, while *Section 3* demonstrates the main results of convergence properties of H_∞ SLAM problem. Next, *Section 4* shows the experimental results of SLAM about both filters performance. Finally, *Section 5* concludes the paper.

2. H_∞ Filter Based-SLAM

2.1. SLAM Mathematical Model

SLAM consists of two general models which are the process model that explains how the robot move through the unknown environment and the measurement model that calculates and measures the relative distance and angle between robot and landmarks for each respective time. This section analyzes both models to describe how a map can be built using both models information. Fig.2 explains about these two models. An assumption of stationary landmarks is made for convenience. The process model is presented as follows.

$$X_{k+1} = f(X_k, \omega_k, v_k, \delta\omega_k, \delta v_k), \quad X_k = [\theta_k \quad x_k \quad y_k \quad L_k^i]^T \tag{1}$$

where θ_k is the robot angle and x_k, y_k are robot x, y positions. L_k^i is the ith landmarks containing its x_i, y_i positions. Note that we assume the landmarks are stationary and therefore is time invariant unlike the robot location. v_k and ω_k are representing the robot velocity and turning
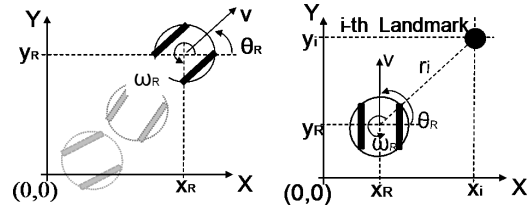
Fig. 2   Process model(left) and measurement model(right) of mobile robot localization

rate respectively. δv_k, $\delta \omega_k$ are the correlated noises for each v_k, and ω_k. T is the transpose of the matrix. On the other hand, the measurement model has the following equation.

$$z_k = \begin{bmatrix} r_i \\ \phi_i \end{bmatrix} = \begin{bmatrix} \sqrt{(y_i - y_k)^2 + (x_i - x_k)^2} + v_{r_i} \\ \arctan\left(\frac{y_i - y_k}{x_i - x_k}\right) - \theta_k + v_{\phi_i} \end{bmatrix} \rightarrow \nabla H_i X_{k+1} + v_k \tag{2}$$

where r_i, ϕ_i are the relative distance and angle between robot and i^{th} landmark while other variables hold the same definitions as described before. v_{r_i} and v_{ϕ_i} are the associated noises to the measurements of relative distance and angle respectively. ∇H_i and v_k are defined as the measurement model in Jacobian representation and the measurement noise with covariance R_k, respectively. Both terms are shown by the following notations.

$$\nabla H_i = \begin{bmatrix} 0 & -\frac{dx}{r} & -\frac{dy}{r} & \frac{dx}{r} & \frac{dy}{r} \\ -1 & \frac{dy}{r^2} & -\frac{dx}{r^2} & -\frac{dy}{r^2} & \frac{dx}{r^2} \end{bmatrix}, \quad v_k = \begin{bmatrix} v_{r_i} \\ v_{\phi_i} \end{bmatrix} \tag{3}$$

where $r = \sqrt{(x_i - x_k)^2 + (y_i - y_k)^2}$, $dx = x_i - x_k$ and $dy = y_i - y_k$. The prediction step yield the following equations.

$$\hat{X}_{k+1} = f(\hat{X}_k, \omega_k, v_k, 0, 0) \tag{4}$$

$$P_{k+1} = \nabla f_X P_k \psi_k^{-1} \nabla f_X^T + \nabla f_{\omega v} \Sigma_k \nabla f_{\omega v}^T \tag{5}$$

where $\hat{X}_k$ is a predicted augmented state and P_k is a state error covariance of the system. As shown above, Σ_k acts as a control noise $(\delta \omega_k, \delta v_k)$ covariance.

$$\nabla f_X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -vt\sin\theta & 1 & 0 & 0 \\ vt\cos\theta & 0 & 1 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad \nabla f_{\omega v} = \begin{bmatrix} \nabla g_{\omega v} \\ 0 \end{bmatrix} \tag{6}$$

where t is a sampling time and I is an identity matrix with an appropriate dimension. ∇f_X is the Jacobian transformation of robot motions and $\nabla f_{\omega v}$ is the linearized process noise. We assume no process noise for landmarks. Therefore the linearized process noise for robot motion is $\nabla g_{\omega v}$. ψ_k is shown by the following expression.

$$\psi_k = (I + \nabla H_i R_k^{-1} \nabla H_i^T P_k - \gamma^{-2} I P_k) \tag{7}$$

where $\gamma > 0$. In H_∞ Filter[7], the ratio of estimation error to the noise errors must be less than γ^2. We show this characteristics later. Using the Jacobian notation for a case of a robot observing one landmark at point A, we use the following notation.

$$\nabla H_A = [-e \quad -A \quad A] \tag{8}$$

The following abbreviation for Eq.(8) is used and is restated as below.

