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# Pointwise convergence of Cesàro and Riesz means on certain function spaces

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**Abstract.** We consider a function space $\mathscr{Q}\mathscr{A}$ on the unit sphere of $\mathbb{R}^3$, which contains $L\log L\log\log\log L$, and prove the spherical harmonics expansions of functions in $\mathscr{Q}\mathscr{A}$ are summable a.e. with respect to the Cesàro means of the critical order 1/2. We also prove that a similar result holds for the Bochner–Riesz means of multiple Fourier series of periodic functions on $\mathbb{R}^d$, $d \geq 2$.

## 1. Introduction

Let

$$Q_d = \{x \in \mathbb{R}^d : -1/2 < x_i \leq 1/2, i = 1, 2, \dots, d\}, \quad x = (x_1, \dots, x_d),$$

be the fundamental cube in the $d$-dimensional Euclidean space $\mathbb{R}^d$. For $f \in L^1(Q_d)$ we consider the Fourier series

$$f(x) \sim \sum a_n e^{2\pi i\langle n,x\rangle}, \quad n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d,$$

where $\langle n, x\rangle = n_1x_1 + \cdots + n_dx_d$ and

$$a_n = \int_{Q_d} f(x)e^{-2\pi i\langle n,x\rangle}\,dx, \quad dx = dx_1 \dots dx_d,$$

is the Fourier coefficient. The Bochner–Riesz means of order $\delta$ of the series are defined by

$$T_R^\delta(f)(x) = \sum_{|n|<R} \left(1 - \frac{|n|^2}{R^2}\right)^\delta a_n e^{2\pi i\langle n,x\rangle},$$

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where $|n| = (n_1^2 + \cdots + n_d^2)^{1/2}$.

According to [2], we define a space $\mathscr{Q}\mathscr{A}(Q_d)$ to be the collection of measurable functions $f$ for which we can find a sequence $\{f_j\}$ of non-negative measurable functions such that

$$|f| \le \sum_{j=1}^{\infty} f_j, \quad N(\{f_j\}) := \sum_{j=1}^{\infty}(1+\log j)\|f_j\|_1 \log\left(\frac{e\|f_j\|_\infty}{\|f_j\|_1}\right) < \infty; \tag{1.1}$$

let $\|f\|_{\mathscr{Q}\mathscr{A}} = \inf N(\{f_j\})$, where the infimum is taken over all such $\{f_j\}$. Then, the space $\mathscr{Q}\mathscr{A}$ is a logconvex quasi-Banach space and a subspace of $L\log L$ (see [2,9]).

Define $T_*^\delta(f)(x) = \sup_{R>0} |T_R^\delta(f)(x)|$. Let $\alpha = (d-1)/2$ (the critical index). Then we shall prove the following.

**Theorem 1.** *There exists a positive constant $C$ such that*

$$\|T_*^\alpha(f)\|_{1,\infty} = \sup_{\lambda>0} \lambda|\{x \in Q_d : T_*^\alpha(f)(x) > \lambda\}| \le C\|f\|_{\mathscr{Q}\mathscr{A}};$$

*consequently,*

$$\lim_{R\to\infty} T_R^\alpha(f)(x) = f(x) \quad a.e. \qquad for\ f \in \mathscr{Q}\mathscr{A}(Q_d).$$

It is known that $L\log L\log\log\log L$ is a proper subspace of $\mathscr{Q}\mathscr{A}$ (see [2]). Thus, Theorem 1 implies the following.

**Theorem 2.** *If $f \in L\log L\log\log\log L(Q_d)$, then*

$$\lim_{R\to\infty} T_R^\alpha(f)(x) = f(x) \quad a.e.$$

The convergence a.e. for $f \in L\log L\log\log L(Q_d)$ was proved in [17].

