**Evaluation modules for the three-point sl2 loop algebra (Finite Groups and Algebraic Combinatorics)**

<table>
<thead>
<tr>
<th>著者</th>
<th>伊藤 達郎</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>数理解析研究所講究録</td>
</tr>
<tr>
<td>出版物</td>
<td></td>
</tr>
<tr>
<td>年</td>
<td>2008-04-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2297/36712">http://hdl.handle.net/2297/36712</a></td>
</tr>
</tbody>
</table>
Overview

1. The tetrahedron algebra realization of the three-point $sl_2$ loop algebra
2. The f.d. irreducible modules
3. The evaluation modules
4. The $S_4$-action on the evaluation modules
5. 24 bases for an evaluation module
6. Realization of the evaluation modules by polynomials in two variables

Warmup: The Lie algebra $sl_2$

Throughout, $F$ will denote an algebraically closed field with characteristic 0.

Recall that $sl_2$ is the Lie algebra over $F$ with a basis $e, f, h$ and Lie bracket

$[h, e] = 2e$,  \hspace{1cm} [h, f] = -2f$

$[e, f] = h$.

The equitable basis for $sl_2$

Define

$x = 2e - h$,  \hspace{1cm} y = -2f - h$,  \hspace{1cm} z = h$.

Then $x, y, z$ is a basis for $sl_2$ and

$[x, y] = 2x + 2y$,
$[y, z] = 2y + 2z$,
$[z, x] = 2z + 2x$.

We call $x, y, z$ the equitable basis for $sl_2$.

The three-point $sl_2$ loop algebra

The three-point $sl_2$ loop algebra is the Lie algebra over $F$ consisting of the vector space

$sl_2 \otimes F[t, t^{-1}, (t-1)^{-1}]$,  \hspace{1cm} \otimes = \otimes_F$

where $t$ is indeterminate, and Lie bracket

$[u \otimes a, v \otimes b] = [u, v] \otimes ab$. 

The tetrahedron algebra $\mathfrak{S}$

Definition (Hartwig + T) The tetrahedron algebra $\mathfrak{S}$ is the Lie algebra over $F$ that has generators

$$\{x_{ij} | i, j \in \mathbb{I}, i \neq j\}$$

and the following relations:

(i) For distinct $i, j \in \mathbb{I}$,

$$x_{ij} + x_{ji} = 0.$$

(ii) For mutually distinct $i, j \in \mathbb{I}$,

$$[x_{hi}, x_{ij}] = 2x_{hi} + 2x_{ij}.$$

(iii) For mutually distinct $i, j, k \in \mathbb{I}$,

$$[x_{hi}, [x_{hi}, x_{jk}]] = 4[x_{hi}, x_{jk}].$$

$\mathfrak{S}$ and the three-point $sl_2$ loop algebra

Theorem (Hartwig + T) There exists an isomorphism of Lie algebras

$$\psi : \mathfrak{S} \rightarrow sl_2 \otimes F[t, t^{-1}, (t-1)^{-1}]$$

that sends

$$z_{12} \mapsto z \otimes 1, \quad z_{13} \mapsto y \otimes 1 + z \otimes (t-1),$$

$$z_{23} \mapsto y \otimes 1, \quad z_{23} \mapsto z \otimes (1-t)^{-1} - z \otimes t^{-1},$$

$$z_{10} \mapsto z \otimes 1, \quad z_{10} \mapsto z \otimes (1-t)^{-1} + y \otimes (1-t)^{-1}$$

where $z, y, z$ is the equitable basis for $sl_2$.

From now on we work with $\mathfrak{S}$.

Finite-dimensional irreducible $\mathfrak{S}$-modules

Our goal is to describe the f.d. irreducible $\mathfrak{S}$-modules.

For these modules there is a special case called an evaluation module.

It turns out that every f.d. irreducible $\mathfrak{S}$-module is a tensor product of evaluation modules.

After some general remarks we focus on the evaluation modules.