$$H_A = \begin{bmatrix} e & A \end{bmatrix}, \quad e = \begin{bmatrix} 0 & -1 \end{bmatrix}^T \implies \nabla H_A = [-H_A \quad A] \tag{9}$$

where

$$A = \begin{bmatrix} \frac{x_i - x_A}{\sqrt{x_i - x_A{}^2 + y_i - y_A{}^2}} & \frac{y_i - y_A}{\sqrt{x_i - x_A{}^2 + y_i - y_A{}^2}} \\ \frac{y_i - y_A}{x_i - x_A{}^2 + y_i - y_A{}^2} & \frac{x_i - x_A}{x_i - x_A{}^2 + y_i - y_A{}^2} \end{bmatrix} \tag{10}$$

(x_i, y_i) and (x_A, y_A) are the ith landmark coordinate and robot position at point A respectively. The initial state covariance $P_0 > 0 (P_0 \in \mathbb{R}^{(3+2N)\times(3+2N)})$ is presented by

$$P_0 = \begin{bmatrix} P_{0v} & 0 \\ 0 & P_{0m} \end{bmatrix} \tag{11}$$

where P_{0v} is the robot initial covariance and P_{0m} is the landmarks initial covariance. N is the number of landmarks.

2.2. H_∞ Filter Properties

This subsection presents the development of HF-Based SLAM considering about its convergence properties. Its algorithm is also included in some literatures[21], [22]. The comparison between H_∞ Filter and Kalman Filter convergence[23] for a robot observing landmarks is evaluated based on some experiments. Hamzah et.al[24]−[26] studied the HF convergence and its application with *Covariance Inflation* to decrease SLAM computation cost. Katayama[7] and D.Simon[9] have presented satisfactory explanations about HF. Referring to them, we first make some assumptions about the noise characteristics.

Assumption 1. $R_k \stackrel{\Delta}{=} D_k D_k^T > 0$

Assumption 2. *Bounded noise energy;* $\Sigma_{t=0}^N \|\zeta_k\|^2 < \infty, \Sigma_{t=0}^N \|\xi_k\|^2 < \infty$

From Eq.(1)-(2), we can derive $D_k = [0 \quad I]$. *Assumption 1* defines that the measurement noise are not correlated to process noise during robot observations. Above assumptions are similar to the standard Kalman Filter assumptions where all components of the measurement vectors are assumed to be corrupted by temporally uncorrelated noise, R_k. Both process noise ζ_k with covariance Q_k and the measurement noise ξ_k with covariance R_k are assumed to be bounded as described by *Assumption 2*. For details of H_∞ Filter, refer to Katayama[7] and Lewis et.al[21].

The differences between Kalman Filter and H_∞ Filter appears in its form of gain, K_k and its state covariance P_k, which integrates both the prediction and updates process. For Kalman Filter, the equation for its gain and covariance are given by,

$$K_k = P_k(I + \nabla H_i^T R_k^{-1} \nabla H_i P_k)^{-1} \tag{12}$$

$$P_{k+1} = \nabla F_k P_k (I + \nabla H_i^T R_k^{-1} \nabla H_i P_k)^{-1} \nabla F_k^T + Q_k \tag{13}$$

On the other hand, H_∞ Filter gain and covariance are expressed by Eqs.(14)-(15).

$$K_k = P_k(I - \gamma^{-2} P_k + \nabla H_i^T R_k^{-1} \nabla H_i P_k)^{-1} \tag{14}$$

$$P_{k+1} = \nabla F_k P_k (I - \gamma^{-2} P_k + \nabla H_i^T R_k^{-1} \nabla H_i P_k)^{-1} \nabla F_k^T + Q_k \tag{15}$$

where γ is an H_∞ norm of the closed-loop system. H_∞ Filter depends on the covariance matrix of error signals, $Q_k \geq 0, R_k > 0$ which are chosen and designed such that to achieve a desired performance. It is observable that if γ becomes bigger in (14) and (15), then these equation are the same as (12), (13) of Kalman Filter.

An H_∞ Filtering problem is defined as following. Given that for $P_0 > 0(P_0 \in \mathbb{R}^{3+2N})$, the state covariance yields

$$P_{k+1} = \nabla f_X P_k \psi_k^{-1} \nabla f_X^T + \nabla f_{\omega v} \Sigma_k \nabla f_{\omega v}^T, \quad P_0 = \Sigma_0 \tag{16}$$

$$\psi_k = I + (\nabla H_i^T R_k^{-1} \nabla H_i - \gamma^{-2} L_k^T L_k) P_k \tag{17}$$

In our case L_k is an identity matrix with an appropriate dimension. The above equations holds a positive definite solution if it satisfies the following equation.

$$P_k = \hat{P}_k^{-1} - \gamma^{-2} L_k^T L_k > 0, \quad k = 0, 1, \ldots, N \tag{18}$$

where

$$\hat{P}_k = (P_k^{-1} + \nabla H_i^T R_k^{-1} \nabla H_i)^{-1} > 0 \tag{19}$$

For $\gamma > 0$, the suboptimal HF is given by below expressions.