If we write $T_N(f) = T_N^\alpha(f)$ when $d = 1$, then $T_{N+1}(f)$ is the $N$th partial sum of the Fourier series of $f$. For $f \in L^2(Q_1)$, there is a result of L. Carleson [5] which shows that $\{T_N f\}$ converges a.e. (see also [7]). Let $T_* f = \sup_{N\ge 1} |T_N f|$. R. Hunt [8] proved the restricted weak type estimates:

$$\sup_{\lambda>0} \lambda|\{x \in Q_1 : T_*(\chi_A)(x) > \lambda\}|^{1/p} \le Cp^2(p-1)^{-1}|A|^{1/p}, \quad 1<p<\infty, \tag{1.2}$$

where $\chi_A$ denotes the characteristic function of a set $A \subset Q_1$. By (1.2) R. Hunt [8] proved the convergence a.e. of $\{T_N f\}$ for $f \in L(\log L)^2(Q_1)$. P. Sjölin [12] showed that (1.2) can be used to prove the convergence a.e. for the class $L\log L\log\log L(Q_1)$. Applying (1.2) more efficiently, N. Yu. Antonov [1] proved that $\{T_N f\}$ converges a.e. if $f \in L\log L\log\log\log L(Q_1)$. Theorem 2 can be regarded as a generalization of this result to higher dimensions.

To prove Theorem 1 for $d \ge 2$ we use the following estimates:

**Lemma 1.** *Let $1 < p \le 2$, $d \ge 2$. Then there exists a constant $C$ independent of $p$ such that*

$$\sup_{\lambda>0} \lambda|\{x \in Q_d : T_*^\alpha(f)(x) > \lambda\}|^{1/p} \le C(p-1)^{-1}\|f\|_p.$$

We write $\delta = \sigma + i\tau$, $\sigma, \tau \in \mathbb{R}$. Lemma 1 was proved in [17] by using the following two results and analytic interpolation.

**Lemma 2.** *Suppose $f \in L^1(Q_d)$, $d \ge 2$ and $\sigma > \alpha$. Then*

$$\|T_*^\delta(f)\|_{1,\infty} \le A_\sigma e^{\pi|\tau|}(\sigma - \alpha)^{-1}\|f\|_1,$$

*where $A_\sigma$ remains bounded as $\sigma \to \alpha$.*

**Lemma 3.** *Suppose that $f \in L^2(Q_d)$, $d \ge 2$. Then*

$$\|T_*^\delta(f)\|_2 \le A_\sigma e^{\pi|\tau|}\|f\|_2, \quad \sigma > 0.$$

See Lemma 12 and Theorem 7 of [15] for Lemmas 2 and 3, respectively.

Sjölin–Soria [13] extended results of [1] to more general settings. We can apply results of [13] to prove Theorem 2 for $d \ge 2$. Indeed, we easily see that Theorem 2 for $d \ge 2$ follows from Lemma 1 and methods of [13, Section 3] (see Remark at the end of Section 3 of [13]). When $d = 1$, Theorem 1 is due to [2]. The result also can be proved by using the estimate (1.2) and Antonov's idea. More precisely, when $d = 1$, Lemma 7 (a key estimate) below is first proved for characteristic functions by applying (1.2) and the transition from characteristic functions to general functions $f$ can be carried out by Antonov's idea. We can prove Theorem 1 by Lemma 1 in the same way in higher dimensions. In fact, our proof of Theorem 1 for $d \ge 2$ is more straightforward; to prove Lemma 7 the application of the idea of Antonov is not needed, since the estimate of Lemma 1 is not restricted to characteristic functions (see Section 2).

We have analogous results for the Cesàro means of spherical harmonics expansions. Let $\mathscr{H}_k$ be the space of the spherical harmonics of degree $k$ on $\Sigma_d$, where $\Sigma_d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$ is the unit sphere in $\mathbb{R}^{d+1}$. We recall that the space $\mathscr{H}_k$ consists of the restrictions to $\Sigma_d$ of harmonic homogeneous polynomials of degree $k$. Let

$$H_k f(x) = \int_{\Sigma_d} Z_x^{(k)}(y) f(y)\, d\mu(y),$$

where $d\mu$ is the Lebesgue surface measure on $\Sigma_d$ normalized as $\mu(\Sigma_d) = 1$ (we also write $|E| = \mu(E)$ for a set $E \subset \Sigma_d$), and $Z_x^{(k)} \in \mathscr{H}_k$ is the zonal harmonic of degree

$k$ with pole $x \in \Sigma_d$:

$$Z_x^{(k)}(y) = \left(\frac{2k}{d-1}+1\right)\frac{\Gamma(d/2)\Gamma(d+k-1)}{\Gamma(d-1)\Gamma(k+d/2)} P_k^{((d-2)/2,(d-2)/2)}(\langle x,y\rangle)$$
$$= \left(\frac{2k}{d-1}+1\right) P_k^{((d-1)/2)}(\langle x,y\rangle).$$