Decompositions

Let $V$ denote a f.d. irreducible $\mathfrak{S}$-module.

By a decomposition of $V$ we mean a sequence $\{V_n\}_{n=0}^{d}$ of nonzero subspaces of $V$ such that

$$V = \sum_{n=0}^{d} V_n$$

(direct sum).

We call $d$ the diameter of the decomposition.

By the shape of this decomposition we mean the sequence $\{\dim(V_n)\}_{n=0}^{d}$. 
The decompositions \([i,j]\)

Hartwig showed:

(i) Each generator \(x_{ij}\) is semisimple on \(V'\).

(ii) There exists an integer \(d \geq 0\) such that for each generator \(x_{ij}\) the set of distinct eigenvalues on \(V'\) is

\[\{2n - d \mid 0 \leq n \leq d\}.

We let \([i,j]\) denote the eigenspace decomposition for \(x_{ij}\) on \(V\) associated with the above ordering of the eigenvalues.

The shape of \(V\)

Hartwig showed that the shape of the decomposition \([i,j]\) is independent of the pair \(i,j\).

We call this common shape the shape of \(V\).

The evaluation modules for \(B\)

We now define the evaluation modules for \(B\).

For \(a \in F\setminus\{0,1\}\) we define a Lie algebra homomorphism

\[\psi : \mathfrak{s}\mathfrak{l}_2 \otimes \mathbb{F}[t, t^{-1}, (t - 1)^{-1}] \to \mathfrak{s}\mathfrak{l}_2\]

\[u \otimes f(t) \to uf(a)\]

For an \(\mathfrak{s}\mathfrak{l}_2\)-module \(V\) we pull back the \(\mathfrak{s}\mathfrak{l}_2\)-module structure via \(\psi\); this turns \(V'\) into a \(B\)-module which we call \(V(a)\).

The evaluation modules for \(B\), cont.

By an evaluation module for \(B\) we mean the module \(V_d(a)\) where

(i) \(d\) is a positive integer;

(ii) \(V_d\) is the irreducible \(\mathfrak{s}\mathfrak{l}_2\)-module with dimension \(d + 1\).

The \(B\)-module \(V_d(a)\) is nontrivial and irreducible.

We call \(a\) the evaluation parameter for \(V_d(a)\).
Characterizing the evaluation modules, I

Theorem For a nontrivial f.d. irreducible $\mathcal{B}$-module $V$ TFAE:

(i) $V$ is isomorphic to an evaluation module for $\mathcal{B}$.

(ii) $V$ has shape $(1,1,\ldots,1)$.

Characterizing the evaluation modules, II

Theorem Let $V$ denote a nontrivial f.d. irreducible $\mathcal{B}$-module.

Then for $a \in F \setminus \{0,1\}$ TFAE:

(i) $V$ is isomorphic to an evaluation module with evaluation parameter $a$.

(ii) Each of the following vanishes on $V$:

\[
\begin{align*}
ax_{01} + (1-a)x_{02} &= x_{03}, \\
ax_{10} + (1-a)x_{13} &= x_{12}, \\
ax_{23} + (1-a)x_{20} &= x_{21}, \\
ax_{32} + (1-a)x_{31} &= x_{30}.
\end{align*}
\]

An $S_4$-action on $\mathcal{B}$-modules

For a $\mathcal{B}$-module $V$ and $a \in S_4$ there exists a $\mathcal{B}$-module structure on $V$, called $V_\sigma$ twisted via $\sigma$, that behaves as follows:

For $u \in \mathcal{B}$ and $v \in V$, the vector $u \cdot v$ computed in $V_\sigma$ twisted via $\sigma$ coincides with the vector $\sigma^{-1}(u) \cdot v$ computed in the original $\mathcal{B}$-module $V$.

Sometimes we abbreviate $\sigma V$ for $V$ twisted via $\sigma$.