$$\hat{c}_k^* = L_k \hat{x}_k, \quad \hat{x}_{k+1}^- = \nabla F_k \hat{x}_k \tag{20}$$

$$\hat{x}_{k+1}^+ = \hat{x}_k + K_k[z_k - \nabla H_i \hat{x}_k], \hat{x}_0 = \bar{x}_0 \tag{21}$$

$$K_k = P_k \nabla H_i (\nabla H_i P_k \nabla H_i^T + R_k)^{-1} \tag{22}$$

where $\hat{c}_k^*$ is the interested state output estimation, $\hat{x}_{k+1}^-$ and $\hat{x}_{k+1}^+$ are the priori and posteriori estimated state, and $\bar{X}_0$ is the estimated initial state with its associated initial state covariance, P_0. In other words, given $\gamma > 0$, an H_∞ Filter attempts to find a solution for $\hat{x}_k$ that satisfies the following equation.

$$\sup_{x_0, \zeta, \xi} \frac{\Sigma_{k=0}^N \|x_k - \hat{x}_k\|}{\|x_0 - \bar{x}_0\|_{P_0^{-1}}^2 + \Sigma_{k=0}^N \|\zeta_k\|_{R_k^{-1}}^2 + \Sigma_{k=0}^N \|\xi_k\|_{Q_k^{-1}}^2} < \gamma^2$$

3. Main Results

To obtain a successful estimation using H_∞ Filter, a solution to the H_∞ Filter must be available[22]. If else, a faulty estimations and *finite escape time* problem is observable. The finite escape time states that an estimation suddenly goes to infinity in finite time horizon.

Assumption 3. $(\nabla F_k, \nabla H_i)$ *is observable and* $(\nabla F_k, \nabla G_k)$ *is controllable.*

From Eqs.(1)-(2), we can derive $\nabla G_k = [I \quad 0]$. This setting is suggested to guarantee that the process noise is uncorrelated to the measurement noise. With respect to Calleja et.al[8], we present the following lemma for the stability of H_∞ Filter. Note that from now on we change ∇H_i to H_i unless stated.

Lemma 1. *The filtering error dynamics for an HF is converging to a zero mean steady state if and only if the matrix* $I - K_k H_i$ *is stable and Eq.(18) is satisfied.* H_i *and* K_k *are defined by Eq.(3) and Eq.(22) respectively.*

Proof. We begin the proof by showing that $(I - K_k H_i)$ is marginally stable in robotic mapping problem. Given that the initial state covariance matrix given by,

$$P_0 = \begin{bmatrix} \sigma_\theta^2 & 0 & 0 \\ 0 & \sigma_{xy}^2 * I_2 & 0 \\ 0 & 0 & \sigma_m^2 * I_{2N} \end{bmatrix} \tag{23}$$

where σ_θ, σ_{xy}, and σ_m are the robot pose angle, robot x_k, y_k positions and landmarks x_i, y_i initial covariance respectively. I is an identity matrix with an appropriate dimension. For a stationary robot whose observing a landmark, the measurement model yields

$$H_i = \begin{bmatrix} r_i \\ \phi_i \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \end{bmatrix} \tag{24}$$

If Eq.(19) is satisfied, and by using Eq.(22) about the filter gain, then the following equation is derived as

$$K_k = P_0 H_i^T (H_i P_k H_i^T + R_k)^{-1}$$
$$= \begin{bmatrix} 0 & -\sigma_\theta^2 s_\theta \\ -(\sigma_{xy}^2 * I_2)s & 0 \\ (\sigma_m^2 * I_{2N})s & 0 \end{bmatrix} \tag{25}$$

where s, s_θ are the innovation variances and $s = ((\sigma_{xy}^2 * I_2) + (\sigma_m^2 * I_{2N}) + R_k)^{-1}$, $s_\theta = (\sigma_\theta^2 + R_k)^{-1}$. Hence,

$$(I - K_k H_i)$$
$$= \begin{bmatrix} I - \sigma_\theta^2 s_\theta & 0 & 0 \\ 0 & I_2 - (\sigma_{xy}^2 * I_2)s & \sigma_{xy}^2 s \\ 0 & -\sigma_m^2 s & I_{2N} - (\sigma_m^2 * I_{2N})s \end{bmatrix} \tag{26}$$

The eigenvalues of the above matrix is obtained as

$$\begin{bmatrix} 1 \\ 1 - \frac{\sigma_{\theta vm}}{2} \mp \frac{1}{2}[\sigma_{\theta m}^2 - \sigma_\theta(\sigma_{xy}^2 * I_2) - \sigma_\theta(\sigma_m^2 * I_{2N})]^{\frac{1}{2}} \end{bmatrix} \tag{27}$$

where $\sigma_{\theta vm} = \sigma_\theta^2 + (\sigma_{xy}^2 * I_2) + (\sigma_m^2 * I_{2N})$ and $s, s_\theta \neq 0$. Shown above, there exist at least an eigenvalue leading to partial stability and there are two other solutions for each sign of the second term. In addition, $(I - K_k H_i)$ secure that there exist a steady state solution for the Differential Riccati Equation(DRE). □

Lewis et.al[21] guaranteed that if $I - K_k H_i$ is stable, then the DRE of HF is converging. This result shows the same structure to Calleja et.al[8] analysis. We also found that most of the results from Dissayanake et.al[3] in linear cases are applicable for HF. These are due to the preceding explanations of both filters stated in Eqs.(12)-(15). The system gain and covariance characteristics explicitly shows HF behavior towards SLAM. To determine this, we investigate

its properties especially about its the convergence characteristics. From this point forward, we present those results consisting about HF convergence properties.

