

# Criterion of applicable models for planar type Cherenkov laser based on quantum mechanical treatments

メタデータ	言語: eng 出版者: 公開日: 2017-10-03 キーワード (Ja): キーワード (En): 作成者: メールアドレス: 所属:
URL	<a href="http://hdl.handle.net/2297/34654">http://hdl.handle.net/2297/34654</a>

# Criterion of Applicable Models for Planar Type Cherenkov Laser Based on Quantum Mechanical Treatments

Minoru Yamada<sup>1,a)</sup> and Hesham Fares<sup>1,2)</sup>

1) Faculty of Electrical and Computer Engineering, Institute of Science and Engineering  
Kanazawa University, Kakuma-machi, Kanazawa, 920-1192, Japan

2) Department of Physics, Faculty of Science, Assiut University, Assiut 71516, Egypt

a) E-mail: mayamada@t.kanazawa-u.ac.jp

## ABSTRACT

A generalized theoretical analysis for amplification mechanism in the planar-type Cherenkov laser is given. An electron is represented to be a material wave having temporal and spatial varying phases with finite spreading length. Interaction between the electrons and the electromagnetic (EM) wave is analyzed by counting the quantum statistical properties. The interaction mechanism is classified into the Velocity and Density Modulation (VDM) model and the Energy Level Transition (ELT) model basing on the relation between the wavelength of the EM wave and the electron spreading length. The VDM model is applicable when the wavelength of the EM wave is longer than the electron spreading length as in the micro-wave region. The dynamic equation of the electron, which is popularly used in the classical Newtonian mechanics, has been derived from the quantum mechanical Schrödinger equation. The amplification of the EM wave can be explained basing on the bunching effect of the electron density in the electron beam. The amplification gain and whose dispersion relation with respect to the electron velocity is given in this paper. On the other hand, the ELT model is applicable for the case that the wavelength of the EM wave is shorter than the electron spreading length as in the optical region. The dynamics of the electron is explained to be caused by the electron transition between different energy levels. The amplification gain and whose dispersion relation with respect to the electron acceleration voltage was derived on the basis of the quantum mechanical density matrix.

## I. INTRODUCTION

The electromagnetic (EM) wave can be emitted or amplified by traveling electron beam as has been realized in the traveling-wave tube, free-electrons laser, and Cherenkov laser [1-15]. We can expect a very wide frequency range of the EM wave, such as from the micro-wave to the X-ray regions, for the operation based on the interaction between the EM wave and the electron beam.

Authors group has presented theoretical analyses to investigate the optical emission and amplification in the planar-type Cherenkov laser basing on the quantum mechanical treatment. In these analyses [16,17], the electron is represented to be spatially spreading wave, and experimentally observed the optical emission with the electron acceleration voltage of around 40 kV [18]. We also estimated the spreading length of an electron wave in our experiment to be 20 to 40  $\mu\text{m}$  by comparing the experimentally obtained emission profile with theoretical analysis [18,19].

On the other hand, there are many theoretical analyses on the interaction between the EM wave and the electron beam. Almost all of these analyses are based on the classical treatment, where the electron is regarded as a spatially localized point particle [1-14,20-22]. Then our quantum mechanical analyses as in Refs.[16-19] seem different from the analyses based on the classical mechanics.

In this paper, we show that both analytical models based on the classical mechanics and the quantum mechanics can be derived from identical quantum statistical treatment. We also confirm that a criterion for the range of applicability of both models is determined by the relation between the wavelength of the EM wave and the spreading length of the single electron. Our analyses are limited to the non-relativistic regime which is well applicable when the electron velocity is slower than  $c/3$ .

The organization of this article is as follows. In Sec. II, excitation of the EM wave by the electron current is formulated basing on the classical Maxwell's equations. In Sec. III, quantum statistical representation of the electron dynamics is given. The electron is represented as a wave which has finite spreading length. The dynamic model for the electron is classified into Velocity and Density Modulation model (VDM model) and Energy Level Transition model (ELT model) according to the relation between the wavelength of the EM wave and the spreading length of the single electron. In Sec. IV, the amplification mechanism in the VDM model is analyzed. The famous dynamic equation of the electron motion in the classical mechanics is derived from the quantum mechanical Schrödinger equation. In Sec. V,

the density matrix method is applied for the ELT model and amplification of the EM wave is analyzed. In Sec. VI, applicable wavelength ranges of both the VDM and ELT models are summarized basing on numerical calculations. Conclusions of this paper are given in Sec. VII.

## II. EXCITATION OF THE ELECTROMAGNETIC WAVE BY THE ELECTRON BEAM CURRENT

Configuration of the planar Cherenkov laser is shown in Fig.1, where a dielectric planar waveguide having high refractive index and an electron gun are set in a vacuum chamber. The electron beam is aligned to be parallel to the surface of the dielectric planar waveguide which is designed to penetrate one part of the guiding EM wave into the vacuum region. If we put the input EM wave, the laser works as an amplifier, and if there is no input light the laser works an EM emitter.

Excitation of the EM wave by the electron beam is formulated from the classical Maxwell's equations. Variation of the electric field  $\mathbf{E}$  is given by

$$\nabla^2 \mathbf{E} - \mu_0 \sigma_i \frac{\partial \mathbf{E}}{\partial t} - \mu_0 \varepsilon_i \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial \mathbf{J}}{\partial t} + \frac{1}{\varepsilon_i} \nabla \rho_c, \quad (1)$$

where  $\mathbf{J}$  is the current density of the electron beam and  $\rho_c$  is the charge density in the electron beam as well as in the waveguide.  $\varepsilon_i$  and  $\sigma_i$  are the dielectric constant and the conductivity in the  $i$ -th layer of the waveguide including the vacuum region, respectively. Solution of  $\mathbf{E}$  is assumed to be given as

$$\mathbf{E} = F(t, z) \mathbf{T}(x, y) e^{j(\omega t - \beta z)} + c.c., \quad (2)$$

where  $\omega$  and  $\beta$  are the angular frequency and the propagation constant of the EM wave, respectively. The effective refractive index  $n_{eff}$  is defined with the propagation constant to be  $\beta = n_{eff} \sqrt{\mu_0 \varepsilon_0} \omega$ .  $j$  is the imaginary unit and  $c.c.$  refers to the complex conjugate of the preceding term.  $\mathbf{T}(x, y)$  is the transverse field distribution of the electric field given as a solution of

$$(\nabla^2 + \mu_0 \varepsilon_i \omega^2) \mathbf{T}(x, y) e^{-j\beta z} = 0, \quad (3)$$

with a normalization condition of

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\mathbf{T}(x, y)|^2 dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|T_y(x, y)|^2 + |T_z(x, y)|^2) dx dy = 1. \quad (4)$$

Since the amplification is achieved through excitation of  $E_z$  component with the electron beam, the Transverse Magnetic (TM) mode is used as the guiding mode in the planar

waveguide. Existing components of the TM mode are  $H_x, E_y$  and  $E_z$ , then the distribution functions  $T_y(x, y)$  and  $T_z(x, y)$  exist but  $T_x(x, y) = 0$ , eventually.  $F(t, z)$  is the field amplitude varying with respect to  $t$  and  $z$  and whose variations are much smoother than those of  $\omega$  and  $\beta$ , respectively.

We substitute Eq. (2) into Eq. (1), multiply both side of Eq. (1) by  $\mathbf{T}^*(x, y)\exp[j(\beta z - \omega t)]$ , perform the spatial integrations along the  $x$  and  $y$  directions, and take the spatial and time averages over  $\Delta z$  and  $\Delta t$  whose values are several periods of  $1/\beta$  and  $1/\omega$ , respectively. Then we obtain an equation for the variation of  $F(t, z)$  as

$$\begin{aligned} & \frac{\partial F(t, z)}{\partial z} + \frac{1}{v_f} \frac{\partial F(t, z)}{\partial t} \\ &= j \frac{\mu_o}{2\beta} \frac{1}{\Delta t} \frac{1}{\Delta z} \int_{t-\Delta t}^t \int_{z-\Delta z}^z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{\partial \mathbf{J}}{\partial t} + \frac{1}{\mu_o \epsilon_o} \nabla \rho_c \right) \cdot \mathbf{T}^*(x, y) e^{j\beta z - j\omega t} dx dy dz dt - \frac{\alpha_{\text{loss}}}{2} F(t, z) \end{aligned} \quad (5)$$

where  $v_f$  is a velocity of the EM wave given by

$$v_f = \frac{\beta}{\mu_o \omega \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \epsilon_i |\mathbf{T}(x, y)|^2 dx dy}, \quad (6)$$

and  $\alpha_{\text{loss}}$  is the guiding loss coefficient given by

$$\alpha_{\text{loss}} = \frac{\mu_o \omega}{\beta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma_i |\mathbf{T}(x, y)|^2 dx dy. \quad (7)$$

By decomposing the field amplitude into an absolute magnitude  $|F(t, z)|$  and a phase  $\phi(t, z)$  as

$$F(t, z) \equiv |F(t, z)| e^{j\phi(t, z)}, \quad (8)$$

and by substituting it into Eq. (5), we get the following equations by comparing the real and the imaginary parts on both sides of the resulted equation:

$$\begin{aligned} \frac{d|F(t, z)|^2}{dz} &= \frac{\partial |F(t, z)|^2}{\partial z} + \frac{1}{v_f} \frac{\partial |F(t, z)|^2}{\partial t}, \\ &= (g - \alpha_{\text{loss}}) |F(t, z)|^2 \end{aligned} \quad (9)$$

with

$$g = -\frac{\mu_o}{\beta} \text{Im} \left[ \frac{1}{F(t, z)} \frac{1}{\Delta t} \frac{1}{\Delta z} \int_{t-\Delta t}^t \int_{z-\Delta z}^z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{\partial \mathbf{J}}{\partial t} + \frac{1}{\mu_o \epsilon_i} \frac{\partial \rho_c}{\partial z} \right) T_z^*(x, y) e^{j\beta z - j\omega t} dx dy dz dt \right] \quad (10)$$

and

$$\begin{aligned}
\Delta\beta &\equiv -\frac{d\phi(t, z)}{dz} = -\frac{\partial\phi(t, z)}{\partial z} - \frac{1}{v_f} \frac{\partial\phi(t, z)}{\partial t} \\
&= -\frac{\mu_0}{2\beta} \text{Re} \left[ \frac{1}{F(t, z)} \frac{1}{\Delta t \Delta z} \int_{t-\Delta t}^t \int_{z-\Delta z}^z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{\partial J}{\partial t} + \frac{1}{\mu_0 \epsilon_i} \frac{\partial \rho_c}{\partial z} \right) T_z^*(x, y) e^{j\beta z - j\omega t} dx dy dz dt \right]
\end{aligned} \tag{11}$$

Here we have supposed that the current density  $\mathbf{J}$  has only  $z$  component, i.e.,  $J$ , and the charge  $\rho_c$  varies along  $z$  direction.

The term  $g$  is the gain coefficient and  $\Delta\beta$  is the change of the propagation constant due to the presence of the electron beam. When  $g$  is a positive value, the EM wave is amplified. The effective propagation constant changes from  $\beta$  to  $\beta + \Delta\beta$ , then the phase velocity  $v_{\text{ph}}$  of the EM wave becomes

$$v_{\text{ph}} = \frac{\omega}{\beta + \Delta\beta}. \tag{12}$$

Although the velocity  $v_f$  introduced in Eq. (6) is not mathematically identical to the phase velocity  $v_{\text{ph}}$ , the value of  $v_f$  is almost same with  $v_{\text{ph}}$  numerically. Also, the value of  $|\Delta\beta|$  is much smaller than  $\beta$ , then we can assume that

$$v_f \approx v_{\text{ph}} \approx \frac{\omega}{\beta} = \frac{c}{n_{\text{eff}}}. \tag{13}$$

for almost cases.

### III. QUANTUM STATISTICAL REPRESENTATION OF ELECTRON DYNAMICS

#### A. Basic definitions

Physical quantities related to the electron dynamics, such as the current density  $J$  and the charge density  $\rho_c$  in the electron beam, should be evaluated as expectation values in the quantum mechanical treatment. When many electrons contribute to the interaction, we further need to count the statistical properties using the so called quantum statistics.

We assign a number  $\nu$  for each electron in a group (an ensemble). The expectation value of any quantum mechanical operator  $A$  in the group of electrons is given by

$$\langle A \rangle = \sum_{\nu} P^{(\nu)} \langle \Psi^{(\nu)}(\mathbf{r}, t) | A | \Psi^{(\nu)}(\mathbf{r}, t) \rangle, \tag{14}$$

where  $P^{(\nu)}$  is the probability to find the  $\nu$ -th electron in the group whereas

$$\sum_{\nu} P^{(\nu)} = 1, \quad (15)$$

$|\Psi^{(\nu)}(\mathbf{r}, t)\rangle$  is the state vector of the  $\nu$ -th electron which satisfies the normalization condition of

$$\langle \Psi^{(\nu)}(\mathbf{r}, t) | \Psi^{(\nu)}(\mathbf{r}, t) \rangle = \int_{V^{(\nu)}} |\Psi^{(\nu)}(\mathbf{r}, t)|^2 d^3\mathbf{r} = 1, \quad (16)$$

where  $\Psi^{(\nu)}(\mathbf{r}, t)$  is a wave function corresponding to the state vector  $|\Psi^{(\nu)}(\mathbf{r}, t)\rangle$ , and

$\int_{V^{(\nu)}} d^3\mathbf{r}$  indicates the spatial integration over the volume  $V^{(\nu)}$  of the  $\nu$ -th electron. The

electron state  $|\Psi^{(\nu)}(\mathbf{r}, t)\rangle$  is expanded with eigen energy states  $|\varphi_m^{(\nu)}(\mathbf{r})\rangle$  in the form of

$$|\Psi^{(\nu)}(\mathbf{r}, t)\rangle = \sum_m c_m^{(\nu)}(\mathbf{r}, t) |\varphi_m^{(\nu)}(\mathbf{r})\rangle e^{-j\omega_m t}. \quad (17)$$

The eigen energy state  $|\varphi_m^{(\nu)}(\mathbf{r})\rangle$  is defined with the eigen energy value of  $W_m = \hbar \omega_m$  for

the principle Hamiltonian  $H_o$  in the relation of

$$H_o |\varphi_m^{(\nu)}(\mathbf{r})\rangle = W_m |\varphi_m^{(\nu)}(\mathbf{r})\rangle = \hbar \omega_m |\varphi_m^{(\nu)}(\mathbf{r})\rangle \quad (18)$$

with

$$H_o = \frac{\hbar^2 \mathbf{p}^2}{2m_0} = -\frac{\hbar^2}{2m_0} \nabla^2, \quad (19)$$

where  $\mathbf{p}$  is the momentum operator and  $m_0$  is the rest mass.