$S_4$ acts on the set of $\mathcal{B}$-modules, with $\sigma$ sending $V$ to $\sigma V$ for all $\sigma \in S_4$ and all $\mathcal{B}$-modules $V$.

An action of $S_4$ on $F \setminus \{0,1\}$

Lemma There exists an action of $S_4$ on the set $F \setminus \{0,1\}$ that does the following.

For $a \in F \setminus \{0,1\}$:

- $(2,0)$ sends $a \mapsto a^{-1}$;
- $(0,1)$ sends $a \mapsto a(a-1)^{-1}$;
- $(1,3)$ sends $a \mapsto a^{-1}$.

The $S_4$-action on $\mathcal{B}$-modules, cont.

The above $S_4$-action on $\mathcal{B}$-modules sends evaluation modules to evaluation modules.

The effect of this action on the evaluation parameter is described in the following two slides.
The effect of $S_4$ on the evaluation parameter

Theorem For an integer $d \geq 1$, $\sigma \in S_4$, and $a \in \mathbb{F}\{0,1\}$ the following are isomorphic:
(i) The $\mathbb{F}$-module $V_d(a)$ twisted via $\sigma$;
(ii) The $\mathbb{F}$-module $V_d(\sigma(a))$.

A subgroup $G$ of $S_4$

Earlier we gave an action of $S_4$ on the set $\mathbb{F}\{0,1\}$.
Let $G$ denote the kernel of this action.
It turns out that $G$ consists of
$(01)(23), (02)(13), (03)(12)$
together with the identity element.

The subgroup $G$ of $S_4$, cont.

Corollary For an integer $d \geq 1$, for $\sigma \in G$, and for $a \in \mathbb{F}\{0,1\}$ the following are isomorphic:
(i) The $\mathbb{F}$-module $V_d(a)$ twisted via $\sigma$;
(ii) The $\mathbb{F}$-module $V_d(\sigma(a))$.

We will return to the subgroup $G$ later in the talk.

The orbits of $S_4$ on $\mathbb{F}\{0,1\}$, cont.

The relative function satisfies this recursion:

Lemma Pick $a \in \mathbb{F}\{0,1\}$ and mutually distinct $i,j,k,\ell \in I$.
Let $\alpha$ denote the $(i,j,k,\ell)$-relative of $a$. Then
- $\alpha^{-1}$ is the $(j,i,k,\ell)$-relative of $a$;
- $\alpha(\alpha - 1)^{-1}$ is the $(i,k,j,\ell)$-relative of $a$;
- $\alpha^{-1}$ is the $(i,j,\ell,k)$-relative of $a$.

The orbits of $S_4$ on $\mathbb{F}\{0,1\}$, cont.

Here is another way to view the relative function.

Lemma For $a \in \mathbb{F}\{0,1\}$ and mutually distinct $i,j,k,\ell \in I$ the following (i), (ii) coincide:
(i) the $(i,j,k,\ell)$-relative of $a$;
(ii) the scalar
\[
\begin{align*}
\frac{i-k}{i-k} & = a, \\
\frac{j-k}{j-k} & = 1, \\
\frac{\ell-k}{\ell-k} & = 2, \\
\frac{\ell-j}{\ell-j} & = \infty.
\end{align*}
\]
The orbits of $S_4$ on $F\setminus\{(0,1)\}$, cont.

Here is an explicit description of the relative function.

**Theorem** Pick $a \in F\setminus\{(0,1)\}$ and mutually distinct $i, j, k, \ell \in I$.

Then the $(i, j, k, \ell)$-relative of $a$ is given in the following table.