Theorem 1. *Assumes that Assumptions 1~3 are satisfied. For $\gamma > 0$, the predicted state error of a robot observing landmarks is gradually decreasing if observations are frequently made by a robot. Moreover, this is achieved if and only if $R_k < \gamma^2$.*

Proof. This property is analyze using Eqs.(18)-(19). Consider a linear case SLAM problem. Let $\nabla f_X = I$ and the observation noise covariance, $R_k > 0$. As the initial state covariance is $P_0 = P_0^T > 0$, from both equations, it shows that if $R_k < \gamma^2$, then $H_i^T R_k^{-1} H_i - \gamma^{-2} I \geq 0$. Hence,

$$P_k \quad = (P_0^{-1} + H_i^T R_k^{-1} H_i - \gamma^{-2} I)^{-1} \leq P_0 \tag{28}$$

$$P_{k+1} = (P_k^{-1} + H_i^T R_k^{-1} H_i - \gamma^{-2} I)^{-1} \leq P_k \tag{29}$$

If the robot observes a landmark for some time, then the updated Eq.(29) simultaneously become smaller and therefore result in nearer estimation to the true value. From the positive semidefinite(PsD) properties, it is known that the map state covariance $P_{k_{mm}}$ also exhibits the same properties.

$$P_{k_{mm}} \quad = (P_{0_{mm}}^{-1} k + H_i^T R_k^{-1} H_i - \gamma^{-2} I)^{-1} \leq P_{0_{mm}} \tag{30}$$

$$P_{k+1_{mm}} = (P_{k_{mm}}^{-1} + H_i^T R_k^{-1} H_i - \gamma^{-2} I)^{-1} \leq P_{k_{mm}} \tag{31}$$

Subsequently, the map state covariance is also decreasing. Therefore, as presented above, the whole state covariance matrix becomes smaller after a sufficient observations if and only if, $H_i^T R_k^{-1} H_i - \gamma^{-2} I \geq 0$. This condition plays an important role in HF -SLAM. □

Theorem 1 had demonstrated that the whole state covariance matrix is decreasing if it is satisfying $H_i^T R_k^{-1} H_i - \gamma^{-2} I \geq 0$. This expression describes its importance in HF estimation. Bolzern et.al[22] suggested some requirements to guarantee HF convergence in some conditions. In *Theorem 6*[22], HF is converging only if for some given positive scalar ε such that the initial state covariance P_0 satisfies the following.

$$0 \leq P_0 \leq P^s + (Y + \varepsilon I)^{-1}$$

where $Y = Y^T$ is the solution of Lyapunov function and P^s is the stabilizing solution. However, technically it is difficult to determine ε. Even more, this equation equivalently means that if no solution is available, then we should increase γ. By increasing γ, a solution is guaranteed to exist as it closer to EKF behavior. Besides that, by calculating P^s online, the equation leads to computational cost. As a result designer must first compute the stabilizing solution with consideration of the noises parameters, initial state covariance while at the same guaranteeing the ratio of estimation error to the noises distribution is less than γ. Thereby, we suggest very general requirements to apply HF-based SLAM which require perceiving only the behavior of updates via Eq.(28) and satisfying $R_k < \gamma^2$ in the system[26].

Corollary 1. *Given that $P_0 = P_0^T > 0$. For each observation, if $R_k > \gamma^2 I$, the whole state covariance becomes a negative definite matrix or indefinite and lead to unstable estimations.*

Proof. Assume that in each observation, $H_i^T R_k^{-1} H_i - \gamma^{-2} I < 0$. For $P_0 > 0$, especially in a case when the robot is stationary, then for $R_k > \gamma^2$, the following can be obtained.

$$P_k \quad = (P_0^{-1} k + H_i^T R_k^{-1} H_i - \gamma^{-2} I)^{-1} < 0 \tag{32}$$

$$P_{k+1} = (P_k^{-1} + H_i^T R_k^{-1} H_i - \gamma^{-2} I)^{-1} > P_k \tag{33}$$

Subsequently, the whole state covariance matrix becomes a negative definite matrix and continues to increase to negative value that is unacceptable behavior in SLAM. □

Corollary 1 generally verifies that the estimation may turn out to be insufficient if *Theorem 1* is not satisfied and become worst if $P_0 \gg 0$ as stated in Eq.(32). Besides, *Corollary 1* explains that HF requires the designer to carefully choose the observation noise

and some other parameters. Moreover, γ must be tuned properly to ensure that a PsD matrix is preserved in each robot observation updates. *Finite Escape Time* is one of the problems of HF. To avoid this problem, some conditions are proposed as follows.

