Non-vanishing Terms of the Jones Polynomial

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Abstract. *We consider the Tutte polynomial for the graph associated to the $(2, 2k+1)$ torus and twist knot. Up to a sign and multiplication by a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $\chi(G; -t, -t^{-1})$. Therefore, the Jones polynomial could be calculated by using the Tutte polynomial for $(2, 2k+1)$ torus and twist knot. The Jones polynomial has a vanishing term if the knot is a $(2, 2k+1)$ torus knot, but there is no vanishing term if the knot is a twist knot. We look for graphs which the associated with 3-tuple of pretzel link have non-vanishing terms in the Jones polynomial. The term Jones polynomial is proven to be non-vanishing by calculated the Tutte polynomial of the given graph.*

Keywords: Graph theory, knot theory, the Tutte polynomial, the Jones polynomial

1 Introduction

A link is a finite family of disjoint, smooth, oriented or unoriented, closed curves in $\mathbb{R}^3$ or equivalently S^3. A knot is a link with one component. Suppose L be an unoriented link, $w(L)$ denotes the writhe of L. We define the normalized bracket polynomial $X(L) = (-A^3)^{-w(L)}\langle L\rangle$. Then, we have the Jones polynomial

$$V_L(t) = (-A^3)^{-w(L)}\langle L\rangle\Big|_{A=t^{-\frac{1}{4}}} \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}] \tag{1}$$

The torus knot $T(2, 2k+1)$

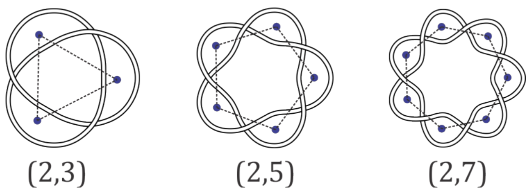

Figure 1: The $(2, 2k+1)$ torus knot, $k = 1, 2, \ldots$

Let G_k be the medial graph of $T(2, 2k+1)$ (Fig. 2).
The Jones polynomial of $T(2, 2k-1)$

$$V_{T(2,2k+1)} = t^k + \sum_{i=1}^{2k}(-1)^{i+1}t^{i+k+1} \tag{2}$$

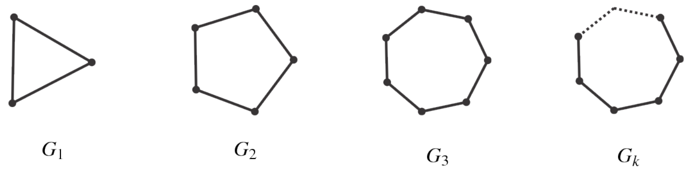

Figure 2: Medial graph of $T(2, 2k + 1)$.

The Jones polynomial of these knot are alternating and has zero coefficient at t^{k+1}. For example:

$$\begin{aligned} k = 1 \Rightarrow V_{T(2,3)}(t) &= -t^4 + t^3 + t \\ k = 2 \Rightarrow V_{T(2,5)}(t) &= -t^7 + t^6 - t^5 + t^4 + t^2 \\ k = 3 \Rightarrow V_{T(2,7)}(t) &= -t^{10} + t^9 - t^8 + t^7 - t^6 + t^5 + t^3 \end{aligned}$$

In this paper, we construct the Jones polynomial of an alternating knot which all coefficients are non-zero. We will briefly review the standard theory of the Tutte polynomial and the connection between the Tutte polynomial and the Jones polynomial.

2 Graph Theory

A graph $G = (V(G), E(G))$ or $G = (V, E)$ consists of two finite sets. $V(G)$ or V is the non-empty vertex set of the graph called vertices and $E(G)$ or E is the edge set of the graph called edges, such that each edge e in E is assigned as an unordered pair of vertices (u, v) called the end vertices of e. A path is a sequence of edges which connect a sequence of vertices which are all distinct from one another. A cycle of a graph G is a subset of the edge set of G that forms a path such that the first node of the path corresponds to the last. An isthmus or a bridge is an edge of graph if and only if it is not contained any cycle. A loop is an edge that connects a vertex to itself.

3 The Tutte Polynomial by Deletion-Contraction

Consider the following recursive definition of the function $\chi_G(x, y)$ of a graph G, x, y are independent variables. Then Tutte polynomal is defined by:

$$\chi(G; x, y) = \begin{cases} 1 & \text{if } E(G) = \emptyset \\ x\chi(G'_e; x, y) & \text{if } e \in E \text{ and } e \text{ is an isthmus} \\ y\chi(G''_e; x, y) & \text{if } e \in E \text{ and } e \text{ is a loop} \\ \chi(G; x, y) = \chi_{G'_e}(x, y) + \chi_{G''_e}(x, y) & \text{if e is neither a loop nor an isthmus} \end{cases}$$

where G'_e denotes the deletion by an edge e of graph G and G''_e denotes the contraction by an edge e of graph G.
Example. If G is a complete graph K_3, then

$$\chi(K_3; x, y) = x^2 + x + y$$

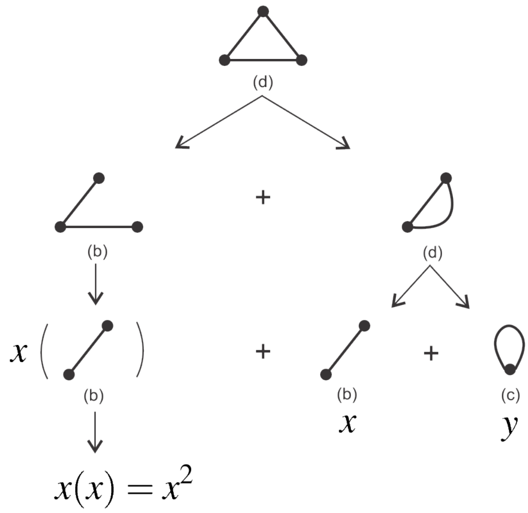

Figure 3: An example of computing the Tutte polynomial of a graph by using deletion and contraction.