Here $P_k^{(\alpha,\beta)}$ is the Jacobi polynomial and $P_k^{(\lambda)}$ is the Gegenbauer polynomial defined by $(1-2tr+r^2)^{-\lambda} = \sum_{k=0}^\infty P_k^{(\lambda)}(t)r^k$. We consider the spherical harmonics expansion $f \sim \sum_{k=0}^\infty H_k f$ and the Cesàro means of order $\delta$ defined by

$$S_n^\delta f = \frac{1}{A_n^{(\delta)}} \sum_{k=0}^n A_{n-k}^{(\delta)} H_k f, \quad n = 0,1,2,\dots, \quad \delta = \sigma + i\tau,$$

where

$$A_k^{(\delta)} = \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta+1)} = \binom{k+\delta}{k}, \quad \sigma > -1 \tag{1.3}$$

(see [19, Chap. III]). We refer to [4, 6, 14, 18] and [16, Chap. IV] for relevant results.

Let $S_*^\delta(f)(x) = \sup_{n>0} |S_n^\delta(f)(x)|$. If we define the space $\mathscr{Q}\mathscr{A}(\Sigma_d)$ analogously to $\mathscr{Q}\mathscr{A}(Q_d)$, we have the following result (we focus on the case $d=2$).

**Theorem 3.** *There exists a positive constant $C$ such that*

$$\sup_{\lambda>0} \lambda |\{x \in \Sigma_2 : S_*^{1/2}(f)(x) > \lambda\}| \le C\|f\|_{\mathscr{Q}\mathscr{A}}$$

*for $f \in \mathscr{Q}\mathscr{A}(\Sigma_2)$, which implies*

$$\lim_{n\to\infty} S_n^{1/2}(f)(x) = f(x) \quad a.e. \qquad for\ f \in \mathscr{Q}\mathscr{A}(\Sigma_2).$$

Theorem 3 implies the following result as Theorem 1 implies Theorem 2.

**Theorem 4.** *If $f \in L\log L\log\log\log L(\Sigma_2)$, then*

$$\lim_{n\to\infty} S_n^{1/2} f(x) = f(x) \quad a.e.$$

See [4] for the convergence a.e. of $\{S_n^{1/2} f\}$ for $f \in L^p(\Sigma_2)$, $p > 1$. The proof of Theorem 3 is similar to that of Theorem 1, with the following estimates:

**Lemma 4.** *Let $1 < p \le 2$. Then we have*

$$\sup_{\lambda>0} \lambda |\{x \in \Sigma_2 : S_*^{1/2}(f)(x) > \lambda\}|^{1/p} \le C(p-1)^{-1}\|f\|_p$$

*for a positive constant $C$ independent of $p$.*

Let

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|\, d\mu(y),$$

where $B(x,r) = \{y \in \Sigma_2 : |y-x| < r\}$, $x \in \Sigma_2$. To prove Lemma 4 we need the following two results.

**Lemma 5.** *Suppose that $f \in L^1(\Sigma_2)$ and $\alpha < \sigma < 1$, where $\alpha = 1/2$. Then*

$$S_*^\delta(f)(x) \le A_\sigma e^{B\tau^2}(\sigma-\alpha)^{-1}(Mf(x) + Mf(-x)).$$

*The constant $A_\sigma$ remains bounded as $\sigma \to \alpha$.*

**Lemma 6.** *Suppose that $f \in L^2(\Sigma_2)$. Then*

$$\|S_*^\delta(f)\|_2 \le A_\sigma e^{B_\sigma \tau^2}\|f\|_2, \quad \sigma > 0.$$

*The constants $A_\sigma$ and $B_\sigma$ are bounded on any compact subinterval of $(0,\infty)$.*

We can find Lemma 6 in [4]. Using Lemmas 5 and 6, we can prove Lemma 4 by analytic interpolation (see Section 4). We shall prove Lemma 5 in Section 3 by applying methods of [10].

## 2. Proof of Theorem 1

We assume that $d \ge 2$. In proving Theorem 1 we use the following result.