In the present analysis, we suppose that the wave function  $\varphi_m^{(\nu)}(\mathbf{r})$  is approximately represented with a boxlike plane wave

$$\varphi_m^{(\nu)}(\mathbf{r}) = \frac{1}{\sqrt{V^{(\nu)}}} e^{j\mathbf{k}_m \cdot (\mathbf{r} - \mathbf{r}_\nu)} \quad (20)$$

with

$$V^{(\nu)} = \ell_x^{(\nu)} \ell_y^{(\nu)} \ell_z^{(\nu)} \quad (21)$$

where  $\mathbf{r}_\nu = (x_\nu, y_\nu, z_\nu)$  is the center position of the  $\nu$ -th electron.  $\ell_x^{(\nu)}$ ,  $\ell_y^{(\nu)}$  and  $\ell_z^{(\nu)}$  are spreading lengths of the confining box in the  $x$ ,  $y$ , and  $z$  directions, respectively, as illustrated in Fig. 2. These lengths can be changed with the electron density in the electron beam. The normalization and orthogonal conditions of the electron wave are written as

$$\langle \varphi_n^{(\mu)}(\mathbf{r}) | \varphi_m^{(\nu)}(\mathbf{r}) \rangle = \int_{V^{(\nu)}} \varphi_n^{(\mu)*}(\mathbf{r}) \varphi_m^{(\nu)}(\mathbf{r}) d^3\mathbf{r} = \delta_{\mu,\nu} \delta_{n,m}. \quad (22)$$

In Eq. (14), the coefficient  $c_m^{(\nu)}(\mathbf{r}, t)$  indicates the contribution of the  $m$ -th energy state in the  $\nu$ -th electron and can be expressed as

$$c_m^{(\nu)}(\mathbf{r}, t) = |c_m^{(\nu)}(\mathbf{r}, t)| e^{j\theta_m^{(\nu)}(\mathbf{r}, t)}. \quad (23)$$

Here, we should note that the coefficient  $c_m^{(\nu)}(\mathbf{r}, t)$  has both spatial and temporal variations which are much smoother than those of  $\varphi_m^{(\nu)}(\mathbf{r})$  and  $\exp(-j\omega_m t)$ , respectively.

Since both the state vector  $|\Psi^{(\nu)}(\mathbf{r}, t)\rangle$  and the eigen energy state  $|\varphi_m^{(\nu)}(\mathbf{r})\rangle$  are normalized as given in Eqs. (16) and (22), we also get the relation of

$$\sum_m |c_m^{(\nu)}(\mathbf{r}, t)|^2 = 1. \quad (24)$$

Equation (24) implies a simple but an important characteristic that is only a single electron exists in the defined space of  $V^{(\nu)}$ . Thus, each electron is assigned with different index  $\nu$ .

Basing on the above discussions, the electron density  $N$  is given by

$$N(\mathbf{r}, t) = \sum_{\nu} P^{(\nu)} |\Psi^{(\nu)}(\mathbf{r}, t)|^2. \quad (25)$$

## B. Excitation term with the beam current

The amplification of the EM wave is derived from the terms  $\partial J / \partial t$  and  $\partial \rho_c / \partial z$  in Eq. (10) multiplied by the complex conjugate of the EM distribution function. Since the current density  $J$  is given as a spatial operator in the quantum mechanics and both the electron and the EM waves have specific spatial distributions, the quantum statistical expectation value must be evaluated over these combined functions.

We now start to determine an expectation value of  $\langle J T_z^*(x, y) e^{j\beta z} \rangle$ . The expectation value of  $\langle (\partial J / \partial t) T_z^*(x, y) e^{j\beta z} \rangle$  shown in Eqs. (10) and (11) is determined from  $\langle J T_z^*(x, y) e^{j\beta z} \rangle$  as shown in later. The current density  $J$  in quantum mechanics is given by

$$J = -\frac{e N_t}{m_o} (p_z + eA_z) \approx -\frac{e N_t}{m_o} p_z = j \frac{e \hbar N_t}{m_o} \frac{\partial}{\partial z}, \quad (26)$$

where  $N_t$  is the total electron density including all energy levels as,

$$N_t(\mathbf{r}, t) = \sum_{\nu} \sum_m P^{(\nu)} |C_m^{(\nu)}(\mathbf{r}, t)|^2 |\varphi_m^{(\nu)}(\mathbf{r})|^2 = \sum_{\nu} \frac{P^{(\nu)}}{V^{(\nu)}}. \quad (27)$$

$p_z = -j \hbar \partial / \partial z$  and  $A_z$  are  $z$ -components of the electron momentum and the EM vector potential, respectively. Since different electrons are not overlapped spatially, expectation value  $\langle J T_z^*(x, y) e^{j\beta z} \rangle$  is written as

$$\begin{aligned}
\langle J T_z^*(x, y) e^{j\beta z} \rangle &= -\frac{e}{m_0} \sum_{\nu} \frac{P^{(\nu)}}{V^{(\nu)}} \langle \Psi^{(\nu)}(\mathbf{r}, t) | p_z T_z^*(x, y) e^{j\beta z} | \Psi^{(\nu)}(\mathbf{r}, t) \rangle \\
&= j \frac{e\hbar}{m_0} \sum_{\nu} \sum_n \sum_m \frac{P^{(\nu)}}{V^{(\nu)}} \\
&\quad \times \int_{V^{(\nu)}} e^{j\omega_n t} \varphi_n^{(\nu)*}(\mathbf{r}) c_n^{(\nu)*}(\mathbf{r}, t) \frac{\partial}{\partial z} \left\{ T_z^*(x, y) e^{j\beta z} c_m^{(\nu)}(\mathbf{r}, t) \varphi_m^{(\nu)}(\mathbf{r}) e^{-j\omega_m t} \right\} d^3\mathbf{r}
\end{aligned} \tag{28}$$

Here, we focus our attention on the two cases of  $n = m$  and  $n \neq m$  regarding the double summations over energy levels  $n$  and  $m$  in Eq. (28). The first case of  $n = m$  corresponds to a dynamic motion of an electron without making any transition from the initial energy level  $m$  to other energy levels. The latter case of  $n \neq m$  indicates an electron transition between different energy levels  $n$  and  $m$ .

In the former case, from Eq. (23), we get a relation of

$$\frac{\partial c_m^{(\nu)}(\mathbf{r}, t)}{\partial z} = \frac{\partial}{\partial z} \left\{ |c_m^{(\nu)}(\mathbf{r}, t)| e^{j\theta_m^{(\nu)}(\mathbf{r}, t)} \right\} = j \frac{\partial \theta_m^{(\nu)}(\mathbf{r}, t)}{\partial z} |c_m^{(\nu)}(\mathbf{r}, t)| e^{j\theta_m^{(\nu)}(\mathbf{r}, t)} \quad \text{when } n = m, \tag{29}$$

where the condition of  $|c_m^{(\nu)}(\mathbf{r}, t)| = 1$  is used because the electron never transits to other energy levels. Then the integrand in Eq. (28) becomes

$$\begin{aligned}
&e^{j\omega_n t} \varphi_n^{(\nu)*}(\mathbf{r}, t) c_n^{(\nu)*}(\mathbf{r}, t) \frac{\partial}{\partial z} \left\{ T_z^*(x, y) e^{j\beta z} c_m^{(\nu)}(\mathbf{r}, t) \varphi_m^{(\nu)}(\mathbf{r}) e^{-j\omega_m t} \right\} \\
&= \left( \beta + \frac{\partial \theta_m^{(\nu)}}{\partial z} + k_m \right) |\varphi_m^{(\nu)}(\mathbf{r})|^2 T_z^*(x, y) e^{j\beta z}
\end{aligned} \tag{30}$$

In the latter case of  $n \neq m$ , the electron dynamics is regarded as an electron transition between different energy levels in the defined space of  $V^{(\nu)}$ . Then the temporal variation of the coefficient  $c_m^{(\nu)}(\mathbf{r}, t)$  has to be taken into account while the spatial distribution of this coefficient is neglected, such as

$$\frac{\partial c_m^{(\nu)}(\mathbf{r}, t)}{\partial z} = 0 \quad \text{when } n \neq m \tag{31}$$

By the help of Eqs. (29)-(31), Equation (28) can be rewritten as

$$\langle J T_z^*(x, y) e^{j\beta z} \rangle = \langle J_{\text{vdm}} T_z^*(x, y) e^{j\beta z} \rangle + \langle J_{\text{elt}} T_z^*(x, y) e^{j\beta z} \rangle, \tag{32}$$

with

$$\langle J_{\text{vdm}} T_z^*(x, y) e^{j\beta z} \rangle = -\frac{e\hbar}{m_0} \sum_{\nu} \frac{P^{(\nu)}}{V^{(\nu)}} \int_{V^{(\nu)}} \left( \beta + \frac{\partial \theta_m^{(\nu)}}{\partial z} + k_m \right) |\varphi_m^{(\nu)}(\mathbf{r})|^2 T_z^*(x, y) e^{j\beta z} d^3\mathbf{r}, \tag{33}$$

and

$$\begin{aligned} \left\langle J_{\text{elt}} T_z^*(x, y) e^{j\beta z} \right\rangle = & -\frac{e}{m_0} \sum_v \sum_{n \neq m} \sum_m \frac{P^{(v)}}{V^{(v)}} c_n^{(v)*}(t) c_m^{(v)}(t) e^{j(\omega_n - \omega_m)t} \\ & \times \int_{V^{(v)}} \varphi_n^{(v)*}(\mathbf{r}) p_z T_z^*(x, y) e^{j\beta z} \varphi_m^{(v)}(\mathbf{r}) d^3\mathbf{r} \end{aligned} \quad (34)$$

The first term  $\left\langle J_{\text{vdm}} T_z^*(x, y) e^{j\beta z} \right\rangle$  corresponds to modulations of the electron velocity and density by the EM field as treated with the classical mechanics. Interaction mechanism induced by this term is named Velocity and Density Modulations model (VDM model) in this paper. The second term  $\left\langle J_{\text{elt}} T_z^*(x, y) e^{j\beta z} \right\rangle$  is caused by the electron transition between different electron energy levels. Then, interaction mechanism induced by the second term is named Energy Level Transition model (ELT model) in this paper.

### C. Effect of the charge term

Charges must exist not only in the electron beam but also in the waveguide in the form of the holes or the positive ions due to the mirror (the image) effect from the electrons in the beam. However, we make a classification here based on properties in the electron beam.

The charge density  $\rho_c$  in the electron beam is given as

$$\rho_c(\mathbf{r}, t) = -e N(\mathbf{r}, t) \quad (35)$$

where  $N(\mathbf{r}, t)$  is the electron density given by a square value of the electron wave function as shown in Eq. (25). Then the electron density is also divided into two components of  $N_{\text{vdm}}$  and  $N_{\text{elt}}$ .  $N_{\text{vdm}}$  is related to the diagonal elements of the energy eigen functions corresponding to the VDM model, while  $N_{\text{elt}}$  is related to the off-diagonal elements in the crossing term of the electron wave functions corresponding the ELT model, such as

$$N(\mathbf{r}, t) = N_{\text{vdm}}(\mathbf{r}, t) + N_{\text{elt}}(\mathbf{r}, t) \quad (36)$$

where

$$N_{\text{vdm}}(\mathbf{r}, t) = \sum_v P^{(v)} \left| \varphi_m^{(v)}(\mathbf{r}) \right|^2 = \sum_v \frac{P^{(v)}}{V^{(v)}} = N_t, \quad (37)$$

and

$$N_{\text{elt}}(\mathbf{r}, t) = \sum_v \sum_{n \neq m} \sum_m P^{(v)} c_n^{(v)*}(t) c_m^{(v)}(t) \varphi_n^{(v)*}(\mathbf{r}) \varphi_m^{(v)}(\mathbf{r}) e^{j(\omega_n - \omega_m)t}. \quad (38)$$

According to the above classification, the term with the charge in Eq. (10) is also divided into two components as

$$\left\langle \frac{\partial \rho_c}{\partial z} T_z^*(x, y) e^{j\beta z} \right\rangle = \left\langle \frac{\partial \rho_{vdm}}{\partial z} T_z^*(x, y) e^{j\beta z} \right\rangle + \left\langle \frac{\partial \rho_{elt}}{\partial z} T_z^*(x, y) e^{j\beta z} \right\rangle, \quad (39)$$

where  $\rho_{vdm}$  and  $\rho_{elt}$  are charge densities in the VDM model and the ELT model, respectively.