<table>
<thead>
<tr>
<th>$(i, j, k, \ell)$</th>
<th>$(i, j, k, \ell)$-relative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 2, 1, 1)$</td>
<td>$(1, 0, 2, 3)$</td>
</tr>
<tr>
<td>$(1, 0, 2, 1)$</td>
<td>$(2, 2, 1, 0)$</td>
</tr>
<tr>
<td>$(0, 2, 1, 3)$</td>
<td>$(3, 1, 2, 0)$</td>
</tr>
<tr>
<td>$(1, 0, 1, 3)$</td>
<td>$(1, 0, 1, 1)$</td>
</tr>
<tr>
<td>$(0, 2, 1, 2)$</td>
<td>$(2, 1, 2, 1)$</td>
</tr>
<tr>
<td>$(1, 0, 1, 2)$</td>
<td>$(2, 2, 1, 1)$</td>
</tr>
<tr>
<td>$(0, 1, 2, 3)$</td>
<td>$(3, 2, 1, 0)$</td>
</tr>
<tr>
<td>$(2, 1, 2, 1)$</td>
<td>$(2, 2, 1, 1)$</td>
</tr>
</tbody>
</table>

**Location of $\eta_i$ ($i \in I$)**

**The basis** $[i, j, k, \ell]$ for $V_d(a)$

**Lemma** For mutually distinct $i, j, k, \ell \in I$ there exists a unique basis $\{u_n\}_n$ for $V_d(a)$ such that:

1. For $0 \leq n \leq d$ the vector $u_n$ is contained in component $n$ of the decomposition $[k, \ell]$.
2. $\eta_i = \sum_{n=0}^d u_n$.

We denote this basis by $[i, j, k, \ell]$.

We have now defined 24 bases for $V_d(a)$.

**The vectors $\eta_i$ ($i \in I$) in $V_d(a)$**

For notational convenience, for $i \in I$ we fix a nonzero vector $\eta_i \in V_d(a)$ which is a common eigenvector for $\{x_j\}_j$ $j \in I, j \neq i$.

**The basis** $[i, j, k, \ell]$ for $V_d(a)$
How the generators \( z_{ij} \) act on the 24 bases

**Theorem** For mutually distinct \( i, j, k, \ell \in I \) and distinct \( r, s \in I \) consider the matrix representing \( z_{rs} \) with respect to the basis \( [i, j, k, \ell] \) of \( V_d(a) \). The entries of this matrix are given in the following table. All entries not displayed are zero.

<table>
<thead>
<tr>
<th>( g_m )</th>
<th>( (n-1) )-entry</th>
<th>( (n) )-entry</th>
<th>( (n-1) )-entry</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a \bar{a} )</td>
<td>2a</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a^2 )</td>
<td>2a</td>
<td>2a</td>
<td>0</td>
</tr>
<tr>
<td>( a^2 \bar{a} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a^3 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a^3 \bar{a} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a^4 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a^4 \bar{a} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a^5 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a^5 \bar{a} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a^6 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a^6 \bar{a} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a^7 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a^7 \bar{a} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a^8 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a^8 \bar{a} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In the above table the scalar \( \alpha \) denotes the \( (i, j, k, \ell) \)-relative of \( a \).

A bilinear form on \( V_d(a) \)

**Lemma** There exists a nonzero bilinear form \( ( , ) \) on \( V_d(a) \) such that

\[
(w, u, v) = -(u, w, v) \quad w, u, v \in V.
\]

The form is nondegenerate.

The form is unique up to multiplication by a nonzero scalar in \( F \).

The form is symmetric (resp. antisymmetric) when \( d \) is even (resp. \( d \) is odd).

We call \( ( , ) \) a standard bilinear form for \( V_d(a) \).

Some transition matrices

We now consider the transition matrices between our 24 bases.

In order to describe these, it is convenient to introduce a certain bilinear form on \( V_d(a) \).

**Theorem** Referring to \( V_d(a) \), pick mutually distinct \( i, j, k, \ell \in I \) and consider the transition matrices from the bases \([i, j, k, \ell],[i, j, k, \ell],[i, j, k, \ell]\) to the bases \([i, j, k, \ell],[i, j, k, \ell],[i, j, k, \ell]\).