Theorem 2. *There exists a solution to HF-SLAM problem if and only if the followings are achieved.*

- *The observation noise, $R_k << \gamma^2 I$.*
- *Theorem 1 is satisfied.*
- *State covariance matrix is always a PsD in each observation.*

If else, the estimation is unreliable and exhibits Finite Escape Time phenomena.

Proof. To examine these conditions, the HF-SLAM algorithm in *Section 2.2* is referred. The observation noise must satisfied $R_k < \gamma^2 I$, as this will guarantee that Eq.(18) exhibit a PsD matrix. To demonstrate this, we consider a uni-dimensional robot case which is a robot with single position information observing a single landmark coordinate to show that the solution for HF-SLAM is exist. From Eq.(17), for $P_0 > 0$ and $\gamma^2 > R_k$, we obtained the following equations.

$$\psi_k = I + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} R_k^{-1} \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} - \begin{bmatrix} \gamma^{-2} & 0 \\ 0 & \gamma^{-2} \end{bmatrix} P_0 \tag{34}$$

$$= \begin{bmatrix} 1 + (R_k^{-1} - \gamma^{-2})P_{vv} & -R_k^{-1}P_{mm} \\ -R_k^{-1}P_{vv} & 1 + (R_k^{-1} - \gamma^{-2})P_{mm} \end{bmatrix} \tag{35}$$

P_{vv} and P_{mm} are the robot and landmarks state covariance respectively. From the properties of a PsD matrix, a determinant of its matrix is non-negative. Therefore, it is sufficient to show that Eq.(35) determinant is non-negative. The determinant of Eq.(35) yields

$$\psi_k = P_{vv}P_{mm} + [2(R_k^{-1} - \gamma^{-2}) - 2R_k^{-1}\gamma^{-2} + \gamma^{-4}]P_{vv}P_{mm} \tag{36}$$

To ease calculation, choose $\gamma^2 >> R_k$. Then we obtained that

$$\psi_k \rightarrow P_{vv}P_{mm} + 2R_k^{-1}P_{vv}P_{mm} \geq 0 \tag{37}$$

As a result, ψ_k is always a PsD. Hence, the linearized state covariance yield

$$\psi_k \quad = I + (H_i^T R_k^{-1} H_i - \gamma^{-2}I)P_k \geq 0 \tag{38}$$

$$P_{k+1} = F_k P_k \psi_k^{-1} F_k + GQ_k G^T \geq 0 \tag{39}$$

Note that, $\gamma^2 I >> R_k$ is a general condition that must be selected to achieve a reliable estimation. If else, it exhibits a non PsD matrix. *Theorem 1* has shown that if $H_i^T R_k^{-1} H_i - \gamma^{-2}I \geq 0$, then the state covariance is decreasing. Finally the state covariance matrix is converging to some steady state value. If Eq.(17) exhibits a non PsD matrix, then the estimation becomes inaccurate as shown by *Corollary 1*. This is also a sign of *finite escape time*. $\qquad\square$

We now moves to investigate the convergence behavior of HF when a robot is observing some landmarks. It is often useful to examine the state covariance matrix since it stands for the whole uncertainties characteristics about a system. S.Huang et.al[3],[23] have analyzed Kalman Filter-Based SLAM convergence properties. In HF, the convergence properties of a robot observing landmarks are still unavailable. This is due to HF is a new approach in SLAM and a very few researchers have applied the filter as a solution to the problem. Hence we obtained the following theorem to understand the HF estimation behavior in SLAM problem especially in unknown noise statistics but bounded.

Theorem 3. *For a stationary robot observing a stationary i^{th} landmark, we assume more than n-times(n > 0) observations are made. For a given $\gamma > 0$, if n goes to infinity, the whole covariance matrix is converging to*

$$P_m^\infty \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{40}$$

Proof. 2D robot with initial covariance matrix P_0 is considered to examine the above theorem and P_0 is given by the following,

$$P_0 = \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_m^2 \end{bmatrix} \tag{41}$$

where $\sigma_v \in \mathbb{R}^3$ is the robot initial state covariance and $\sigma_m \in \mathbb{R}^{2i}$ is the landmark initial state covariance. Assume that the stationary robot is observing one landmark i. From Eq.(17), when the robot is observing i landmark n times, and if $R_k = \sigma_r^2 I$, then we obtain the following. Here, I is an identity matrix with an appropriate dimension.