In the electron beam, the terms  $\rho_{vdm}$  and  $\rho_{elt}$  are given by

$$\left\langle \frac{\partial N_{vdm}}{\partial z} T_z^*(x, y) e^{j\beta z} \right\rangle = \sum_{\nu} P^{(\nu)} \int_{V^{(\nu)}} \frac{\partial N_{vdm}}{\partial z} |\varphi_m^{(\nu)}|^2 T_z^*(x, y) e^{j\beta z} d^3\mathbf{r} \quad (40)$$

and

$$\begin{aligned} \left\langle \frac{\partial N_{elt}}{\partial z} T_z^*(x, y) e^{j\beta z} \right\rangle &= jN_t \sum_{\nu} \sum_{n \neq m} \sum_m P^{(\nu)} c_n^{(\nu)*}(t) c_m^{(\nu)}(t) e^{j(\omega_n - \omega_m)t} (k_m - k_n) \\ &\times \int_{V^{(\nu)}} \varphi_n^{(\nu)*}(\mathbf{r}) \varphi_m^{(\nu)}(\mathbf{r}) T_z^*(x, y) e^{j\beta z} d^3\mathbf{r} \end{aligned} \quad (41)$$

According to the above categorizations of the current density and the electron density, the gain coefficient  $g$  and the change of the propagation constant  $\Delta\beta$  defined in Eq. (10) and (11), respectively, are calculated as a sum of two components

$$g = g_{vdm} + g_{elt}, \quad (42)$$

with

$$g_{vdm} = -\frac{\mu_0}{\beta} \text{Im} \left[ \frac{1}{F(t, z)} \frac{1}{\Delta t \Delta z} \int_{t-\Delta t}^t \int_{z-\Delta z}^z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle \left\{ \frac{\partial J_{vdm}}{\partial t} + \frac{1}{\mu_o \varepsilon_i} \frac{\partial \rho_{vdm}}{\partial z} \right\} T_z^*(x, y) e^{j\beta z} \right\rangle e^{-j\omega t} dx dy dz dt \right] \quad (43)$$

$$g_{elt} = -\frac{\mu_0}{\beta} \text{Im} \left[ \frac{1}{F(t, z)} \frac{1}{\Delta t \Delta z} \int_{t-\Delta t}^t \int_{z-\Delta z}^z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle \left\{ \frac{\partial J_{elt}}{\partial t} + \frac{1}{\mu_o \varepsilon_i} \frac{\partial \rho_{elt}}{\partial z} \right\} T_z^*(x, y) e^{j\beta z} \right\rangle e^{-j\omega t} dx dy dz dt \right] \quad (44)$$

and

$$\Delta\beta = \Delta\beta_{vdm} + \Delta\beta_{elt} \quad (45)$$

with

$$\Delta\beta_{vdm} = -\frac{\mu_0}{2\beta} \text{Re} \left[ \frac{1}{F(t, z)} \frac{1}{\Delta t \Delta z} \int_{t-\Delta t}^t \int_{z-\Delta z}^z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle \left\{ \frac{\partial J_{vdm}}{\partial t} + \frac{1}{\mu_o \varepsilon_i} \frac{\partial \rho_{vdm}}{\partial z} \right\} T_z^*(x, y) e^{j\beta z} \right\rangle e^{-j\omega t} dx dy dz dt \right] \quad (46)$$

$$\Delta\beta_{\text{elt}} = -\frac{\mu_0}{2\beta} \text{Re} \left[ \frac{1}{F(t, z)} \frac{1}{\Delta t \Delta z} \int_{t-\Delta t}^t \int_{z-\Delta z}^z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle \left\{ \frac{\partial J_{\text{elt}}}{\partial t} + \frac{1}{\mu_o \epsilon_i} \frac{\partial \rho_{\text{elt}}}{\partial z} \right\} T_z^*(x, y) e^{i\beta z} \right\rangle e^{-j\omega t} dx dy dz dt \right] \quad (47)$$

### C. Criterion for application of the VDM and ELT models

Here, we examine applicable range of the VDM and ELT models. Since we assume a finite spreading length  $\ell_z^{(\nu)}$  for a single electron, the energy levels are characterized by the relation of

$$k_m \ell_z^{(\nu)} = 2m\pi. \quad (48)$$

Then, as illustrated in Fig. 3, separations of the electron wave-numbers and the energy levels are

$$\Delta k = k_m - k_{m-1} = \frac{2\pi}{\ell_z^{(\nu)}}, \quad (49)$$

and

$$\Delta W = W_m - W_{m-1} = \frac{\hbar^2}{2m_o} (k_m^2 - k_{m-1}^2) \approx \frac{\hbar^2 k_m}{m_o} \Delta k. \quad (50)$$

As will be shown in Sec. V for the ELT model, the conditions required to induce the electron transition between different energy levels are to match the photon energy  $\hbar\omega$  and the propagation constant  $\beta$  with the separations of the energy and the wave-number of the electron, respectively, satisfying the energy and the momentum conservation rules. Then, the EM wave, whose photon energy and the propagation constant are smaller than the energy and wave-number separations, respectively, such as  $\hbar\omega < \Delta W$  and  $\beta < \Delta k$ , can not induce the electron transition in the ELT model.

However, the EM wave having  $\hbar\omega < \Delta W$  and  $\beta < \Delta k$  can modulate the electron velocity around the initial energy level. This type of interaction is analyzed in the VDM model.

The above mentioned discussions are also understood by comparing the spatial distribution of the electrons with that of the EM wave as illustrated in Fig. 4. The EM wave with the higher photon energy has shorter wavelength. The condition of  $\beta > \Delta k$  required to cause the electron transition in the ELT model corresponds to the condition of  $\lambda < \ell_z^{(\nu)}$ . Then the phase of the EM wave varies within the spreading length of a single electron  $\ell_z^{(\nu)}$ . Typical

example of the EM wave to be analyzed by the ELT model is the optical wave, because the wavelength is  $\lambda \approx 1 \mu\text{m}$  while the spreading length  $\ell_z^{(\nu)}$  of a single electron is expected to be several tens of  $\mu\text{m}$  [18,19].

The EM wave with the lower photon energy has longer wavelength ( $\lambda > \ell_z^{(\nu)}$ ). Then an electron can be treated as a point particle in comparison with the wavelength of the EM wave corresponding to the classical mechanics. Typical example to be applied the VDM model is the micro-wave region whose wavelength  $\lambda$  is longer than 1 mm. The so called electron bunching is caused in the VDM model.

Classifications and applicability of present models are summarized in Fig. 5.

#### IV. VELOCITY AND DENSITY MODULATION (VDM) MODEL

##### A. Derivation of the classical dynamic equation from the Schrödinger equation

An important feature of this section is the introduction of a phase angle  $\theta_m^{(\nu)}(z, t)$  which is a function of time  $t$  and position  $z$ . This is because the modulation of the electron velocity is characterized by the term  $\partial \theta_m^{(\nu)} / \partial z$  in Eq. (29). In this subsection, a dynamic equation for the electron motion is derived from the Schrödinger equation showing that obtained results well coincide with those derived by the classical Newtonian mechanics.

In this model, since the  $\nu$ -th electron is assumed to occupy an energy level whereas electron transitions between different energy levels are not counted, the notation of the energy level  $m$  can be dropped and replaced with  $\nu$  without loss of generality. Therefore, the electron wave function in Eq. (17) is expressed as

$$|\Psi^{(\nu)}(\mathbf{r}, t)\rangle = e^{-j\omega_\nu t + j\theta_\nu(\mathbf{r}, t)} |\varphi_\nu(\mathbf{r})\rangle. \quad (51)$$

This wave function must follow the Schrödinger equation in the form of

$$j\hbar \frac{\partial |\Psi^{(\nu)}(\mathbf{r}, t)\rangle}{\partial t} = \left( H_0 + H_{\text{int}} - j \frac{\Gamma}{2} \right) |\Psi^{(\nu)}(\mathbf{r}, t)\rangle, \quad (52)$$

where  $H_0$  is the principle Hamiltonian have been given in Eq. (19).  $H_{\text{int}}$  is the interaction Hamiltonian showing interactions between the electron and the EM wave

$$H_{\text{int}} = \frac{e}{2m_0} (p_z A_z + A_z p_z) - eU \quad (53)$$

where  $A_z$  is  $z$  component of the EM vector potential and  $U$  is the scalar potential.

$\Gamma$  is an operator indicating the relaxation effect of the electron wave [23]. The

expectation value of  $\Gamma$  is characterized with the relaxation time  $\tau$  for the electron wave, being given by

$$\langle \Psi^{(\nu)} | \Gamma | \Psi^{(\nu)} \rangle = \hbar / \tau. \quad (54)$$

The left-hand side of Eq. (52) is rewritten as

$$j\hbar \frac{\partial |\Psi^{(\nu)}\rangle}{\partial t} = \hbar \left( \omega_\nu - \frac{\partial \theta_\nu}{\partial t} \right) |\Psi^{(\nu)}\rangle. \quad (55)$$

The term with the principle Hamiltonian on the right-hand side of Eq. (52) is

$$\begin{aligned} H_o |\Psi^{(\nu)}\rangle &= -\frac{\hbar^2}{2m_o} \frac{\partial^2 |\Psi^{(\nu)}\rangle}{\partial z^2} \\ &= \frac{\hbar^2}{2m_o} \left( k_\nu + \frac{\partial \theta_\nu}{\partial z} \right)^2 |\Psi^{(\nu)}\rangle \quad . \\ &\approx \frac{\hbar^2}{2m_o} \left( k_\nu^2 + 2k_\nu \frac{\partial \theta_\nu}{\partial z} \right) |\Psi^{(\nu)}\rangle \end{aligned} \quad (56)$$

Here, to trace the phase variation, we put the vector potential  $A_z$  using

$$A_z(\mathbf{r}, t) \equiv A_o e^{j(\omega t - \beta z)} + c.c. \quad (57)$$

By applying  $|\Psi^{(\nu)}\rangle$  to Eq. (53), the term of the interaction Hamiltonian becomes

$$\begin{aligned} H_{\text{int}} |\Psi^{(\nu)}\rangle &= \frac{e\hbar}{m_o} \left( k_\nu + \frac{\partial \theta_\nu}{\partial z} \right) \left\{ A_o e^{j(\omega t - \beta z)} + c.c. \right\} |\Psi^{(\nu)}\rangle \\ &\quad - \frac{e\hbar\beta}{2m_o} \left\{ A_o e^{j(\omega t - \beta z)} - c.c. \right\} |\Psi^{(\nu)}\rangle - eU |\Psi^{(\nu)}\rangle \quad . \end{aligned} \quad (58)$$

It is worth noting here that the varying phase  $\theta_\nu(\mathbf{r}, t)$  of the electron wave affects both the principle Hamiltonian and the interaction Hamiltonian.

By substituting Eqs. (55)-(58) into Eq. (52) and using the relations of  $\hbar\omega_\nu = \hbar^2 k_\nu^2 / 2m_o$  and  $k_\nu \gg \partial \theta_\nu / \partial z$  and  $\beta$ , we get

$$\left( \frac{\partial \theta_\nu}{\partial t} + \frac{\hbar k_\nu}{m_o} \frac{\partial \theta_\nu}{\partial z} \right) |\Psi^{(\nu)}\rangle = \left\{ -\frac{ek_\nu}{m_o} A_z + \frac{e}{\hbar} U + \frac{\Gamma}{2\hbar} \right\} |\Psi^{(\nu)}\rangle, \quad (59)$$

Here, we take one more spatial derivative  $\partial / \partial z$  to Eq. (59) and drop the terms with  $\partial |\Psi^{(\nu)}\rangle / \partial z$  by using relation of

$$\frac{\partial |\Psi^{(\nu)}\rangle}{\partial z} = j \left( k_\nu + \frac{\partial \theta_\nu}{\partial z} \right) |\Psi^{(\nu)}\rangle \quad . \quad (60)$$

Then, we get the following equation

$$\left( \frac{\partial^2 \theta_\nu}{\partial z \partial t} + \frac{\hbar k_\nu}{m_0} \frac{\partial^2 \theta_\nu}{\partial z^2} \right) \left| \Psi^{(\nu)} \right\rangle = \left\{ -\frac{e k_\nu}{m_0} \frac{\partial A_z}{\partial z} + \frac{e}{\hbar} \frac{\partial U}{\partial z} + \frac{j}{2\hbar} \frac{\partial \Gamma}{\partial z} \right\} \left| \Psi^{(\nu)} \right\rangle. \quad (61)$$

We now reform Eq.(61) to a dynamic equation giving variation of the electron velocity.

