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On the critical case of Okamoto's continuous non-differentiable functions

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Abstract: In a recent paper in this Proceedings, H. Okamoto presented a parameterized family of continuous functions which contains Bourbaki's and Perkins's nowhere differentiable functions as well as the Cantor-Lebesgue singular function. He showed that the function changes its differentiability from 'differentiable almost everywhere' to 'non-differentiable almost everywhere' at a certain parameter value. However, differentiability of the function at the critical parameter value remained unknown. For this problem, we prove that the function is non-differentiable almost everywhere at the critical case.

Key words: Continuous non-differentiable function; the law of the iterated logarithm.

1. Introduction. We consider a parameterized family of continuous functions which were presented by H. Okamoto [3, 4]. This function can be regarded as a generalization of Bourbaki's [1] and Perkins's [5] nowhere differentiable functions as well as of the Cantor-Lebesgue singular function.

Okamoto's function is constructed as the limit of a sequence $\{f_n\}_{n=0}^{\infty}$ of piecewise linear and continuous functions. For a fixed parameter $a \in (0, 1)$, each function in the sequence is defined as follows:

- (i) $f_0(x) = x$,
- (ii) $f_{n+1}(x)$ is continuous on $[0, 1]$,
- (iii) $f_{n+1}\left(\frac{k}{3^n}\right) = f_n\left(\frac{k}{3^n}\right)$,
 $f_{n+1}\left(\frac{3k+1}{3^{n+1}}\right) = f_n\left(\frac{k}{3^n}\right) + a\left[f_n\left(\frac{k+1}{3^n}\right) - f_n\left(\frac{k}{3^n}\right)\right]$,
 $f_{n+1}\left(\frac{3k+2}{3^{n+1}}\right) = f_n\left(\frac{k}{3^n}\right) + (1-a)\left[f_n\left(\frac{k+1}{3^n}\right) - f_n\left(\frac{k}{3^n}\right)\right]$,
 $f_{n+1}\left(\frac{k+1}{3^n}\right) = f_n\left(\frac{k+1}{3^n}\right)$,
for $k = 0, 1, \dots, 3^n - 1$,

- (iv) $f_{n+1}(x)$ is linear in each subinterval

$$\frac{k}{3^{n+1}} \leq x \leq \frac{k+1}{3^{n+1}} \quad \text{for}$$

$$k = 0, 1, \dots, 3^{n+1} - 1.$$

Figure 1 shows the operation from f_n to f_{n+1} . Okamoto's function $F_a(x)$ is then defined as

$$F_a(x) = \lim_{n \rightarrow \infty} f_n(x).$$

He noticed that $F_a(x)$ is continuous on $[0, 1]$ and coincides with some known functions when a takes particular values. For example, the cases $a = 5/6$ and $a = 2/3$ correspond to nowhere-differentiable functions defined by Perkins [5] and Bourbaki [1] respectively. Also, if $a = 1/2$, F_a is the Cantor-Lebesgue singular function which is non-decreasing and has zero derivative almost everywhere (Fig. 2).

2. Differentiability of F_a . In the paper [3], H. Okamoto proved that $F_a(x)$ has the following features:

- (i) If $a < a_0$, then $F_a(x)$ is differentiable almost everywhere.
- (ii) If $a_0 < a < 2/3$, then $F_a(x)$ is non-differentiable almost everywhere.
- (iii) If $2/3 \leq a < 1$, then $F_a(x)$ is nowhere differentiable.

Here, the constant $a_0 (= 0.5592\dots)$ is the unique real root of

$$54a^3 - 27a^2 = 1.$$

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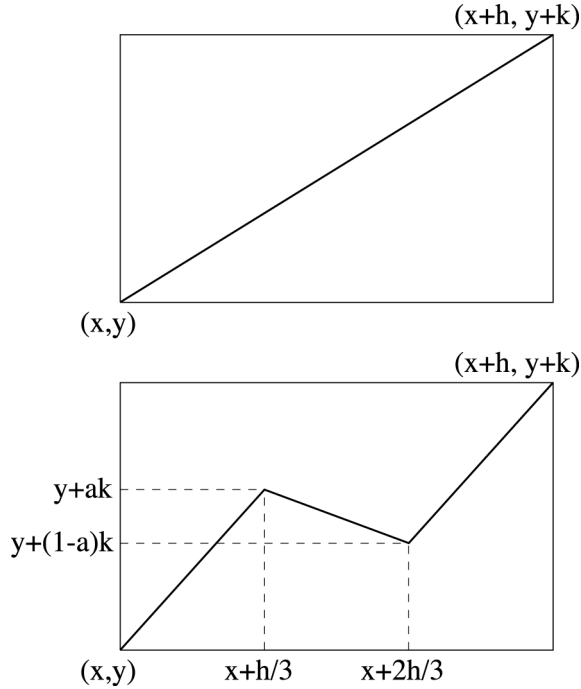


Fig. 1. The operation from f_n to f_{n+1} . Before the operation (top) and after the operation (bottom). This operation is performed in each subinterval $[k/3^n, (k+1)/3^n]$.