(i) The first transition matrix is diagonal with \( (r, r) \)-entry

\[
(\alpha^r_1, \alpha^r_2) \quad (r, r)
\]

for \( 0 \leq r \leq d \), where \( \alpha \) is the \( (i, j, k, \ell) \)-relative of \( a \).

(ii) The second transition matrix is lower triangular with \( (r, s) \)-entry

\[
\binom{\alpha^r_s}{\alpha^r_1} (1 - \alpha)^s
\]

for \( 0 \leq s \leq r \leq d \), where \( \alpha \) is the \( (i, j, k, \ell) \)-relative of \( a \).

(iii) The third transition matrix is the matrix \( Z \).

The matrix \( Z \)

The following matrix will play a role in our discussion.

For an integer \( d \geq 0 \) let \( Z = Z(d) \) denote the matrix in \( \mathbb{M}_{d+1}(F) \) with entries

\[
Z_{ij} = \begin{cases} 
1, & \text{if } i + j = d; \\
0, & \text{otherwise}.
\end{cases}
\]

We observe

\[
Z^2 = I.
\]

Realizing the evaluation modules for \( \mathfrak{B} \) using polynomials in two variables

Let \( x_0, x_1 \) denote commuting indeterminates.

Let \( F[x_0, x_1] \) denote the \( F \)-algebra of all polynomials in \( x_0, x_1 \) that have coefficients in \( F \).

We abbreviate \( \mathfrak{A} = F[x_0, x_1] \).

We often view \( \mathfrak{A} \) as a vector space over \( F \).

For an integer \( d \geq 0 \) let \( \mathfrak{A}_d \) denote the subspace of \( \mathfrak{A} \) consisting of the homogeneous polynomials in \( x_0, x_1 \) that have total degree \( d \).

Thus \( \{x_0^d x_1^s \}_{s=0}^d \) is a basis for \( \mathfrak{A}_d \).
Realizing the evaluation modules

Note that

\[ \mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_d \quad \text{(direct sum)} \]

and that

\[ \mathcal{A}_r \mathcal{A}_s = \mathcal{A}_{r+s} \quad (r, s \geq 0). \]

We fix mutually distinct \( \beta_i \in \mathbb{F} \) (\( i \in \mathbb{I} \)).

Then there exist unique \( z_2, z_3 \in \mathcal{A} \) such that

\[ \sum_{i \in \mathbb{I}} z_i = 0, \quad \sum_{i \in \mathbb{I}} \beta_i z_i = 0. \]

Comments on the \( z_i \) (\( i \in \mathbb{I} \))

Lemma For mutually distinct \( i, j, k, l \in \mathbb{I} \) we have

\[ z_k = \frac{\beta_l - \beta_k}{\beta_l - \beta_k} z_i + \frac{\beta_l - \beta_j}{\beta_l - \beta_k} z_j, \]

\[ z_l = \frac{\beta_k - \beta_l}{\beta_k - \beta_l} z_i + \frac{\beta_k - \beta_j}{\beta_k - \beta_l} z_j. \]

Some bases for \( \mathcal{A} \)

Lemma For distinct \( i, j \in \mathbb{I} \) the elements

\[ z_i, z_j \]

form a basis for \( \mathcal{A} \).

Example: Some bases for \( \mathcal{A}_3 \)

Some bases for \( \mathcal{A}_d \)

Lemma For an integer \( d \geq 0 \) and distinct \( i, j \in \mathbb{I} \) the elements \( \{z_{d-n}^i \}_{n=0}^{d} \) form a basis for \( \mathcal{A}_d \).

Derivations of \( \mathcal{A} \)

Our next goal is to display a \( \mathbb{F} \)-module structure on \( \mathcal{A} \).

We will use the following terms.

By a derivation of \( \mathcal{A} \) we mean an \( \mathbb{F} \)-linear map \( D: \mathcal{A} \rightarrow \mathcal{A} \) such that

\[ D(uv) = D(u)v + uD(v) \quad (u, v \in \mathcal{A}). \]
Theorem There exists a unique $\mathfrak{B}$-module structure on $\mathcal{A}$ such that:

(i) each element of $\mathfrak{B}$ acts as a derivation on $\mathcal{A}$;

(ii) $x_{ij}x_i = -x_i$ and $x_{ij}x_j = x_j$ for distinct $i, j \in \mathbb{I}$.