$$\begin{aligned} \psi_k &= I + n(H_i^T R_k^{-1} H_i - \gamma^{-2} L_k^T L_k) P_k \\ &= I + n \begin{bmatrix} R_k^{-1} - \gamma^{-2} I & R_k^{-1} \\ R_k^{-1} & R^{-1k} - \gamma^{-2} I \end{bmatrix} \begin{bmatrix} \sigma_{v_k}^2 & \sigma_{vm_k} \\ \sigma_{mv_k} & \sigma_{m_k}^2 \end{bmatrix} \\ &= \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \end{aligned} \tag{42}$$

where each element in Eq.(42) is shown by

$$\rho_{11} = I + n(R_k^{-1} - \gamma^{-2})\sigma_{v_k}^2, \qquad \rho_{12} = (nR_k^{-1})\sigma_{m_k}^2$$
$$\rho_{21} = (nR_k^{-1})\sigma_{v_k}^2, \qquad\qquad \rho_{22} = I + n(R_k^{-1} - \gamma^{-2})\sigma_{m_k}^2$$

Finding the inverse matrix of Eq.(42) by the Matrix Inversion Lemma yields

$$\psi_k^{-1} = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \tag{43}$$

where

$$\psi_{11} = [1 + n(R_k^{-1} - \gamma^{-2})\sigma_v^2 - nR_k^{-1}\sigma_m^2 \times (1 + n(R_k^{-1} - \gamma^{-2}))^{-1} R_k^{-1} \sigma_v^2]^{-1}$$
$$\psi_{12} = -\psi_{11} nR_k^{-1}\sigma_m^2 [1 + n(R_k^{-1} - \gamma^{-2})]^{-1}$$
$$\psi_{21} = \psi_{12}^T$$
$$\psi_{22} = [1 + n(R_k^{-1} - \gamma^{-2})]^{-1} + [1 + n(R_k^{-1} - \gamma^{-2})]^{-1} \times nR_k^{-1}\sigma_m^2 \psi_{11} nR_k^{-1}\sigma_m^2 [1 + n(R_k^{-1} - \gamma^{-2})]^{-1}$$

As $n \to \infty$ and for convenience assume that the correlation terms are very small and therefore, can be neglected. It then yields

$$\psi_\infty^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{44}$$

Substituting Eq.(44) into Eq.(16), finally we obtain that if $n-$observations are made, the state covariance P^∞ becomes as

$$\begin{aligned} P^\infty &= P_k \psi_\infty^{-1} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

□

It is recognizable that in the limit, the whole state covariance matrix is decreasing. We discovered that, the designer must satisfy *Theorem 1, Theorem 2*. Otherwise, the results may not be expected. Although above result encourages good estimation for the robot and landmark positions, the true landmark location is still unknown[23]. Even more, similar to Kalman Filter, if bigger magnitude of noises are applied to the filter, then oscillations becomes slightly bigger. Hence it may give an impact to the overall estimations.

In normal application, the robot has to observe more than a landmark to increase its confidence about its surroundings. We further study the convergence properties when the non-moving robot was observing two stationary landmarks. Let σ_v be robot state covariance and σ_m as landmark state covariance. Besides, assume that $\sigma_{vm} = \sigma_{mv}, \sigma_{v\bar{m}} = \sigma_{\bar{m}v}$ as the correlation between robot and landmarks. Then the following rule is propose.

Proposition 1. *For a stationary robot observing two stationary landmarks, m and $\bar{m}$ with $\gamma > 0$, as more n times observations are made, the whole covariance matrix is converging to*

$$P_m^\infty = \begin{bmatrix} \sigma_v^2 & \sigma_v\sigma_m & \sigma_v\sigma_{\bar{m}} \\ \sigma_m\sigma_v & \sigma_m^2 & \sigma_m\sigma_{\bar{m}} \\ \sigma_{\bar{m}}\sigma_v & \sigma_{\bar{m}}\sigma_m & \sigma_{\bar{m}}^2 \end{bmatrix} \tag{45}$$

In the limit, the covariance matrix finally becomes

$$P_m^\infty = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{46}$$

Proof. The proof is similar to the proof of *Theorem 3* and it is omitted. □

These results are extendable only for a single robot observing a number of landmarks. Note that, the map covariance for landmark 1 and landmark 2 are different and varied to each other, especially whenever the initial covariance matrix is not equal to zero and if and only if the uncertainties are differ from each other. As a conclusion, the uncertainties of those two landmarks are differ to each other and therefore contributes about of its dependence to the initial covariance.

4. Experimental Results

4.1. Stationary robot observing landmarks

Experiments have been conducted by using an e-puck robot to observe some point landmarks in order to evaluate above proposed analysis. We assume that the landmarks are not occluded to each other and are available in sensor coverage. The process noise is assumed to be very small than the measurement noise and both noises are able to represent our environment conditions. Table 1 shows the experimental parameters.