As for the case $a = a_0$, it remained open whether $F_a(x)$ is differentiable almost everywhere or non-differentiable almost everywhere. In this case, we proved that $F_a(x)$ is non-differentiable almost everywhere.

3. Main result. The main result of this article is the following

Theorem 1. *If $a = a_0$, then $F_a(x)$ is non-differentiable almost everywhere in $[0, 1)$.*

In order to prove this theorem, we need some definitions and a preliminary lemma concerning with the law of the iterated logarithm [2].

Definitions. Let

$$x = \sum_{n=1}^{\infty} \frac{\xi_n(x)}{3^n}, \quad \xi_n(x) \in \{0, 1, 2\},$$

denote the ternary expansion of $x \in [0, 1)$. If x is a rational number of the form $k/3^n$, we use the ternary expansion ending in all 0's (instead of the one ending in all 2's). We also use the following notations:

$$c(k) = \begin{cases} 1, & (k = 0 \text{ or } k = 2), \\ -2, & (k = 1), \end{cases}$$

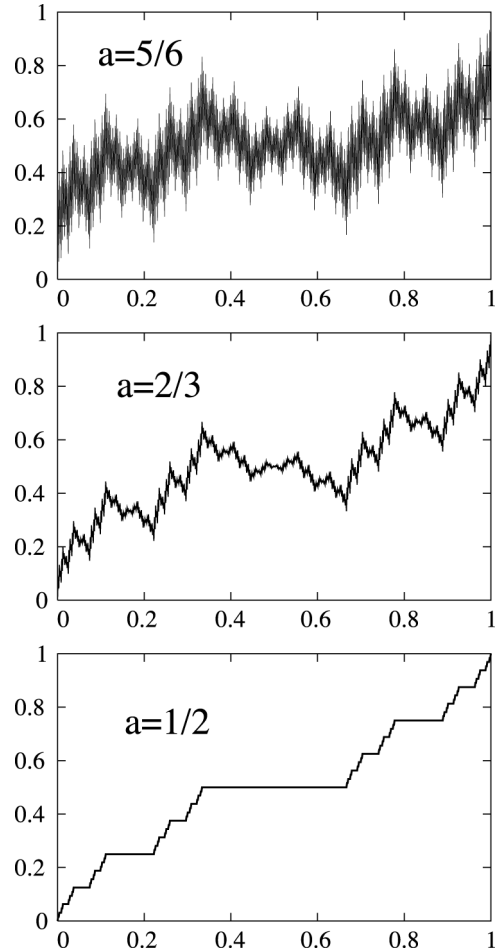


Fig. 2. The graph of Perkins's function (top), Bourbaki's function (middle) and the Cantor-Lebesgue singular function (bottom).

and

$$S_n(x) = \sum_{k=1}^n c(\xi_k(x)),$$

$$T_n(x) = 1(\xi_{n+1}(x) = 1) \cdot S_n(x),$$

where $1(A)$ is the indicator function that takes the value one if argument A is true and zero otherwise.

With these definitions, we have the following lemma:

Lemma 1.

$$\limsup_{n \rightarrow \infty} \frac{T_n(x)}{\sqrt{n}} \geq 1$$

holds for almost every $x \in [0, 1)$.

Proof. Since the $c(\xi_n)$ are i.i.d. random variables with mean 0 and variance 2 with respect to Lebesgue measure on $(0, 1)$, the law of the iterated logarithm [2] implies that

$$\limsup_{n \rightarrow \infty} \frac{S_n(x)}{\sqrt{4n \log \log n}} = 1$$

and

$$\liminf_{n \rightarrow \infty} \frac{S_n(x)}{\sqrt{4n \log \log n}} = -1$$

almost everywhere in $(0, 1)$.

Thus in particular, the events $S_n(x)/\sqrt{n} \geq 1$ and $S_n(x)/\sqrt{n} \leq -1$ both happen infinitely often. Each time $S_n(x)/\sqrt{n}$ exits the interval $[1, \infty)$, it must do so with a value of

$$c(\xi_{n+1}(x)) = -2$$

(the only negative value). Thus,

$$\frac{T_n(x)}{\sqrt{n}} \geq 1$$

happens infinitely often as well. \square

We now complete the proof of the main theorem.