The eigenvectors for the $x_{ij}$ on $\mathcal{A}$

Lemma for distinct $i, j \in \mathbb{I}$ and integers $r, s \geq 0$ the element $x_i^r x_j^s$ is an eigenvector for $x_{ij}$ with eigenvalue $s - r$.

The irreducible $\mathfrak{B}$-submodules of $\mathcal{A}$

Proposition Referring to the $\mathfrak{B}$-module $\mathcal{A}$,

(i) For $d \geq 0$ the subspace $\mathcal{A}_d$ is an irreducible $\mathfrak{B}$-submodule of $\mathcal{A}$.

(ii) The $\mathfrak{B}$-module $\mathcal{A}_0$ is trivial.

(iii) For $d \geq 1$ the $\mathfrak{B}$-module $\mathcal{A}_d$ is isomorphic to $V_d(a)$ where

$$a = \frac{b_0 - b_1 b_2 - b_3}{b_0 - b_3 b_2 - b_1}.$$

The decomposition $[i, j]$ for $\mathcal{A}_d$

Earlier in the talk we described the $\mathfrak{B}$-module $V_d(a)$.

We now consider how things look from the point of view of $\mathcal{A}_d$.

Proposition For an integer $d \geq 0$ and for distinct $i, j \in \mathbb{I}$ the decomposition $[i, j]$ on $\mathcal{A}_d$ is described as follows.

For $0 \leq n \leq d$ the $n$th component is spanned by $x_i^{d-n} x_j^n$.

The elements $\eta_i$ ($i \in \mathbb{I}$) for $\mathcal{A}_d$

For an integer $d \geq 1$ and $i \in \mathbb{I}$ the element $x_i^d$ is a scalar multiple of $\eta_i$.

Recall $\eta_i$ is defined up to scalar multiplication.

For the rest of the talk we choose $\eta_i = x_i^d$.

The basis $[i, j, k, \ell]$ for $\mathcal{A}_d$

Proposition For an integer $d \geq 1$ and for mutually distinct $i, j, k, \ell \in \mathbb{I}$ the basis $[i, j, k, \ell]$ of $\mathcal{A}_d$ is described as follows.

For $0 \leq n \leq d$ the $n$th component is

$$x_i^{d-n} x_j^n (\ell - \beta_i)^{d-n} (k - \beta_i)^n$$

$$= \frac{(\ell - \beta_i)^{d-n} (k - \beta_i)^n}{(\ell - \beta_j)^{d-n} (k - \beta_j)^n}.$$
The group $G$ revisited

We saw earlier that if we twist the $\mathbb{S}$-module $V_d(a)$ via an element of $G$ then the result is isomorphic to $V_d(a)$.

We now explain this fact using $A$.

Some automorphisms of $A$

Lemma For mutually distinct $i, j, k, \ell \in I$ there exists a unique automorphism of $A$ that sends

- $z_i \mapsto \beta_j - \beta_k$
- $z_j \mapsto \beta_i - \beta_k$
- $z_k \mapsto \beta_i - \beta_j$
- $z_\ell \mapsto \beta_i - \beta_k$

Some automorphisms of $A$

Theorem The following hold for $a \in G$:

(i) There exists an automorphism $g_r$ of $A$ that sends $z_r$ to a scalar multiple of $z_{\alpha(r)}$ for all $r \in I$.

(ii) For $u \in \mathbb{S}$ the equation

\[ \sigma(u) = g_u g_{u^{-1}} \]

holds on $A$.

(iii) The map $g_r$ is an isomorphism of $\mathbb{S}$-modules from $A$ to $A$ twisted via $\sigma$.

THE END