The velocity  $v_\nu$  of the  $\nu$ -th electron is given with the expectation value of the momentum as

$$\langle \Psi^{(\nu)} | p_z | \Psi^{(\nu)} \rangle = \hbar \langle \Psi^{(\nu)} | \left( k_\nu + \frac{\partial \theta_\nu}{\partial z} \right) | \Psi^{(\nu)} \rangle \equiv m_0 v_\nu \quad (62)$$

Then, we get

$$\langle \Psi^{(\nu)} | \frac{\partial \theta_\nu}{\partial z} | \Psi^{(\nu)} \rangle = \frac{m_0}{\hbar} (v_\nu - \bar{v}_\nu) \quad (63)$$

with

$$\bar{v}_\nu = \frac{\hbar k_\nu}{m_0}. \quad (64)$$

On the other hand, the spatial derivative of the vector potential given by Eq.(57) can be rewritten with the temporal derivative, such as

$$\frac{\partial A_z}{\partial z} = -\frac{\beta}{\omega} \frac{\partial A_z}{\partial t} = -\frac{1}{v_{ph}} \frac{\partial A_z}{\partial t}. \quad (65)$$

Then, the first two terms in the right side of Eq. (61) is rewritten with the electric field component  $E_z$  as

$$-\frac{\hbar k_\nu}{m_0} \frac{\partial A_z}{\partial z} + \frac{\partial U}{\partial z} = \frac{\bar{v}_\nu}{v_{ph}} \frac{\partial A_z}{\partial t} + \frac{\partial U}{\partial z} \approx -E_z \quad (66)$$

giving the interaction between the electron and the EM wave, where we used the relation of  $\bar{v}_\nu \approx v_{ph}$ . Expectation value of these terms shown in Eq. (66) become

$$\langle \Psi^{(\nu)} | \left( -\frac{e k_\nu}{m_0} \frac{\partial A_z}{\partial z} + \frac{e}{\hbar} \frac{\partial U}{\partial z} \right) | \Psi^{(\nu)} \rangle = -\frac{e}{\hbar} \langle \Psi^{(\nu)} | E_z | \Psi^{(\nu)} \rangle \approx -\frac{e}{\hbar} \text{Sinc} \left( \frac{\beta \ell_z^{(\nu)}}{2} \right) E_z \quad (67)$$

where  $\text{Sinc}(x) = \text{Sin}(x)/x$  is the so-called Sinc function. In deriving Eq. (67), we assume that spatial distributions of the EM wave in the transverse x-y direction are sufficiently smooth but the variation along z direction is not neglected and is counted with the Sinc function.

$$\begin{aligned} \langle \Psi^{(\nu)} | T_z(x, y) e^{-j\beta z} | \Psi^{(\nu)} \rangle &= \int_{V^{(\nu)}} T_z(x, y) e^{-j\beta z} |\varphi_\nu(\mathbf{r})|^2 d^3\mathbf{r} \\ &\approx T_z(x, y) \text{Sinc} \left( \frac{\beta \ell_z^{(\nu)}}{2} \right) e^{-j\beta z} \end{aligned} \quad (68)$$

Finally we examine the last term in the right side of Eq. (61), that is the relaxation effect on the electron wave. Since the relaxation time operator  $\Gamma$  introduced in Eq. (52) is defined for the temporal variation of the electron wave, the spatially averaged value of  $\partial \Gamma / \partial z$  must

be zero, giving the following relation

$$\begin{aligned}
\langle \Psi^{(\nu)} | \frac{\partial}{\partial z} \Gamma | \Psi^{(\nu)} \rangle &= \int_{V^{(\nu)}} \Psi^{(\nu)*}(\mathbf{r}, t) \frac{\partial}{\partial z} \{ \Gamma \Psi^{(\nu)}(\mathbf{r}, t) \} d^3 \mathbf{r} \\
&= \int_{V^{(\nu)}} \Psi^{(\nu)*}(\mathbf{r}, t) \left\{ \frac{\partial \Gamma}{\partial z} + j \Gamma \left( k_\nu + \frac{\partial \theta_\nu}{\partial z} \right) \right\} \Psi^{(\nu)}(\mathbf{r}, t) d^3 \mathbf{r} . \\
&= 0
\end{aligned} \tag{69}$$

Hence, we get

$$\langle \Psi^{(\nu)} | \frac{\partial \Gamma}{\partial z} | \Psi^{(\nu)} \rangle = -j \left( k_\nu + \frac{\partial \theta_\nu}{\partial z} \right) \frac{\hbar}{\tau} . \tag{70}$$

By multiplying  $\langle \Psi^{(\nu)} |$  to Eq.(61) and by using above derived relations, we get a dynamic equation of the electron motion as

$$\frac{\partial \mathbf{v}_\nu}{\partial t} + \bar{\mathbf{v}}_\nu \frac{\partial \mathbf{v}_\nu}{\partial z} = -\frac{e}{m_0} \text{Sinc} \left( \frac{\beta \ell_z^{(\nu)}}{2} \right) E_z - \frac{\mathbf{v}_\nu}{\tau_\nu} , \tag{71}$$

where  $\tau_\nu = 2\tau$  is the relaxation time of the electron velocity. Equation (71) is almost same as the well-known dynamic equation directly obtained by the Newtonian classical mechanics. The exceptional difference of Eq. (71) is the term  $\text{Sinc}(\beta \ell_z^{(\nu)} / 2)$ . This term results from taking into account the spatial average of the electric field over the finite length of the electron wave  $\ell_z^{(\nu)}$ , while the electron in the classical mechanics is assumed to be a point particle (i.e.,  $\ell_z^{(\nu)} = 0$ ) giving  $\text{Sinc}(\beta \ell_z^{(\nu)} / 2) = 1$ .

## B. Dynamics of the electron velocity and density

Here, we examine the term  $\langle J_{\text{vdm}} T_z^*(x, y) e^{j\beta z} \rangle$  in Eq. (33) showing further correspondence with the classical treatment. In Eq. (33),  $(\beta + \partial \theta_n^{(\nu)} / \partial z + k_n) \hbar / m_0$  is approximately assumed to be  $\mathbf{v}_\nu$  since  $\partial \theta_n^{(\nu)} / \partial z + k_n \gg \beta$ . We define an effective velocity  $\mathbf{v}$  averaged over all electrons with the electron density as

$$\sum_{\nu} P^{(\nu)} \mathbf{v}_\nu = \mathbf{v} \tag{72}$$

and

$$\sum_{\nu} \frac{P^{(\nu)}}{V^{(\nu)}} \mathbf{v}_\nu = N_{\text{vdm}} \mathbf{v} . \tag{73}$$

Then, Eqs. (71) and (33) becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \bar{v} \frac{\partial \mathbf{v}}{\partial z} = -\frac{e}{m_0} \text{Sinc}\left(\frac{\beta \ell}{2}\right) E_z - \frac{\mathbf{v}}{\tau_v} \quad (74)$$

and

$$\langle J_{vdm} T_z^*(x, y) e^{j\beta z} \rangle = -e N_{vdm} v \text{Sinc}\left(\frac{\beta \ell}{2}\right) T_z^*(x, y) e^{j\beta z}, \quad (75)$$

where  $\ell$  is an averaged value of  $\ell_z^{(v)}$  for the group of the electron.

The contributing term to the gain coefficient  $g_{vdm}$  in Eq. (43) and the change of the propagation constant  $\Delta\beta_{vdm}$  in Eq. (46) will be given by

$$\langle \frac{\partial J_{vdm}}{\partial t} T_z^*(x, y) e^{j\beta z} \rangle = -e \left( N_{vdm} \frac{\partial v}{\partial t} + v \frac{\partial N_{vdm}}{\partial t} \right) \text{Sinc}\left(\frac{\beta \ell}{2}\right) T_z^*(x, y) e^{j\beta z}. \quad (76)$$

In the VDM model, the electrons run with the average velocity  $\bar{v}$  and are subjected to the electric force from the EM wave as given by Eq. (74). Then we put the electron velocity in the form of

$$\mathbf{v} = \bar{v} + [u(t) e^{(j\beta\bar{v}-1/\tau_v)t-j\beta z} + c.c.], \quad (77)$$

where  $u(t)$  is the amplitude of the modulated component. By substituting Eq. (77) into Eq. (74), we obtain

$$u(t) = -\frac{e}{m_0} \frac{F(t, z) T_z(x, y)}{(-j\Omega + 1/\tau_v)} \text{Sinc}\left(\frac{\beta \ell}{2}\right) [e^{(-j\Omega+1/\tau_v)t} - 1], \quad (78)$$

where

$$\Omega = \beta\bar{v} - \omega. \quad (79)$$

$\Omega$  is a relative EM wave frequency as seen by the electrons. Here, we have supposed that the interaction starts at  $t = 0$ . Then the velocity becomes

$$\mathbf{v} = \bar{v} - \left[ \frac{e}{m_0} \frac{F(t, z) T_z(x, y)}{(-j\Omega + 1/\tau_v)} \text{Sinc}\left(\frac{\beta \ell}{2}\right) \left\{ e^{j\omega t} - e^{(j\beta\bar{v}-1/\tau_v)t} \right\} e^{-j\beta z} + c.c. \right] \quad (80)$$

The modulation of the electron velocity due to the presence of the electric field induces a corresponding modulation in the electron density. The continuity equation of the electron is obtained from the Maxwell's equations of  $\nabla \cdot \nabla \times \mathbf{H} = \nabla \cdot \{ \partial \mathbf{D} / \partial t + (\sigma_o / \varepsilon_o) \mathbf{D} + \mathbf{J} \} = 0$  and  $\nabla \cdot \mathbf{D} = \rho_c$  to be

$$\frac{\partial N_{vdm}}{\partial t} = -\frac{\partial}{\partial z} (N_{vdm} v) - \frac{N_{vdm}}{\tau_N}, \quad (81)$$

where  $\tau_N (= \varepsilon_o / \sigma_o)$  is a relaxation time for temporal variation of the electron density.

The electron density  $N_{vdm}$  can be expressed to have temporal and spatial variations in

the form of

$$N_{vdm} = \bar{N} + [n(t) e^{(j\beta\bar{v}-1/\tau_N)t-j\beta z} + c.c.], \quad (82)$$

where  $n(t)$  is the amplitude of the density modulation. By substituting Eq. (82) into Eq. (81), with the help of Eq. (80), we obtain

$$n(t) = -j \frac{e\beta}{m_0} \frac{\bar{N}F(t, z)T_z(x, y)}{(-j\Omega + 1/\tau')} \text{Sinc}\left(\frac{\beta\ell}{2}\right) \left\{ \frac{e^{(-j\Omega+1/\tau')t} - 1}{(-j\Omega + 1/\tau')} - t \right\}. \quad (83)$$

and then,

$$N_{vdm} = \bar{N} - \left[ j \frac{e\beta}{m_0} \frac{\bar{N}F(t, z)T_z(x, y)}{(-j\Omega + 1/\tau')} \text{Sinc}\left(\frac{\beta\ell}{2}\right) \left\{ \frac{e^{j\omega t} - e^{(j\beta\bar{v}-1/\tau')t}}{(-j\Omega + 1/\tau')} - te^{(j\beta\bar{v}-1/\tau')t} \right\} e^{j\beta z} + c.c. \right] \quad (84)$$

Here, we have supposed more approximation for the sake of simplicity that the relaxation times of the electron velocity and the density are almost identical, i.e.,  $\tau_N = \tau_V = \tau'$ .

By substituting Eqs. (80) and (84) to Eq. (76), we get

$$\left\langle \frac{\partial J_{vdm}}{\partial t} T_z^*(x, y) e^{j\beta z} \right\rangle = - \frac{e^2 \bar{v} \beta \omega \bar{N} F(t, z)}{m_0} \text{Sinc}^2\left(\frac{\beta\ell}{2}\right) |T_z(x, y)|^2 Y(\bar{v}, t) e^{j\omega t} \quad (85)$$

with

$$Y(\bar{v}, t) = \tau'^2 \cdot \frac{1 - \{1 - (j\Omega\tau' - 1)(t/\tau')\} e^{(j\Omega\tau'-1)(t/\tau')}}{(j\Omega\tau' - 1)^2}. \quad (86)$$

In derivation of Eq.(85), the relation of  $|N_{vdm} \partial v / \partial t| \ll |v \partial N_{vdm} / \partial t|$  is used. That is the modulation of the electron density is more important than that of the velocity for the interaction as called be the bunching.

Equation (86) is a dispersion function for the electron velocity, where the imaginary and the real parts give the gain coefficient  $g_{vdm}$  and the change in the propagation constant  $\Delta\beta$ , respectively. The dispersion function is written in more unified form by taking a normalized form of  $Y(\bar{v}, t)/\tau'^2$  whose numerical examples are shown in Fig. 6(a) and (b). The dispersion function is characterized not only by the electron velocity but also by the time  $t$  from the start of the interaction and the relaxation time  $\tau'$ . The gain show peaks when the electron velocity  $\bar{v}$  is slightly faster than the EM phase velocity  $v_{ph} = \omega / \beta = c / n_{eff}$ .

### C. Effect of the charge distribution in the VDM model

The effect of the space charge on the gain coefficient in the VDM model is given in Eq. (40). In the right side of Eq. (40), the volume  $V^{(\nu)}$  is regarded to be an averaged volume with

relation of  $\sum_{\nu} P^{(\nu)} = 1$ . Then, by substitution of Eq. (84) to Eq. (40), we get

$$\left\langle \frac{\partial N_{\text{vdm}}}{\partial z} T_z^*(x, y) e^{j\beta z} \right\rangle = -\frac{e\beta^2 \bar{N}F(t, z)}{m_0} \text{Sinc}^2\left(\frac{\beta\ell}{2}\right) |T_z(x, y)|^2 Y(\bar{\nu}, t) e^{j\omega t} . \quad (87)$$

The charge effect is represented with almost same form with the term caused by the current density  $J_{\text{vdm}}$  given in Eq. (85). However, the charge effect gives an opposite sign in the amplification gain  $g_{\text{vdm}}$  of Eq. (43) in comparison with the term caused by the electron current density  $J_{\text{vdm}}$ .

We need to pay attention that the positive ions or the holes must be induced in the waveguide especially at surface of the metal or the semiconductor when these materials are used to cancel the electric flux from the electron beam as well as to release electrons accumulated at the waveguide surface. These induced positive charges work to reduce the charge effect of the electron beam. Therefore, we define a coefficient  $\kappa$  to characterize the degradation of the charge effect as

$$\kappa = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle \frac{1}{\epsilon_i} \frac{\partial \rho_c}{\partial z} T_z^*(x, y) e^{j\beta z} \right\rangle dx dy}{\iint_{\text{beam}} \left\langle \frac{1}{\epsilon_o} \frac{\partial \rho_c}{\partial z} T_z^*(x, y) e^{j\beta z} \right\rangle dx dy} . \quad (88)$$

where  $\iint_{\text{beam}} dx dy$  indicates to count the electron charges in the beam as given by Eq. (87),

while  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy$  means to count all possible charges including the positive ions or the holes in the waveguide.  $\kappa$  takes a value between 0 to 1, where the smaller value leads to the more efficient operation.

#### D. Gain coefficient and the change of the propagation constant in the VDM model

According to the above mentioned definitions and discussions, the gain coefficient  $g_{\text{vdm}}$  and the change of the propagation constant  $\Delta\beta_{\text{vdm}}$  in the VDM model are given as

$$g_{\text{vdm}} = \left\{ 1 - \kappa \left( \frac{c}{\bar{\nu}} \right)^2 \right\} \frac{e\mu_0 J_0 \omega}{m_0} \xi_{\text{vdm}} \text{Sinc}^2\left(\frac{\beta\ell}{2}\right) \text{Im}\{Y(\bar{\nu}, t)\} \quad (89)$$

$$\Delta\beta_{\text{vdm}} = \left\{ 1 - \kappa \left( \frac{c}{\bar{\nu}} \right)^2 \right\} \frac{e\mu_0 J_0 \xi_{\text{vdm}} \omega}{2m_0} \text{Sinc}^2\left(\frac{\beta\ell}{2}\right) \text{Re}\{Y(\bar{\nu}, t)\} \quad (90)$$

where  $J_0 = e\bar{N}\bar{\nu}$  is the average current density and  $\xi_{\text{vdm}}$  is a coupling coefficient,

$$\xi_{\text{vdm}} = \iint_{\text{beam}} |T_z(x, y)|^2 dx dy , \quad (91)$$

where  $\iint_{beam} dx dy$  means the spatial integration over the cross-sectional area of the electron beam.