Proof of theorem 1. We first note that $F_a(x)$ has the following representation:

$$F_a(x) = \sum_{k=1}^{\infty} \Psi_k(x),$$

$$\Psi_k(x) = \prod_{l=1}^{k-1} p(\xi_l(x)) \cdot q(\xi_k(x)),$$

where

$$p(0) = a, \quad p(1) = 1 - 2a, \quad p(2) = a,$$

$$q(0) = 0, \quad q(1) = a, \quad q(2) = 1 - a.$$

In what follows, we assume that $a = a_0$ and x satisfies

$$\limsup_{n \rightarrow \infty} \frac{T_n(x)}{\sqrt{n}} \geq 1.$$

From the definition of $T_n(x)$, we can take an increasing sequence $\{r_n\}$ which satisfies

$$\sum_{k=1}^{r_n} c(\xi_k(x)) \geq \sqrt{r_n}, \quad \xi_{r_n+1}(x) = 1, \quad n = 1, 2, 3, \dots$$

Here we define $\{x_n\}$ by

$$x_n = \sum_{k=1}^{r_n} \frac{\xi_k(x)}{3^k}.$$

Then,

$$x - x_n = \sum_{k=r_n+1}^{\infty} \frac{\xi_k(x)}{3^k} \geq \frac{1}{3^{r_n+1}} > 0$$

and we have the following evaluation:

$$\begin{aligned} \left| \frac{F_a(x) - F_a(x_n)}{x - x_n} \right| &= \left| \frac{\sum_{k=r_n+1}^{\infty} \Psi_k(x)}{\sum_{k=r_n+1}^{\infty} \xi_k(x)/3^k} \right| \\ &\geq \frac{\left| \sum_{k=r_n+1}^{\infty} \Psi_k(x) \right|}{1/3^{r_n}} \\ &= 3^{r_n} \prod_{l=1}^{r_n} |p(\xi_l(x))| \cdot \left| q(\xi_{r_n+1}(x)) \right| \\ &\quad + \sum_{k=r_n+2}^{\infty} \prod_{l=r_n+1}^{k-1} p(\xi_l(x)) \cdot q(\xi_k(x)) \Big| \\ &\geq 3^{r_n} \prod_{l=1}^{r_n} |p(\xi_l(x))| \left(q(\xi_{r_n+1}(x)) \right. \\ &\quad \left. - \sum_{k=r_n+2}^{\infty} \prod_{l=r_n+1}^{k-1} |p(\xi_l(x))| q(\xi_k(x)) \right) \\ &\geq 3^{r_n} \prod_{l=1}^{r_n} |p(\xi_l(x))| \\ &\quad \left(q(1) - \sum_{k=1}^{\infty} |p(1)| \max_{0 \leq l \leq 2} |p(l)|^{k-1} \max_{0 \leq l \leq 2} q(l) \right) \\ &= 3^{r_n} \prod_{l=1}^{r_n} |p(\xi_l(x))| \left(a - (2a - 1) \sum_{k=1}^{\infty} a^k \right) \\ &= \frac{a(2 - 3a)}{1 - a} \exp \left(\sum_{l=1}^{r_n} \log |3p(\xi_l(x))| \right). \end{aligned}$$

Using the following relations:

$$\log |3p(0)| = \log |3p(2)| = \log(3a),$$

$$\log |3p(1)| = \log |3(1 - 2a)| = \log \left| \frac{27a^2 - 54a^3}{9a^2} \right|$$

$$= \log \left| -\frac{1}{9a^2} \right| = -2 \log(3a),$$

we obtain

$$\begin{aligned} \left| \frac{F_a(x) - F_a(x_n)}{x - x_n} \right| &\geq \frac{a(2 - 3a)}{1 - a} \exp \left(\log(3a) \sum_{l=1}^{r_n} c(\xi_l(x)) \right) \\ &\geq \frac{a(2 - 3a)}{1 - a} (3a)^{\sqrt{r_n}}. \end{aligned}$$

It follows that

$$\lim_{n \rightarrow \infty} \left| \frac{F_a(x) - F_a(x_n)}{x - x_n} \right| = \infty.$$

Namely, $F_a(x)$ is non-differentiable at x . From the previous lemma, we know that

$$\limsup_{n \rightarrow \infty} \frac{T_n(x)}{\sqrt{n}} \geq 1$$

holds almost everywhere in $[0, 1)$, and so, we can conclude that $F_a(x)$ is non-differentiable almost everywhere in $[0, 1)$.

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