Table 1   EXPERIMENTAL PARAMETERS

Process noise distribution,Q	$1 \times 10^{-7} * diag(I_3)$
Observation noise distribution,R	0.05 (for each r_i, θ_i)
Observation noise(uniform noise),R	$\begin{bmatrix} R_{\theta_{max}} = 0.1 \\ R_{\theta_{min}} = -0.08 \\ R_{distance_{max}} = 0.3 \\ R_{distance_{min}} = 0.1 \end{bmatrix}$
Initial covariance, P_0(stationary)	$P_{vv} = 1 \times 10^{-5} * I_3$ $P_{mm} = 1 \times 10^5 * I_{2m}$
Initial covariance, P_0(moving)	$P_{vv} = 6 \times 10^{-3} * I_3$ $P_{mm} = 6 \times 10^{-2} * I_{2m}$
γ(stationary)	0.9
γ(moving)	8
Sampling time	0.1s

For a stationary robot case, two landmarks are defined at two positions at $(50, 0)$ and $(60, 0)$ in millimeters(mm) respectively while the robot is at $(0,0)$. The landmarks are assume as point landmarks with respect to a global coordinate system. See Fig.3 that presents the experiment environment. From Fig.4, it is easy to identify that the HF estimates very well for its position and landmarks. From the sensor's measurements, HF performs well in both landmark inference with reliable convergence results and thus proves *Theorem 3*. Besides that, no *finite escape time* phenomena is perceived during experiment. As the robot is stationary at (0,0), Fig.4(a) equivalently means that the estimation error is also decreasing and approximating the true value of (0,0) even though the associated figure is not mentioned here. Note that, HF estimation converges faster than KF especially for the landmarks estimations. The same characteristics can be interpreted from Fig.4(b) for the landmarks estimation error where both landmark 1 and 2 estimation error are monotonically decreasing as more observations are made.

On the other hand, when $\gamma << R$, HF performance is incapable of achieving better results than Kalman Filter(see Fig.5). This result is caused by ineffective estimation about robot localization and landmark's estimation. Consequently, this proves *Theorem 1* and *Theorem 2*
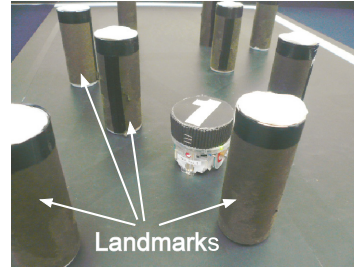
Fig. 3   Stationary Epuck observing landmarks
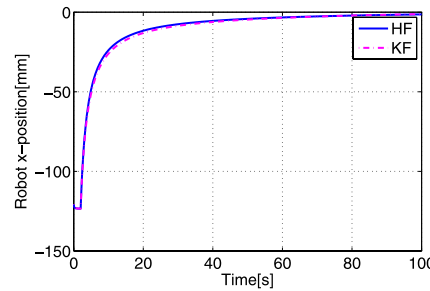

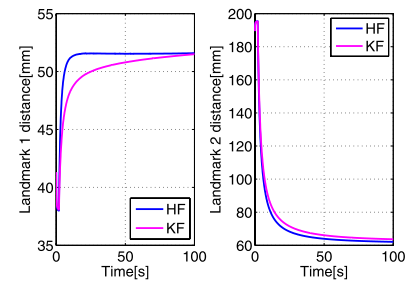Fig.4(a) Robot position estimation

Fig.4(b) Landmark 1 estimation

Fig. 4   HF estimation about robot and landmarks position

when some conditions are not satisfied. The situation also synonymously means that the estimation errors are keep increasing during observations. In linear case, this is happened especially when the initial state covariance $P_0 >> 0$ and if (28) or (29) exhibits a negative definite matrix where $R_k^{-1} - \gamma^{-2}I < 0$. Therefore, this situation result in $P_0^{-1} + R_k^{-1} - \gamma^{-2}I < 0^{(26)}$. Remark that commonly in SLAM, the initial state covariance holds large value as the robot does not know where it is located in the environment before performing any given tasks.

4.2. Moving robot observing landmarks

The experiment for a case of moving robot observing landmarks is conducted to evaluate the consistency of HF in SLAM in both gaussian and non-gaussian noises characteristics. Figs.6-7 demonstrates that HF is more robust especially in **robot pose angle estimation** than EKF. Even if the resulted covariance is slightly higher than EKF state covariance, HF has the best solution for the respective case. These results also agrees with preceding findings regarding the state covariance update(See Bolzern et.al[22] and Eqs.(12)-(15) for better comprehension). Furthermore, these results shows that HF converges faster than EKF about the position and consistent to our previous results[24]. The EKF robot path estimation is become inconsistent especially when robot turns(See Fig.6(a) and Fig.7). Subsequently, the landmarks estimation becomes faulty due to EKF inconsistent estimation. Remark that robot angle is important to achieve better performance in SLAM[23]. A slight difference in angle estimation leads to different estimation for the landmarks positions.