A numerical example of the gain coefficient in the VDM model is shown in Fig.7 for a fixed value of  $\Omega \tau' = 1.0$  s, and other assumed parameters are given in the figure. The gain coefficient varies with the time up to  $t/\tau' \approx 3$  as a transient phenomenon then reaches to the steady state. Other numerical examples will be shown in Fig. 12 together with those for the ELT model.

## V. ENERGY LEVEL TRANSITION (ELT) MODEL

### A. Quantum Mechanical Density Matrix

The interaction mechanism caused by the electron transition between energy levels can be well analyzed using the quantum mechanical density matrix by which the statistical behavior of the electron group is taken into account [16,23,24].

In this section, we suppose the volume of all electrons  $V^{(\nu)}$  is identical as  $V^{(\nu)} = V = \ell^3$  assuming a balanced state and the energy levels are also common for all electrons. Then we can remove the index  $\nu$  from the eigen function to be

$$|\varphi_m^{(\nu)}(\mathbf{r})\rangle \equiv |\varphi_m(\mathbf{r})\rangle \equiv |m\rangle. \quad (92)$$

By rewriting  $N_t = 1/V$ , Eq. (34) becomes

$$\langle J_{\text{elt}} T_z^*(x, y) e^{j\beta z} \rangle = -\frac{e N_t}{m_o} \sum_{\nu} \sum_{n \neq m} \sum_m P^{(\nu)} c_n^{(\nu)*}(t) c_m^{(\nu)}(t) e^{j(\omega_n - \omega_m)t} \langle n | p_z T_z^*(x, y) e^{j\beta z} | m \rangle. \quad (93)$$

Here we define a matrix  $\rho$  whose matrix element is given by

$$\rho_{mn} = \sum_{\nu} P^{(\nu)} c_n^{(\nu)*}(t) c_m^{(\nu)}(t) e^{j(\omega_n - \omega_m)t}. \quad (94)$$

$\rho$  is different from the previously introduced charge density  $\rho_c$ , and is called the quantum mechanical density matrix [24]. Equation (94) implies that the matrix  $\rho$  can be given by

$$\rho = \sum_{\nu} |\Psi^{(\nu)}\rangle P^{(\nu)} \langle \Psi^{(\nu)}|. \quad (95)$$

Since the diagonal element  $\langle m | p_z T_z^*(x, y) e^{j\beta z} | m \rangle = 0$ , Eq. (93) is reduced to

$$\begin{aligned}
\langle J_{\text{elt}} T_z^*(x, y) e^{j\beta z} \rangle &= -\frac{e N_t}{m_o} \sum_n \sum_m \langle m | \rho | n \rangle \langle n | p_z T_z^*(x, y) e^{j\beta z} | m \rangle \\
&= -\frac{e N_t}{m_o} \sum_m \langle m | \rho p_z T_z^*(x, y) e^{j\beta z} | m \rangle, \\
&= -\frac{e N_t}{m_o} \text{Tr} \{ \rho p_z T_z^*(x, y) e^{j\beta z} \}
\end{aligned} \tag{96}$$

Here,  $\text{Tr}\{ \}$  means to take all diagonal elements in  $\{ \}$  and sum up them as is called the trace operation[24].

The dynamic equation of the density matrix is given by [16,23]:

$$\frac{d\rho}{dt} = \frac{1}{j\hbar} [H_0 + H_{\text{int}}, \rho] - \frac{1}{2} \{ (\rho - \tilde{\rho}) \Gamma + \Gamma (\rho - \tilde{\rho}) \} + A, \tag{97}$$

where  $\tilde{\rho}$  is the electron density at thermal equilibrium,  $A$  indicates electron supports, and  $\Gamma$  is the relaxation operator as has been given in Eqs. (52) and (54).

To analyze Eq. (97), we represent the interacting term with the electric field. The interaction Hamiltonian has been given in Eq. (53) and the vector potential is given by Eq. (57). The momentum operator in  $H_{\text{int}}$  is  $p_z = -j\hbar \partial / \partial z$ . Since the wave-number of the electron is much larger than the propagation constant of the EM wave  $\beta$ , a matrix element of  $H_{\text{int}}$  is simply rewritten as

$$\langle m | H_{\text{int}} | n \rangle = e \left\{ \frac{\hbar k_n}{m_o} \langle m | A_z | n \rangle - \langle m | U | n \rangle \right\} \tag{98}$$

Now we put the scalar potential to be

$$U(\mathbf{r}, t) \equiv U_o e^{j(\omega t - \beta z)} + c.c.. \tag{99}$$

Then the electric field  $E_z$  is represented with the vector and the scalar potentials as

$$E_z(\mathbf{r}, t) = F(t, z) T_z(x, y) e^{j(\omega t - \beta z)} + c.c. = -j\omega \left( A_o - \frac{1}{v_{ph}} U_o \right) e^{j(\omega t - \beta z)} + c.c. \tag{100}$$

Since the phase velocity  $v_{ph}$  of the EM wave is almost same as the electron velocity  $v_n = \hbar k_n / m_o$ , Eq. (98) is written as

$$\langle m | H_{\text{int}} | n \rangle = e \frac{\hbar k_n}{m_o} \langle m | \left\{ \frac{j}{\omega} F(t, z) T_z(x, y) e^{j(\omega t - \beta z)} + c.c. \right\} | n \rangle \tag{101}$$

Therefore, the exchange term between  $H_{\text{int}}$  and  $\rho$  in Eq. (97) becomes

$$\langle m | (H_{\text{int}} \rho - \rho H_{\text{int}}) | n \rangle = \left[ \frac{j e \hbar F(t, z)}{2 m_o \omega} (k_n + k_m) T_{mn} (\rho_{mn} - \rho_{mm}) e^{j\omega t} + c.c. \right], \tag{102}$$

where  $T_{mn}$  is an off-diagonal matrix element relating to the EM wave and the electron wave functions, being given by

$$\begin{aligned}
T_{mn} &\equiv \langle m | T_z(x, y) e^{-j\beta z} | n \rangle \\
&= \frac{1}{V} \int_V T_z^*(x, y) e^{j(k_n - k_m - \beta)z} d^3\mathbf{r} \quad ,
\end{aligned} \tag{103}$$

and

$$T_{mn}^* \equiv \langle m | T_z^*(x, y) e^{j\beta z} | n \rangle \quad . \tag{104}$$

The dynamic equation for the off-diagonal elements of the density matrix is

$$\begin{aligned}
\frac{d\rho_{mn}}{dt} &= \left( j\omega_{nm} - \frac{1}{\tau} \right) \rho_{mn} \\
&+ \left[ \frac{e(k_n + k_m) T_{mn} (\rho_{nn} - \rho_{mm})}{2m_0 \omega} F(t, z) e^{j\omega t} + c.c. \right],
\end{aligned} \tag{105}$$

where

$$\omega_{nm} = (W_n - W_m) / \hbar \quad . \tag{106}$$

From Eq. (105), we obtain the temporal variation of the density matrix element as

$$\rho_{mn}(t) = \frac{e(k_n + k_m) T_{mn} (\rho_{nn} - \rho_{mm})}{2m_0 \omega [j(\omega - \omega_{nm}) + 1/\tau]} F(t, z) \left\{ e^{j\omega t} - e^{(j\omega_{nm} - 1/\tau)t} \right\}. \tag{107}$$

This equation is applicable for all energy levels having the relation of  $n > m$ . Then Eq. (96) becomes

$$\langle J_{elt} T_z^*(x, y) e^{j\beta z} \rangle = - \frac{e\hbar N_t}{m_0} \sum_{n>m} \sum_m \rho_{mn}(t) k_m T_{nm}^* \tag{108}$$

and

$$\begin{aligned}
\langle \frac{\partial J_{elt}}{\partial t} T_z^*(x, y) e^{j\beta z} \rangle &= - \frac{e\hbar N_t}{m_0} \sum_{n>m} \sum_m \frac{\partial \rho_{mn}(t)}{\partial t} k_m T_{nm}^* \\
&= -j \frac{e^2 \hbar N_t}{2m_0^2} \sum_{n>m} \sum_m \frac{(k_n + k_m) k_n |T_{mn}|^2 (\rho_{nn} - \rho_{mm})}{[j(\omega - \omega_{nm}) + 1/\tau]} F(t, z) \left\{ e^{j\omega t} - e^{(j\omega_{nm} - 1/\tau)t} \right\}
\end{aligned} \tag{109}$$

As can be shown from Eqs. (103) and (109), the interaction between the electron wave and the EM wave is the most expected under the conditions of  $\omega \approx \omega_{nm} = (W_n - W_m) / \hbar$  and  $\beta \approx k_n - k_m$ . These conditions correspond to the energy and momentum conservation rules for the electron transition, respectively.

The interaction mechanism is illustrated in Fig. 8. The electron wave  $\varphi_n(\mathbf{r})$  at the initial state has the spatial phase variation of  $e^{jk_n z}$ , and that of the final state  $\varphi_m(\mathbf{r})$  has  $e^{jk_m z}$ . Then the mixed wave  $\varphi_m^*(\mathbf{r})\varphi_n(\mathbf{r})$  has a beating vibration of  $e^{j(k_n - k_m)z}$ . When the beating spatial variation of the mixed wave coincides with the spatial variation of the EM wave  $e^{-j\beta z}$  satisfying the momentum conservation rule given in Eq. (103), the electron transition is induced. Note that the electron transition occurs when the wave-number of the beating wave

matches with the wave-number of the EM wave, while the de Broglie wavelength given by  $2\pi/k_n$  itself is much shorter than the wavelength of the EM wave  $\lambda$ . We can also describe a similar relation between the temporal variation of the mixed electron wave  $e^{-j(\omega_n-\omega_m)t}$  and that of the EM wave  $e^{j\omega t}$ , resulting in the energy conservation rule given in Eq. (109).

## B. Designation of the energy levels

We suppose here that the incident electron is accelerated with velocity  $\bar{v} = v_b = \hbar k_b / m_o$  corresponding to the energy level  $b$ , as illustrated in Fig. 9. When the electron transits to a lower energy  $a$  the emission and the amplification of the EM wave are generated. On the other hand, when the electron transits to an upper energy level  $c$ , the EM wave is absorbed. Since there are dense energy levels, the final energy levels are chosen to satisfy relations of  $\omega_{ba} = \omega$  for the amplification and  $\omega_{cb} = \omega$  for the absorption. We also suppose that electron populations in these final energy levels are zero, because the thermal distribution of the electron beam is much narrower than the photon energy ( $K_B T \ll \hbar\omega$ ),

$$\rho_{aa} = \rho_{cc} = 0. \quad (110)$$

Then Eq. (109) is rewritten in more simple form as

$$\begin{aligned} & \left\langle \frac{\partial J_{elt}}{\partial t} T_z^*(x, y) e^{j\beta z} \right\rangle \\ &= -j \frac{e^2 \hbar N_t \tau}{m_o^2} \sum_b k_b^2 \rho_{bb} \left\{ |T_{ab}|^2 - |T_{cb}|^2 \right\} F(t, z) (1 - e^{-t/\tau}) e^{j\omega t} \end{aligned} \quad (111)$$

where

$$\begin{aligned} |T_{ab}|^2 &= \left| \langle a | T_z(x, y) e^{-j\beta z} | b \rangle \right|^2 \\ &= \left| \frac{1}{\ell^2} \int_{y-\ell/2}^{y+\ell/2} \int_{x-\ell/2}^{x+\ell/2} T_z(x', y') dx' dy' \right|^2 \text{Sinc}^2 \left[ (k_b - k_a - \beta) \frac{\ell}{2} \right]. \end{aligned} \quad (112)$$

Since the averaged current density in the ELT model is written as

$$J_o = e N_t \sum_b \rho_{bb} v_b \quad (113)$$

Equation (111) is rewritten as

$$\begin{aligned} & \left\langle \frac{\partial J_{elt}}{\partial t} T_z^*(x, y) e^{j\beta z} \right\rangle \\ &= -j \frac{e J_o \bar{v} \tau}{\hbar} \left\{ |T_{ab}|^2 - |T_{cb}|^2 \right\} F(t, z) (1 - e^{-t/\tau}) e^{j\omega t}. \end{aligned} \quad (114)$$

## C. Effect of the charge distribution in the ETL model

The space charge term of Eq. (41) is also represented with the density matrix as

$$\left\langle \frac{\partial N_{elt}}{\partial z} T_z^*(x, y) e^{j\beta z} \right\rangle = -jN_t \sum_{n>m} \sum_m (k_n - k_m) \rho_{nm}(t) T_{nm}^* \quad (115)$$

By supposing similar energy levels and notation with the last sub-section, this equation becomes

$$\left\langle \frac{\partial N_{elt}}{\partial z} T_z^*(x, y) e^{j\beta z} \right\rangle = -j \frac{J_o \tau}{\hbar v_{ph}} \{ |T_{ab}|^2 - |T_{cb}|^2 \} F(t, z) (1 - e^{-t/\tau}) \quad (116)$$

However, this space charge term also gives an opposite sign in the amplification gain  $g_{elt}$  in comparison with the term caused by the electron current density  $J_{elt}$ . We have to take into account induced positive charges in the waveguide similar as the case of VDM model by embedding the coefficient  $\kappa$  defined in Eq. (88).