**Lemma 7.** *Suppose that $f \in L^\infty(Q_d)$, $f \neq 0$. Then*

$$\|T_*^\alpha(f)\|_{1,\infty} \le C\|f\|_1 \log\Big(\frac{e\|f\|_\infty}{\|f\|_1}\Big).$$

**Proof.** By homogeneity we may assume that $\|f\|_\infty = 1$. For $\lambda > 0$, let $m(\lambda) = \inf_{1<p\le 2} \lambda^{-p}(p-1)^{-p}$. Then, observing that $\|f\|_p^p \le \|f\|_1$, by Lemma 1 we have

$$|\{x \in Q_d : T_*^\alpha(f)(x) > \lambda\}| \le C \min\left(1, m(\lambda)\|f\|_1\right).$$

This will imply the conclusion, if we note that $m(\lambda) = \lambda^{-2}$ when $\lambda \ge e^{-2}$ and $m(\lambda) \sim \lambda^{-1}\log(1/\lambda)$ when $\lambda < e^{-2}$. ∎

Let $f \in \mathscr{Q}\mathscr{A}(Q_d)$. To prove Theorem 1, we may assume that $f \ge 0$. For any $\epsilon > 0$ there exists a sequence $\{f_j\}$ of non-negative bounded functions such that $f = \sum f_j$ and $N(\{f_j\}) \le \|f\|_{\mathscr{Q}\mathscr{A}} + \epsilon$ (see [2, p. 149]). Since $L^{1,\infty}$ is a logconvex quasi-Banach space (see [9]) and $T_*^\alpha$ is a sublinear operator, using Lemma 7 we have

$$\begin{aligned}\|T_*^\alpha(f)\|_{1,\infty} &\le C\sum_j(1+\log j)\|T_*^\alpha(f_j)\|_{1,\infty}\\ &\le C\sum_j(1+\log j)\|f_j\|_1\log\left(\frac{e\|f_j\|_\infty}{\|f_j\|_1}\right)=CN(\{f_j\})\le C(\|f\|_{\mathscr{Q}\mathscr{A}}+\epsilon).\end{aligned}$$

Letting $\epsilon\to 0$, we get the conclusion.

## 3. Proof of Lemma 5

Let

$$S_n^{(\delta,\lambda)}(\cos v)=(A_n^{(\delta)})^{-1}\sum_{k=0}^n A_{n-k}^{(\delta)}2(k+\lambda)P_k^{(\lambda)}(\cos v),$$

where $0<\lambda<1$, $0\le v\le\pi$, $0<\sigma<1$, $\delta=\sigma+i\tau$. Then, $S_n^{(\delta,1/2)}(\langle x,y\rangle)$ is the kernel of the operator $S_n^\delta$. In [10, p. 121], $S_n^{(\delta,\lambda)}(\cos v)$ was represented by the contour integrals as follows:

$$\frac{1}{2}A_n^{(\delta)}S_n^{(\delta,\lambda)}(\cos v)=\frac{1}{2\pi i}\int_{L_1}\varphi(z)\,dz+\frac{1}{2\pi i}\int_{L_2}\varphi(z)\,dz+\frac{1}{2\pi i}\int_{L_3}\varphi(z)\,dz,\quad(3.1)$$

where

$$\varphi(z)=\frac{\lambda(1+z)z^{n+\delta+2\lambda}}{(z-1)^\delta(1-2z\cos v+z^2)^{\lambda+1}}.$$

Let

$$\begin{aligned}i_n^{(\delta,\lambda)}(v)&=\frac{\lambda\sin(\delta\pi)}{\pi}\int_0^1\frac{u^{n+\delta+2\lambda}}{(1-u)^\delta(1-2u\cos v+u^2)^{\lambda+1}}\,du,\\ \mathscr{I}_n^{(\delta,\lambda)}(v)&=\frac{\exp\left(-i\left[(n+\lambda+(\delta+1)/2)v-(\lambda+\delta+1)\pi/2\right]\right)}{(2\sin v)^\lambda(2\sin(v/2))^{\delta+1}}\frac{\sin(\lambda\pi)}{\pi}\times\\ &\quad\times\int_0^1\frac{u^{-\lambda}(1-u)^{n+\delta+2\lambda}}{(1-u\tau(v/2))^{\delta+1}(1-u\tau(v))^\lambda}\,du,\\ \mathscr{J}_n^{(\delta,\lambda)}(v)&=\frac{\exp\left(i\left[(n+\lambda+(\delta+1)/2)v-(\lambda+\delta+1)\pi/2\right]\right)}{(2\sin v)^\lambda(2\sin(v/2))^{\delta+1}}\frac{\sin(\lambda\pi)}{\pi}\times\\ &\quad\times\int_0^1\frac{u^{-\lambda}(1-u)^{n+\delta+2\lambda}}{(1-u\tau(-v/2))^{\delta+1}(1-u\tau(-v))^\lambda}\,du,\end{aligned}$$