Considering uniform noise statistic, HF still surpassed EKF estimations(again refer to Fig.7 when robot turns). The HF robot path estimation gives better result than EKF with better robot pose angle estimation. Observes the figure at the first time when robot turns and when robot finished estimating its surroundings. In addition, in an experiment when the robot initial covariance is 1×10^2 while landmarks initial covariance is 1×10^4, HF again shows good performance with $\gamma = 12$(see Fig.8). In most of the observations, HF estimation surpasses EKF estimation. At the beginning of estimation EKF shows big error while HF estimation keeps near to the true value for both robot path and landmarks locations. Thus, these results
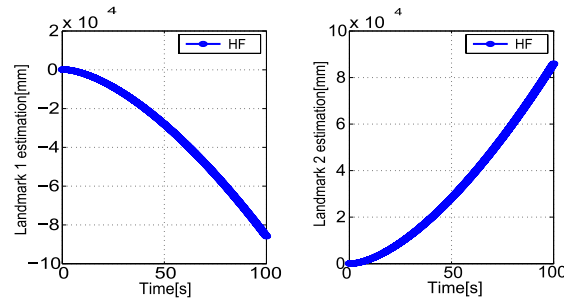
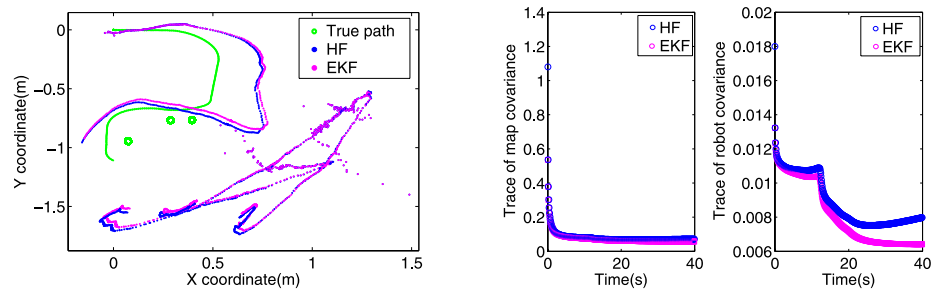Fig. 5   Landmark estimation: Effect when $R \gg \gamma^2$

Fig.6(a) Biased gaussian noise

Fig.6(b) State Covariance behavior

Fig. 6   Built map under HF Estimation under biased gaussian noise and its associated
state covariance

inspires further development for HF while at the same time guaranteeing HF as a solution to
the SLAM problem.

4.3. Discussion

The results have shown that HF is very useful and competent to EKF performance especially in a case of non-Gaussian noise characteristics. Instead of using Particle Filter(PF), which suffers from computational cost and requires larger data storage, HF can be a best candidate for SLAM problem as it holds almost the same characteristics to Kalman Filter with lower computational cost and memory storage. Furthermore, in our best knowledge, PF-based SLAM still is incapable to be employed successfully online. Thus, the implementation of HF in SLAM is a very considerable approach. Nevertheless, its performance in a case of an environment with unknown noise characteristics is a notable achievement, which enable a simpler approach than the PF.

5. Conclusions

It has been shown that HF is able to be a complement filter for unknown noise statistics in SLAM. Besides that, HF is applicable in linear and non-linear system that has variety of noises and useful for SLAM problem. We demonstrated that state covariance of HF is converging if the robot keeps observing any landmarks. However, there are some circumstances to be considered to ensure that the estimation results achieved the expected performance. If these conditions are ignored, then the estimation result in erroneous estimation and probably exhibits *Finite Escape Time* problem which is undesirable situation in SLAM. In addition, the experimental results showed that HF is a good selection for an environment with unknown noise statistics. Even though it seems that the estimation is almost the same to Kalman Filter, HF improved better about the robot angle estimation which is very important in SLAM as stated by S.Huang et.al. However, designer should design an appropriate level of weighting noise Q_k, and R_k to achieve a certain level of performance in HF. In addition, by applying the
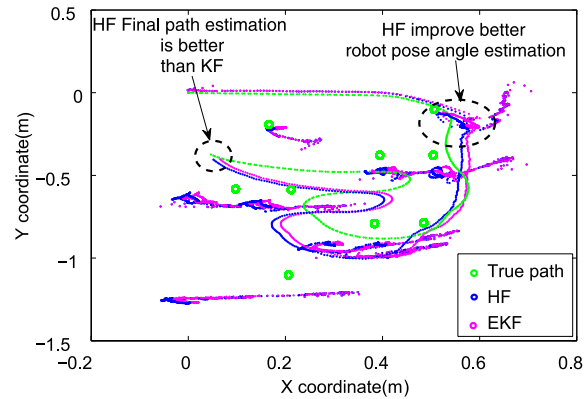
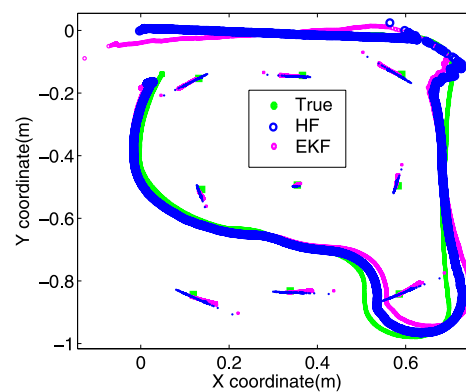Fig. 7 HF and EKF Filter built map under uniform noise characteristics

Fig. 8 Estimation with bigger initial covariance

HF-SLAM, the computation cost is preserved and better than the PF approach which suffers from computational cost and difficult to apply in real-time SLAM.

Acknowledgements

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