#### D. The gain and the change of the propagation constant in the ELT model

Basing on the above mentioned derivations, the gain and the change of the propagation constant in the ELT model become

$$g_{elt} = \left\{ 1 - \kappa \left( \frac{c}{\bar{v}} \right)^2 \right\} \frac{e \mu_o J_o \tau \bar{v}}{\hbar \beta} \xi_{elt} D(\bar{v}, \ell) [1 - e^{-t/\tau}], \quad (117)$$

and

$$\Delta\beta_{elt} = 0. \quad (118)$$

Here,  $\xi_{elt}$  is the spatial coupling coefficient between the EM field and the electron wave given by

$$\xi_{elt} = \iint_{beam} \left| \frac{1}{\ell^2} \int_{y-\ell/2}^{y+\ell/2} \int_{x-\ell/2}^{x+\ell/2} T_z(x', y') dx' dy' \right|^2 dx dy. \quad (119)$$

and  $D(\bar{v}, \ell)$  is a dispersion function in the ELT model given as

$$\begin{aligned} D(\bar{v}, \ell) &= \text{Sinc}^2[\{k_b - k_a - \beta\}\ell/2] - \text{Sinc}^2[\{k_c - k_b - \beta\}\ell/2] \\ &= \text{Sinc}^2 \left[ \left\{ \frac{\sqrt{2m_0}}{\hbar} (\sqrt{eV_b} - \sqrt{eV_b - \hbar\omega}) - \beta \right\} \frac{\ell}{2} \right] - \text{Sinc}^2 \left[ \left\{ \frac{\sqrt{2m_0}}{\hbar} (\sqrt{eV_b + \hbar\omega} - \sqrt{eV_b}) - \beta \right\} \frac{\ell}{2} \right], \end{aligned} \quad (120)$$

where  $V_b$  is the acceleration voltage of the electron beam related with the initial velocity as

$$eV_b = \frac{m_o \bar{v}^2}{2}. \quad (121)$$

Since we supposed dense energy levels satisfying  $\omega_{ba} = \omega$  and  $\omega_{cb} = \omega$ , the gain coefficient is proportional to the relaxation time  $\tau$  and the change in the propagation constant becomes zero  $\Delta\beta_{elt} = 0$  as in Eq.(118) in the ELT model.

As shown in Eq.(120), the spreading length  $\ell$  is understood as the coherent length of the electron wave. Numerical examples of the dispersion function  $D(\bar{v}, \ell)$  are shown in Figs.10(a) and (b). Fig.10(a) is for the case when  $\ell = 40\mu\text{m}$  and Fig.10(b) is for the case when  $\ell = 1\text{cm}$ . When  $\ell$  is long enough as shown in Fig.10(b), the maximum value of the dispersion function approaches to  $D(\bar{v}, \ell)_{\text{max}} \approx 1$ . On the other hand, as shown in Fig.10(a), when  $\ell$  becomes very small as in the real situation such as  $\ell = 40\mu\text{m}$ , the dispersion function approaches to 0.

Numerical example of the time variation of the gain coefficient  $g_{elt}$  is shown in Fig.11. The gain coefficient reaches to the steady state after a time longer than several times of  $\tau$ .

## VI. APPLICATION OF THE TWO MODELS IN WIDER WAVELENGTH RANGE

The variations of the gain coefficients  $g_{vdm}$  in Eq. (89) and  $g_{elt}$  in Eq. (117) with the wavelength of the EM wave  $\lambda$  are shown in Fig. 12. In these examples, the relaxation times and the spatial coupling coefficients are supposed to be identical in two models for a direct comparison, i.e.,  $\tau' = \tau = 10^{-9}$  sec and  $\xi_{vdm} = \xi_{elt} = 0.1$ . The examined range of the EM wavelength is from  $0.1\mu\text{m}$  to 10 cm in this figure. Peak values of the dispersion functions  $\text{Im}\{Y(\bar{v}, t)\}$  and  $D(\bar{v}, \ell)$  in the steady states have been traced for each  $\ell$ . Applicable ranges of the two models are characterized by the spreading length of the electron wave  $\ell$ .  $g_{vdm}$  shows sufficient values in the case of  $\lambda \gg \ell$ , while  $g_{elt}$  is effective for the case of  $\lambda \ll \ell$ . The shorter  $\ell$  is more profitable for  $g_{vdm}$ , while the longer  $\ell$  is more profitable for  $g_{elt}$ .

## VII. CONCLUSIONS

A generalized theoretical analysis for amplification mechanism in the planar-type Cherenkov laser is given. An electron is represented to be a material wave having temporal and spatial varying phases with finite spreading length. Interaction between the electrons and the electro-magnetic (EM) wave is analyzed by counting the quantum statistical properties.

The interaction mechanism is classified into the Velocity and Density Modulation (VDM) model and the Energy Level Transition (ELT) model basing on the relation between the wave-length of the EM wave and the electron spreading length. The VDM model is applicable when the wavelength of the EM wave is longer than the electron spreading length as in the micro-wave region. The dynamic equation of the electron, which is popularly used in the classical Newtonian mechanics, has been derived from the quantum mechanical Schrödinger equation. The amplification of the EM wave can be explained basing on the bunching effect of the electron density in the electron beam. The amplification gain and whose dispersion relation with respect to the electron velocity is given in this paper. On the other hand, the ELT model is applicable for the case that the wavelength of the EM wave is shorter than the electron spreading length as in the optical region. The dynamics of the electron is explained to be caused by the electron transition between different energy levels. The amplification gain and whose dispersion relation with respect to the electron acceleration voltage was derived on the basis of the quantum mechanical density matrix.

In both VDM and ELT models, the effect of the electron charge in the electron beam work to reduce the amplification gain by the electron beam. This reduction effect can be degraded by induced positive ions or holes at the surface of metallic or semiconductor waveguides (or inside the waveguide).

## REFERENCES

- [1] J. R. Pierce and L. M. Field, Proc. of the I.R.E., 108 (1947).
- [2] J. R. Pierce, Proc. of the I.R.E., 111 (1947).
- [3] H. Motz, J. Appl. Phys. **22**, 527 (1951).
- [4] D. A. G. Deacon, Phys. Rev Lett. **38**, 892 (1977).
- [5] Josip Šoln, J. Appl. Phys. **54**, 2036 (1983).
- [6] W. Case, “Basic Theory of Cerenkov Radiators”, in “High Power Microwave Sources”, eds. V. L. Granastein and I. Aexeff, Boston, MA : Artech, **412** (1987).
- [7] W. Main, and R. Cherry, Appl. Phys. Lett. **55**, 1498 (1989).
- [8] E. Garate, R. Cook, P. Heim, R. Layman, and J. Walsh, J. Appl. Phys. **58**, 627 (1985).
- [9] A. S. Shlapakovskii and K. A. Chirko, IEEE Trans. Plasma Sci. **22**, 544 (1994).
- [10] W. Main, R. Cherry, and E. Garate, Trans. Plasma Sci. **18**, 507 (1990).
- [11] E. Fisch, A. K. Henning, and J. Walsh, IEEE J. of Quantum Electron. **27**, 753 (1991).
- [12] J. Walsh, C. Shaughnessy, R. Layman, G. Dattoli, G-P. Gallerano, and A. Renieri, Nucl. Instr. and Meth. **A272**, 132 (1988).
- [13] E. P. Garate, C. H. Shaughnessy, and J. E. Walsh, IEEE J. of Quantum Electron. **23**, 1627 (1987).
- [14] A. Yariv and C. C. Shih, Opt. Commun. **24**, 233 (1978).
- [15] I. J. Owens and J. H. Brownell, J. Appl. Phys. **97**, 104915 (2005).
- [16] M. Yamada, IEEE J. of Quantum Electron **35**, 147 (1999).
- [17] M. Asada and M. Yamada, J. Appl. Phys. **95**, 5123 (2004).
- [18] Y. Kuwamura, M. Yamada, R. Okamoto, T. Kanai, and H. Fares, J. Appl. Phys. **104**, 103105 (2008).
- [19] H. Fares, M. Yamada, Y. Kuwamura, I. Matsumoto, and T. Kanai, IEEE J. of Quantum Electron. **46** 981 (2010).
- [20] E. P. Garate, J. Walsh, C. Shaughnessy, B. Johnson, and S. Moustazis, Nucl. Instr. and Meth. , **A259**, 125 (1987).
- [21] E. P. Garate, C. H. Shaughnessy, and J. E. Walsh, IEEE J. of Quantum Electron., **23**, 1627 (1987).
- [22] H. Fares, M. Yamada, and Y. Kuwamura, Jpn. J. Appl. Phys. **49**, 096402. (2010).
- [23] M. Yamada and Y. Suematsu, J. Appl. Phys. **52**, 2653 (1981).
- [24] D. Marcuse, “Principles of Quantum Electronics”, (Academic Press, New York, 1980).

## Figure Captions

FIG.1 Schematic illustration of the Cherenkov laser utilizing a dielectric planer waveguide and traveling electron beam.

FIG.2 Supposed shape of a single electron. The electron wave is approximately represented with a boxlike plane wave.

FIG.3 Energy levels of a single electron. The energy levels are not continuous but are given with discrete levels.

FIG. 4 Spatial distributions of the electrons and the EM waves. The spreading length of an electron is longer than the wavelength of the optical wave but is shorter than the wavelength of the micro-wave.

FIG.5 Summary of the classifications and applicability of the VDM and ELT models.

FIG.6 Normalized dispersion function in VDM model. (a) is the imaginary part giving the gain coefficient and (b) is the real part giving the change of the propagation constant. The dispersion function is characterized not only by the electron velocity but also by the interaction time  $t$  from start of the interaction and the relaxation time  $\tau'$ .

FIG.7 Numerical example of the gain coefficient in the VDM model. The gain coefficient varies with the time up to  $t/\tau' \approx 3$  as a transient phenomenon then reaches to the steady state.

FIG.8 Illustration of interaction mechanism in ELT model. The mixed wave of the initial and the states  $\varphi_m^* \varphi_n$  synchronizes with the EM wave, giving the electron transition from the initial state to the final state when  $k_n - k_m = \beta$ .

FIG.9 Illustration of the electron transition. The initial energy level of the electron beam is set to be  $b$ . When the electron transits to a lower energy level  $a$ , the EM wave is emitted or amplified. When the electron transits to a higher energy level  $b$ , the EM wave is absorbed. The energy conservation and the momentum conservation rules should be satisfied during the electron transition.

FIG.10 Numerical examples of the dispersion function in the ELT model. (a) for  $\ell = 40\mu\text{m}$  and (b) for  $\ell = 1\text{cm}$ .

FIG.11 Numerical example of the time variation of the gain coefficient in the ELT model The peak values of the dispersion function are traced for each  $\ell$

FIG.12 Wavelength dispersions of the gain coefficients.  $g_{vdm}$  shows sufficient values in the case of  $\lambda \gg \ell$ , while  $g_{elt}$  is effective for the case of  $\lambda \ll \ell$ .