where $\tau(v)=(1+i\cot v)/2$. Then, according to (3.1), it follows that

$$\begin{aligned}\frac{1}{2}A_n^{(\delta)}S_n^{(\delta,\lambda)}(\cos v)=(n+\lambda)\mathscr{I}_n^{(\delta,\lambda)}(v)-(\delta+1)\mathscr{I}_{n-1}^{(\delta+1,\lambda)}(v)+i_{n+1}^{(\delta,\lambda)}(v)+\\ +i_n^{(\delta,\lambda)}(v)+(n+\lambda)\mathscr{J}_n^{(\delta,\lambda)}(v)-(\delta+1)\mathscr{J}_{n-1}^{(\delta+1,\lambda)}(v)\end{aligned} \tag{3.2}$$

(see [10]). Put

$$K(n,\delta,\lambda,v)=\frac{4(n+\lambda)}{\Gamma(\lambda)}C(n,\delta,\lambda)\frac{\cos\left[(n+\lambda+(\delta+1)/2)v-(\lambda+\delta+1)\pi/2\right]}{(2\sin v)^{\lambda}(2\sin(v/2))^{\delta+1}},$$
$$L(n,\delta,\lambda,v)=\frac{-4(\delta+1)}{\Gamma(\lambda)}C(n,\delta,\lambda)\frac{\cos\left[(n+\lambda+\delta/2)v-(\lambda+\delta+2)\pi/2\right]}{(2\sin v)^{\lambda}(2\sin(v/2))^{\delta+2}},$$

where

$$C(n,\delta,\lambda)=\frac{\Gamma(n+\delta+2\lambda+1)}{\Gamma(n+\delta+\lambda+2)};$$

and also

$$\begin{aligned}R_1(n,\delta,\lambda,v)&=2(n+\lambda)\mathscr{I}_n^{(\delta,\lambda)}(v)+2(n+\lambda)\mathscr{J}_n^{(\delta,\lambda)}(v)-K(n,\delta,\lambda,v),\\ R_2(n,\delta,\lambda,v)&=-2(\delta+1)\mathscr{I}_{n-1}^{(\delta+1,\lambda)}(v)-2(\delta+1)\mathscr{J}_{n-1}^{(\delta+1,\lambda)}(v)-L(n,\delta,\lambda,v),\\ R_3(n,\delta,\lambda,v)&=2i_{n+1}^{(\delta,\lambda)}(v)+2i_n^{(\delta,\lambda)}(v).\end{aligned}$$

Then (3.2) implies that

$$\begin{aligned}S_n^{(\delta,\lambda)}(\cos v)=(A_n^{(\delta)})^{-1}(K(n,\delta,\lambda,v)+L(n,\delta,\lambda,v)+R_1(n,\delta,\lambda,v)+\\ +R_2(n,\delta,\lambda,v)+R_3(n,\delta,\lambda,v)).\end{aligned} \tag{3.3}$$

We need the following results.

**Lemma 8.** *Let $x>-1$, $y\in\mathbb{R}$. Then $|A_n^{(x+iy)}|\ge|A_n^{(x)}|$ and $|A_n^{(x+iy)}|\le e^{c(x)y^2}A_n^{(x)}$, where $c(x)=(1/2)\sum_{k=1}^{\infty}(x+k)^{-2}$ and $A_n^{(x+iy)}$ is as in (1.3).*

**Lemma 9.** *Suppose $0<\lambda<1$, $0<\sigma<1$. Let $C(n,\delta,\lambda)$ be as above. Then*

$$|C(n,\delta,\lambda)|\le C(n+1)^{\lambda-1},$$

*where the constant $C$ is independent of $\delta$ and $\lambda$.*

Lemma 8 is in [3]. Lemma 9 can be proved by using the formula

$$\lim_{\mathrm{Re}(z)\ge c>0,|z|\to\infty}\frac{\Gamma(z)}{\sqrt{2\pi}e^{-z}z^{z-1/2}}=1.$$

Let $|\pi/2 - v| \le (\pi/2)(n/(n+1))$. By [10, pp. 130–133] and Lemma 9 we have

$$\begin{aligned} |R_1(n,\delta,\lambda,v)| &\le Ce^{B|\tau|}\frac{C(n,\sigma,\lambda)}{\Gamma(\lambda)|n+\sigma+\lambda+2|}\frac{n+1}{(\sin v)^{\lambda+1}(\sin(v/2))^{\sigma+1}} \\ &\le Ce^{B|\tau|}\frac{(n+1)^{\lambda-1}}{(\sin v)^{\lambda+1}(\sin(v/2))^{\sigma+1}}, \\ |R_2(n,\delta,\lambda,v)| &\le Ce^{B|\tau|}\frac{C(n,\sigma,\lambda)}{\Gamma(\lambda)|n+\sigma+\lambda+2|}\frac{1}{(\sin v)^{\lambda+1}(\sin(v/2))^{\sigma+2}} \\ &\le Ce^{B|\tau|}\frac{(n+1)^{\lambda-1}}{(\sin v)^{\lambda+1}(\sin(v/2))^{\sigma+1}}. \end{aligned}$$

Also, by [10, pp. 122–123] and estimates similar to the one in Lemma 9

$$\begin{aligned} |R_3(n,\delta,\lambda,v)| &\le C\frac{|\sin(\delta\pi)|\Gamma(1-\sigma)}{(\sin(v/2))^{2(\lambda+1)}}\Big(\frac{\Gamma(n+\sigma+2\lambda+1)}{\Gamma(n+2\lambda+2)}+\frac{\Gamma(n+\sigma+2\lambda+2)}{\Gamma(n+2\lambda+3)}\Big) \\ &\le C(n+1)^{\sigma-1}\frac{|\sin(\delta\pi)|\Gamma(1-\sigma)}{(\sin(v/2))^{2(\lambda+1)}}. \end{aligned}$$

Since $|A_n^{(\delta)}| \ge |A_n^{(\sigma)}|$ and $A_n^{(\sigma)} \sim (n+1)^\sigma$ (see Lemma 8 and [19, Chap. III]), if $|\pi/2 - v| \le (\pi/2)(n/(n+1))$, we have

$$|R_j(n,\delta,\lambda,v)/A_n^{(\delta)}| \le Ce^{B|\tau|}\frac{(n+1)^{\lambda-1-\sigma}}{(\sin v)^{\lambda+1}(\sin(v/2))^{\sigma+1}}, \quad j=1,2, \tag{3.4}$$

$$|R_3(n,\delta,\lambda,v)/A_n^{(\delta)}| \le C\frac{|\sin(\delta\pi)|\Gamma(1-\sigma)}{(n+1)(\sin(v/2))^{2(\lambda+1)}}. \tag{3.5}$$

By Lemma 9 we have

$$|K(n,\delta,\lambda,v)/A_n^{(\delta)}| \le Ce^{(\pi/2)|\tau|}\frac{(n+1)^{\lambda-\sigma}}{(\sin v)^{\lambda}(\sin(v/2))^{\sigma+1}}. \tag{3.6}$$

Similarly,

$$|L(n,\delta,\lambda,v)/A_n^{(\delta)}| \le C(1+|\tau|)e^{(\pi/2)|\tau|}\frac{(n+1)^{\lambda-\sigma-1}}{(\sin v)^{\lambda}(\sin(v/2))^{\sigma+2}}. \tag{3.7}$$

We also need the following.

**Lemma 10.** *Let* $0<\lambda<1$, $0<\sigma<1$, $\delta=\sigma+i\tau$, $0\le v\le\pi$. *Then*

$$|S_n^{(\delta,\lambda)}(\cos v)| \le Ce^{c\tau^2}(n+1)^{2\lambda+1}.$$

**Proof.** By [18, p. 168], we have $|P_n^{(\lambda)}| \le C A_n^{(2\lambda-1)}$. Using this and Lemma 8, we see that

$$\begin{aligned}
|S_n^{(\delta,\lambda)}(\cos v)| &\le C|A_n^{(\delta)}|^{-1} \sum_{m=0}^{n} |A_{n-m}^{(\delta)}|(m+\lambda)A_m^{(2\lambda-1)} \\
&\le C\lambda |A_n^{(\delta)}|^{-1} \sum_{m=0}^{n} \frac{m+\lambda}{m+2\lambda} |A_{n-m}^{(\delta)}| A_m^{(2\lambda)} \\
&\le Ce^{c\tau^2} |A_n^{(\sigma)}|^{-1} \sum_{m=0}^{n} |A_{n-m}^{(\sigma)}| A_m^{(2\lambda)} \\
&\le Ce^{c\tau^2} |A_n^{(\sigma)}|^{-1} A_n^{(\sigma+2\lambda+1)} \le Ce^{c\tau^2}(n+1)^{2\lambda+1}.
\end{aligned}$$