FIG. 1

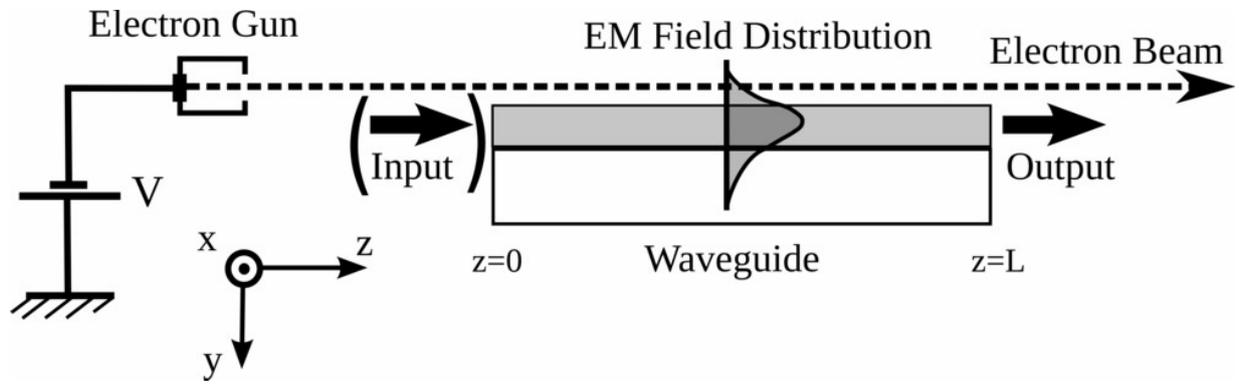


FIG. 2

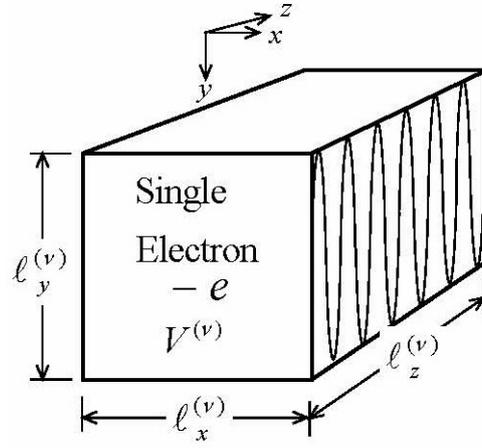


FIG. 3

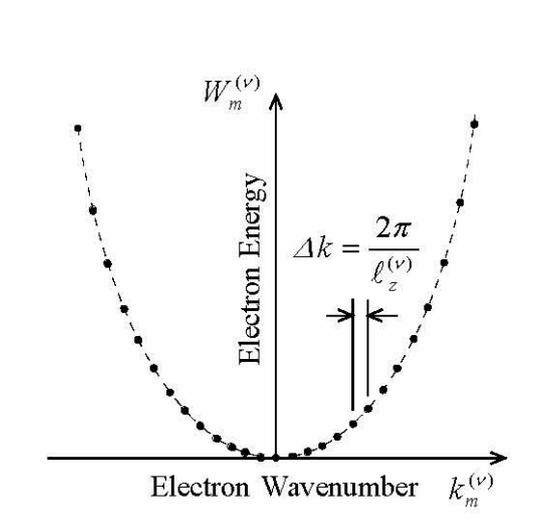


FIG.4

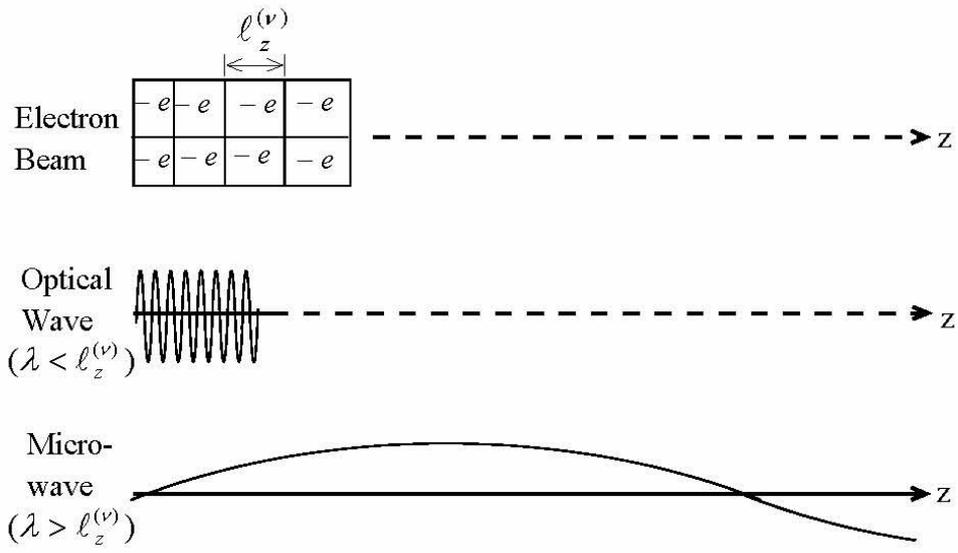


FIG.5

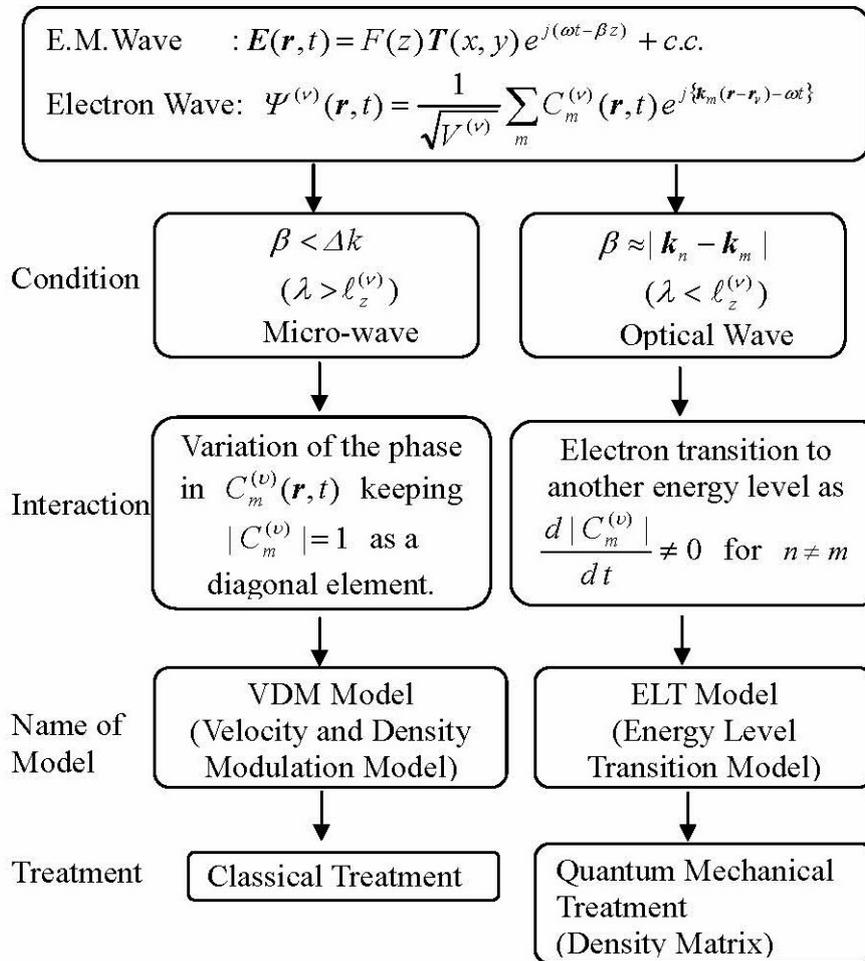


FIG. 6

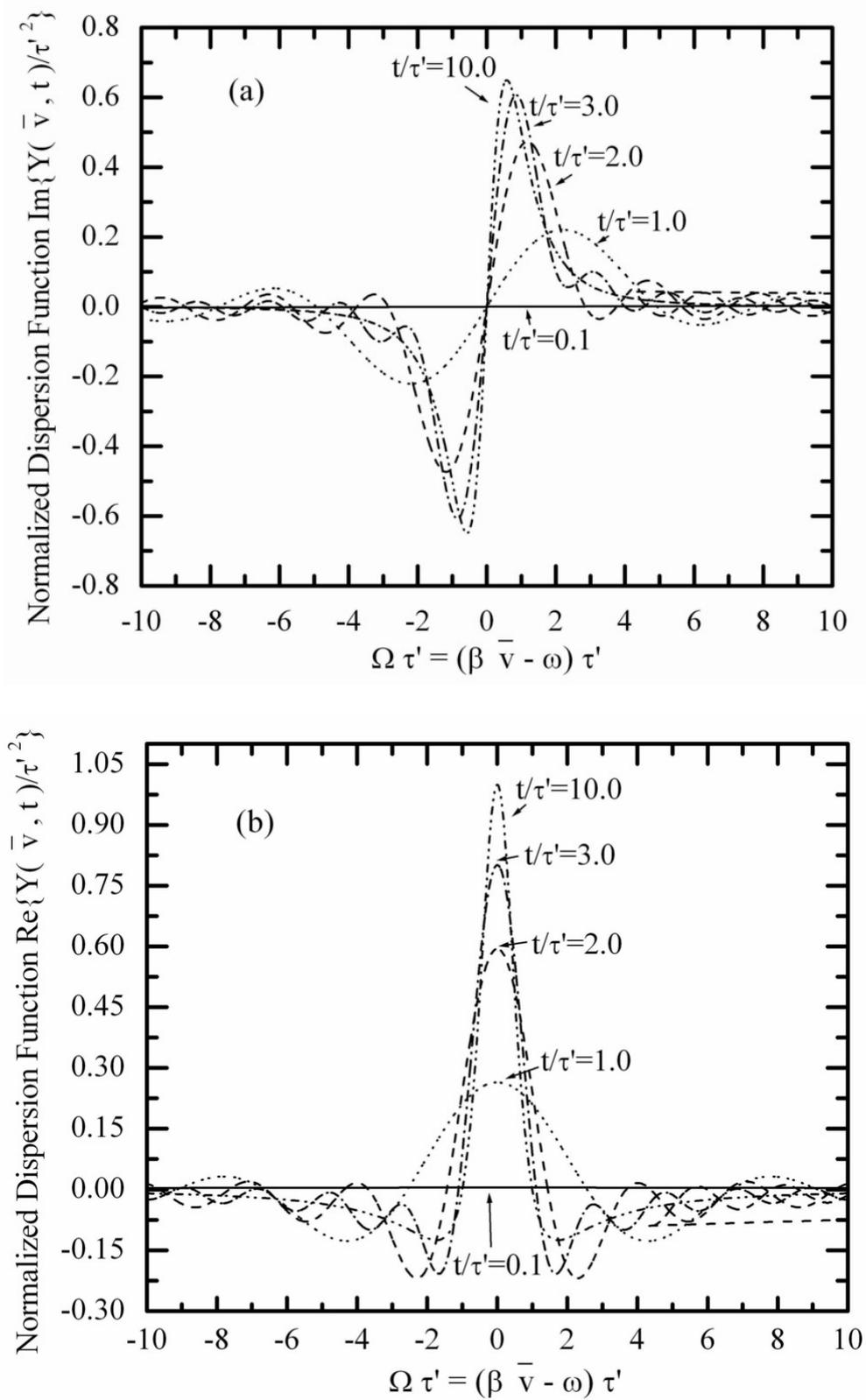