∎

By (3.3)–(3.7) and Lemma 10, we have

$$|S_n^{(\delta,\lambda)}(\cos v)| \le Ce^{B\tau^2}(n+1)^{\lambda-\sigma}((n+1)^{-1}+\sin v)^{-\lambda-\sigma-1}, \tag{3.8}$$

where $0 \le v \le \pi$, $\lambda = 1/2$, $1/2 < \sigma < 1$. Suppose $\langle x,y\rangle = \cos v$, $x,y \in \Sigma_2$. Then $\sin v \sim |x-y|$ if $\langle x,y\rangle \ge 0$ and $\sin v \sim |x+y|$ if $\langle x,y\rangle \le 0$. Thus (3.8) implies

$$\begin{aligned}
&|S_n^{(\delta,\lambda)}(\langle x,y\rangle)| \\
&\le \begin{cases} Ce^{B\tau^2}(n+1)^{\lambda-\sigma}((n+1)^{-1}+|x-y|)^{-\lambda-\sigma-1}, & \text{if } \langle x,y\rangle \ge 0, \\ Ce^{B\tau^2}(n+1)^{\lambda-\sigma}((n+1)^{-1}+|x+y|)^{-\lambda-\sigma-1}, & \text{if } \langle x,y\rangle \le 0. \end{cases}
\end{aligned} \tag{3.9}$$

Since $S_n^\delta f(x) = \int_{\Sigma_2} S_n^{(\delta,1/2)}(\langle x,y\rangle) f(y)\,d\mu(y)$, the conclusion of Lemma 5 easily follows from (3.9).

**Remark.** In fact, we can prove estimates of the type in [6, Theorem (3.21)], partly improving (3.9). We do not need those estimates here; for our purpose (3.9) suffices.

## 4. Proofs of Lemmas 4, 6 and Theorem 3

We first prove Lemma 6.

**Proof of Lemma 6.** When $\delta > 0$, we have $\|S_*^\delta(f)\|_2 \le A_\delta \|f\|_2$ (see [4, Lemma (3.5)]). If $\delta = \sigma + i\tau$, $\sigma > 0$, $\tau \in \mathbb{R}$, we write

$$S_n^\delta(f) = (A_n^\delta)^{-1} \sum_{k=0}^{n} A_k^{(\sigma-\epsilon)} A_{n-k}^{(\epsilon-1+i\tau)} S_k^{\sigma-\epsilon}(f),$$

where $0 < \epsilon < \sigma$. Using Lemma 8 as in [4], we have $S_*^\delta(f) \le e^{c(\epsilon-1)\tau^2} S_*^{\sigma-\epsilon}(f)$. Combining these results, we reach the conclusion of Lemma 6.

∎

**Proof of Lemma 4.** Let $1 < p < 2$, $1/p = (1-\theta)/2 + \theta$, $\alpha = (1-\theta)c + \theta b$, where $c = \alpha - (1/2)(1/p - 1/2)$, $b = \alpha + (1/2)(1 - 1/p)$, $\alpha = 1/2$. We note that $\theta = 2(1/p - 1/2)$, $1/4 \le c \le \alpha$, $\alpha \le b \le 3/4$.

Define $T_z f = S_0^{\delta(z)} f$, $\delta(z) = (1-z)c + zb$, $0 \le \sigma \le 1$, $z = \sigma + i\tau$, $\tau \in \mathbb{R}$. Here $S_0^\delta$ is a linear operator approximating $S_*^\delta$ defined by $S_0^\delta f(x) = S_{n(x)}^\delta f(x)$, where $n(x)$ is a suitable non-negative mapping from $\Sigma_2$ to $\mathbb{Z}$, so that $\{T_z\}$ is an analytic family of linear operators which is admissible in the sense of [11] (see also [16, Chap. V, Section 4]).