FIG. 7

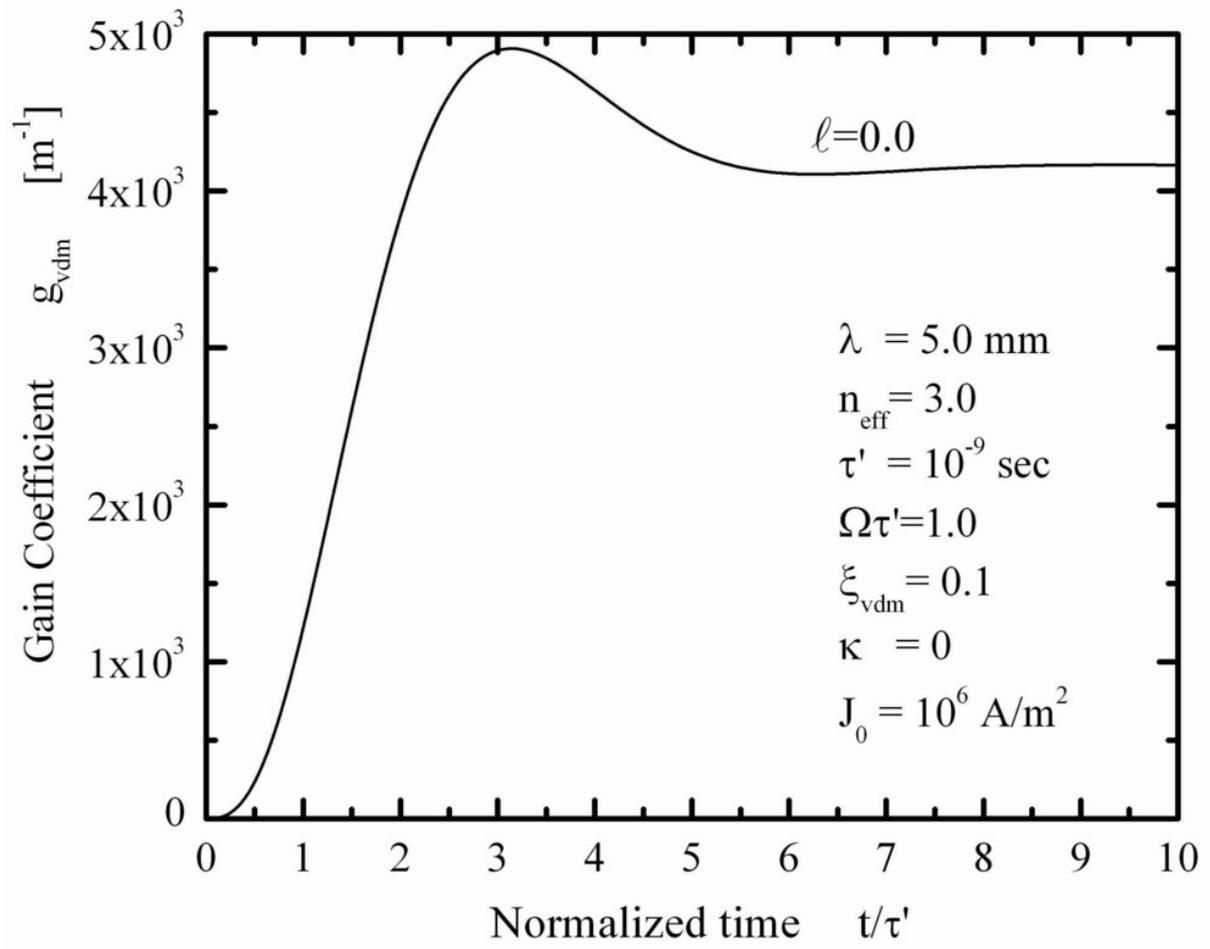


FIG. 8

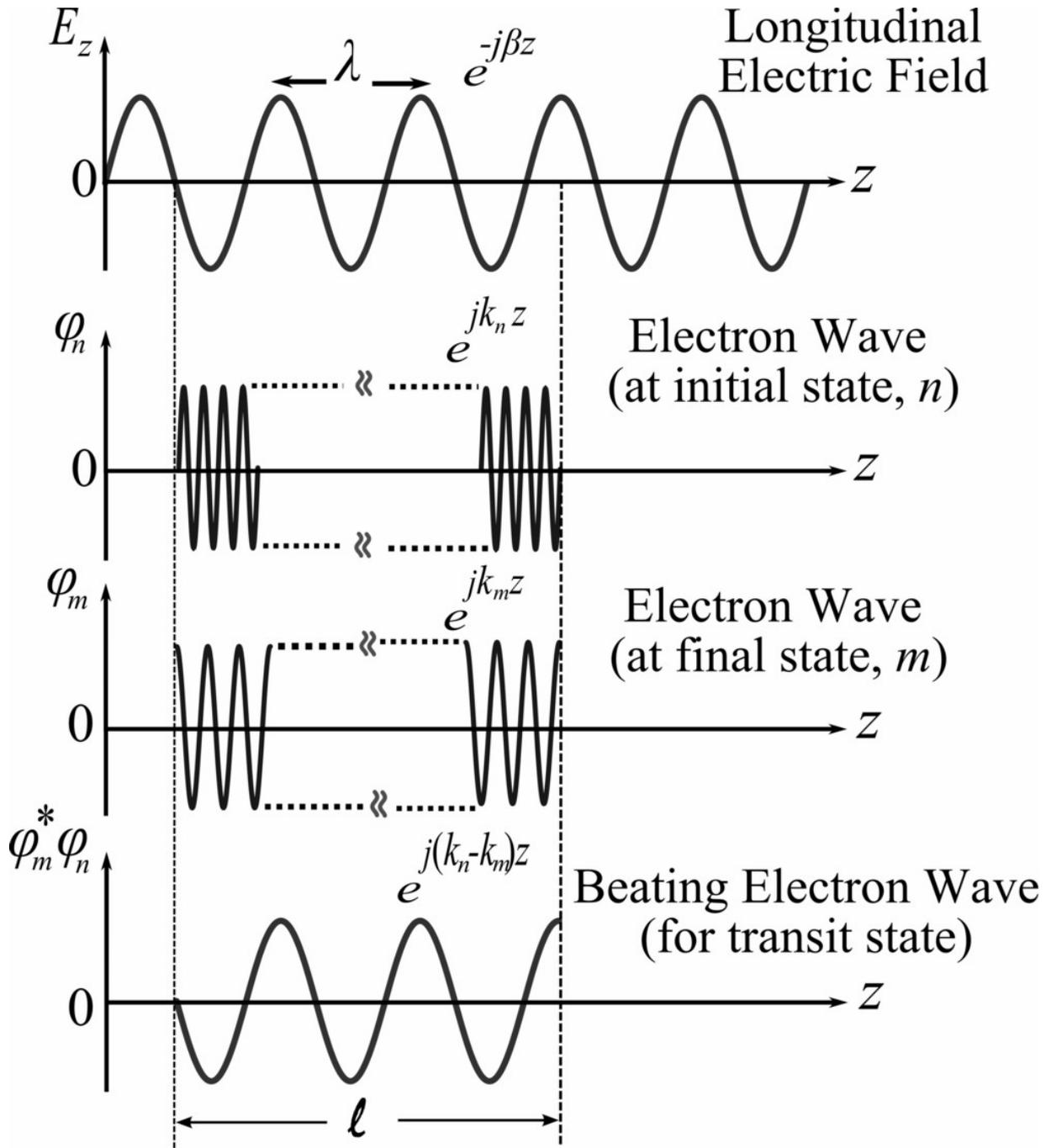


FIG. 9

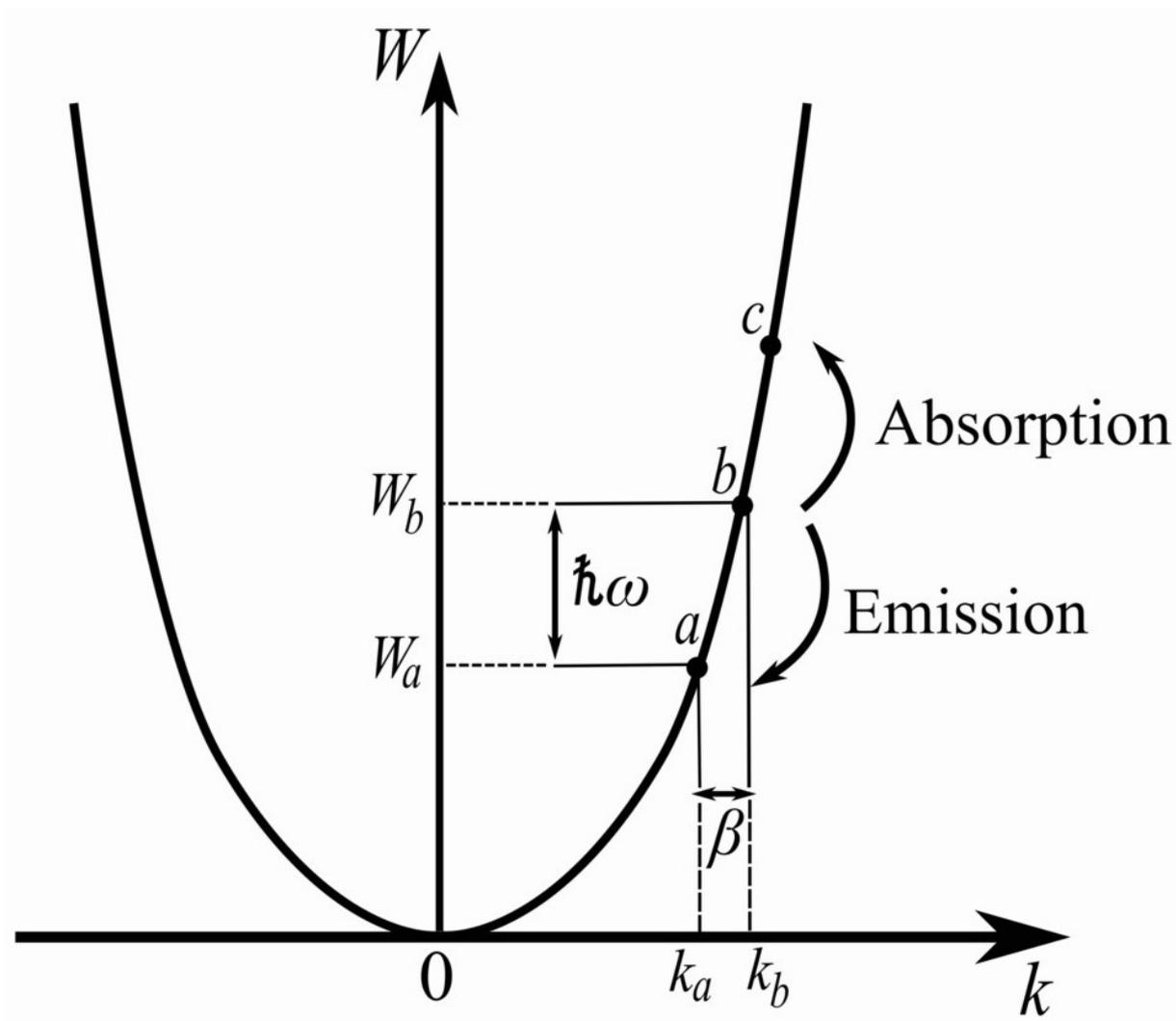


FIG. 10

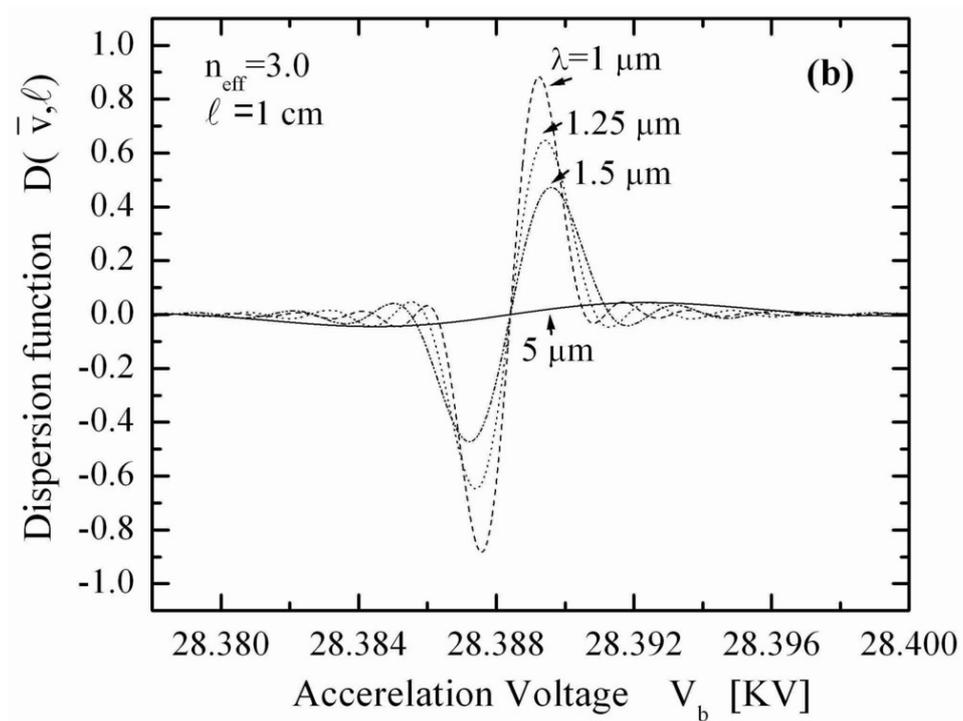
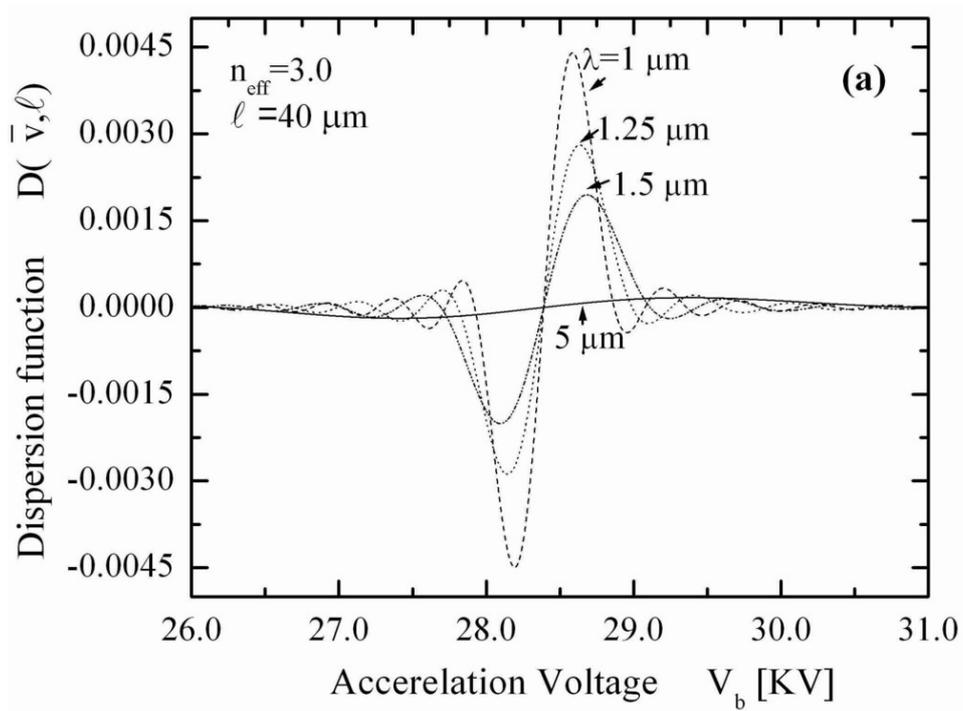


FIG.11

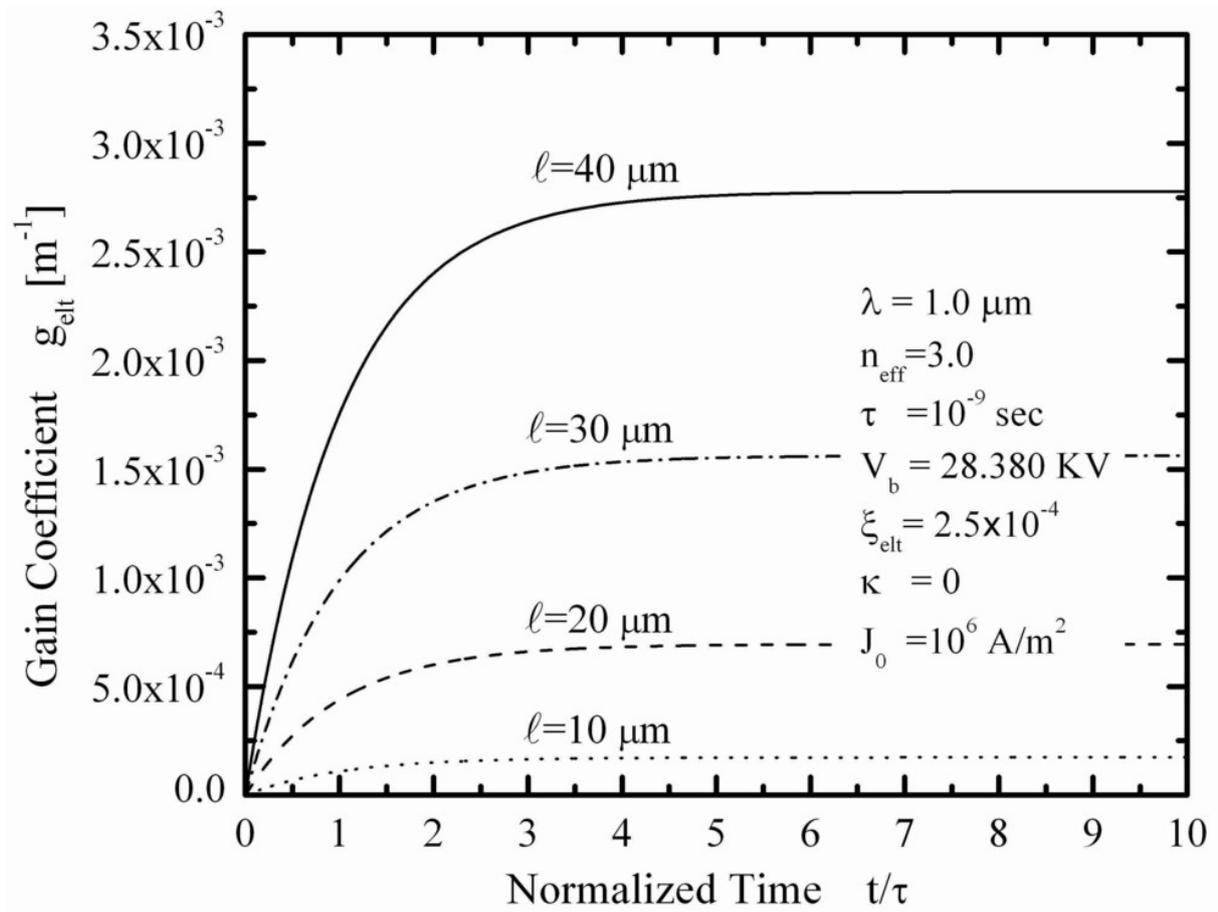


FIG.12