We apply the analytic interpolation theorem on the Lorentz spaces $L^{p,q}$ due to [11]. Note that $\mathrm{Re}(\delta(i\tau)) = c \in [1/4, 1/2]$. Thus Lemma 6 implies

$$\|T_{i\tau} f\|_{2,2} \le C_0 e^{B_0 \tau^2} \|f\|_{2,2} \tag{4.1}$$

for some $B_0, C_0 > 0$. By Lemma 5 and the $L^1 - L^{1,\infty}$ boundedness of the maximal operator $M$ we have

$$\|T_{1+i\tau} f\|_{1,\infty} \le C_1 (p-1)^{-1} e^{B_1 \tau^2} \|f\|_{1,1} \tag{4.2}$$

for some $B_1, C_1 > 0$, since $\mathrm{Re}(\delta(1+i\tau)) = b$. Interpolating between (4.1) and (4.2), we get

$$\|S_0^\alpha f\|_{p,p'} = \|T_\theta f\|_{p,p'} \le A_\theta \|f\|_{p,p},$$

where

$$A_\theta \le C(p-1)^{-\theta} \le C(p-1)^{-1}.$$

Therefore

$$\|S_0^\alpha f\|_{p,\infty} \le C \|S_0^\alpha f\|_{p,p'} \le C(p-1)^{-1} \|f\|_p,$$

from which Lemma 4 follows. ■

To prove Theorem 3, we note that by Lemma 4, similarly to the case of $T_*^\alpha$, we can prove

$$\|S_*^{1/2} f\|_{1,\infty} \le C \|f\|_1 \log\left(\frac{e\|f\|_\infty}{\|f\|_1}\right) \tag{4.3}$$

if $f \in L^\infty(\Sigma_2)$, $f \ne 0$. Also, as in the case of $T_*^\alpha$, the estimate (4.3) readily implies $\|S_*^{1/2} f\|_{1,\infty} \le C \|f\|_{\mathscr{Q}A}$, from which the almost everywhere convergence follows.

## References


[1] N. Yu. Antonov, Convergence of Fourier series, *East J. Approx.*, **2** (1996), 187–196.

[2] J. Arias-de-Reyna, Pointwise convergence of Fourier series, *J. London Math Soc. (2)*, **65** (2002), 139–153.

[3] R. Askey and I. I. Hirschman, Mean summability for ultraspherical polynomials, *Math. Scand.*, **12** (1963), 167–177.

[4] A. Bonami and J.-L. Clerc, Sommes de Cesàro et multiplicateurs des développments en harmonique sphérique, *Trans. Amer. Math. Soc.*, **183** (1973), 223–263.

[5] L. Carleson, On convergence and growth of partial sums of Fourier series, *Acta Math.*, **116** (1966), 135–157.

[6] L. Colzani, M. H. Taibleson and G. Weiss, Maximal estimates for Cesàro and Riesz means on spheres, *Indiana Univ. Math. J.*, **33** (1984), 873–889.

[7] C. Fefferman, Pointwise convergence of Fourier series, *Ann. of Math.*, **98** (1973), 551–572.

[8] R. Hunt, On the convergence of Fourier series, *Orthogonal Expansions and their Continuous Analogues (Edwardsville, IL, 1967)*, Southern Illinois Univ. Press, 1968, 235–255.

[9] N. J. Kalton, Convexity type and the three space problem, *Studia Math.*, **69** (1981), 247–287.

[10] E. Kogbetliantz, Recherches sur la sommabilité des séries ultrasphériques par la méthode des moyennes arithmétiques, *J. Math. Pures Appl.*, **3** (1924), 107–187.

[11] Y. Sagher, On analytic families of operators, *Israel J. Math.*, **7** (1969), 350–356.

[12] P. Sjölin, An inequality of Paley and convergence a.e. of Walsh–Fourier series, *Ark. Mat.*, **7** (1968), 551–570.

[13] P. Sjölin and F. Soria, Remarks on a theorem by N. Yu. Antonov, *Studia Math.*, **158** (2003), 79–97.

[14] C. D. Sogge, Oscillatory integrals and spherical harmonics, *Duke Math. J.*, **53** (1986), 43–65.

[15] E. M. Stein, Localization and summability of multiple Fourier series, *Acta Math.*, **100** (1958), 93–147.

[16] E. M. Stein and G. Weiss, *Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, 1971.

[17] G. Sunouchi, On the summability almost everywhere of the multiple Fourier series at the critical index, *Kodai Math. J.*, **8** (1985), 1–4.

[18] G. Szegö, *Orthogonal Polynomials*, 4th ed., Amer Math. Soc. Coll. Publ. No. 23, Amer. Math. Soc., Providence, R. I. , 1975.

[19] A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge Univ. Press, Cambridge – London – New York – Melbourne, 1977.

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