Theorem 1. (Thistlethwaite [4]) *Suppose $\phi(t)$ be a sign and multiplication by power of t, Let L be an unoriented link and G be a planar graph associated with L. Then, $V_L(t) = \phi(t)\chi(G; -t, -t^{-1})$*

Example. Let K be a trefoil knot, G is a medial graph of knot K.

$$\chi(G; x, y) = x^2 + x + y$$
$$\chi(G; -t, -t^{-1}) = t^2 - t - t^{-1}$$

$$V_K(t) = t + t^3 - t^4$$
$$V_K(t) = (-t^2)\chi(G; -t, -t^{-1})$$

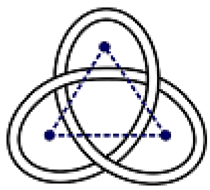

Figure 4: Trefoil knot with its medial graph.

4 The Jones Polynomial of 3-Tuple Pretzel Link

Let $G_{(p,q,r)}$ be a connected planar graph with three number of faces. p, q, and r are the number of vertices of graph $G_{(p,q,r)}$ (Fig. 5). The graph $G_{(p,q,r)}$ is associated with 3-tuple pretzel link.

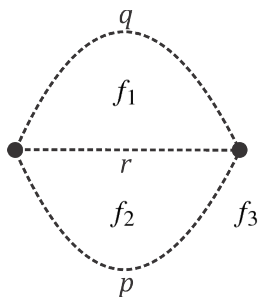

Figure 5: Graph $G_{(p,q,r)}$ with three number of faces f_1, f_2, and f_3

We have the Tutte polynomial of $G_{(p,q,r)}$

$$\chi(G_{(p,q,r)};x,y)=(y-1)^2+(y-1)\sum_{i=0}^{p}x^i+\sum_{j=0}^{q}x^j\left(y-1+\sum_{i=0}^{p}x^i\right)+\sum_{k=0}^{r}\left(x^k y+\sum_{l=0}^{q+p}x^{k+l+1}\right) \quad (3)$$

By changing variable $x=-t$ and $y=-t^{-1}$ from **(3)** we get

$$\begin{aligned}\chi(G_{(p,q,r)};-t,-t^{-1}) &= t^{-2}+2t^{-1}+1+(-t^{-1}-1)\sum_{i=0}^{p}(-t)^i\\ &+\sum_{j=0}^{q}(-t)^j\left(-t^{-1}-1+\sum_{i=0}^{p}(-t)^i\right)\\ &+\sum_{k=0}^{r}\left((-t)^{k-1}+\sum_{l=0}^{q+p}(-t)^{k+l+1}\right)\end{aligned} \quad (4)$$

We simplify the Tutte polynomial of $(-t,-t^{-1})$ of graph $G_{(p,q,r)}$

$$\begin{aligned}\chi(G_{(p,q,r)};-t,-t^{-1}) &= \frac{1}{t^2(t+1)^2}\\ &\times[1+t+\{2-(-t)^p-(-t)^q-(-t)^r\}t^2\\ &+\{1-(-t)^p-(-t)^q-(-t)^r\}t^3\\ &+\{1-(-t)^p-(-t)^q-(-t)^r\}t^4\\ &+(-t)^{p+q+r+5}]\end{aligned} \quad (5)$$

So by theorem 1 we get the Jones polynomial of link associated with graph $G_{(p,q,r)}$

$$\begin{aligned}V_L(t) &= \frac{\phi(t)}{t^2(t+1)^2}\\ &\times[1+t+\{2-(-t)^p-(-t)^q-(-t)^r\}t^2\\ &+\{1-(-t)^p-(-t)^q-(-t)^r\}t^3\\ &+\{1-(-t)^p-(-t)^q-(-t)^r\}t^4\\ &+(-t)^{p+q+r+5}]\end{aligned} \quad (6)$$

Where $\phi=(-t^{\frac{3}{4}})^w(t^{-\frac{1}{4}(p+q+r-1)})$ [3]

5 Statement of Results

Theorem 2. *All coefficients in* $\chi(G; -t, -t^{-1})$ *are non-zero*

Proof. Let $\sum_i^n (-1)^i a_i t^i$ be a Laurent polynomial with alternating sign and non-vanishing term ($a_i \neq 0$).

Consider $\sum_i^n (-1)^i a_i t^i \sum_i^m (-1)^i b_i t^i = \sum_i^{m+n} (-1)^i c_i t^i$ and $\sum_i^m \sum_i^n (-1)^i a_i t^i = \sum_i^{m+n} (-1)^i d_i t^i$.

By using the symmetricity of Tutte polynomial, we will show all coefficients of the polynomial with alternating sign in **(4)** are non-zero if at least p, q, r is greater than zero, assume that $p \leq q \leq r$. Let $r > 0$, we get the Tutte polynomial of $(-t, -t^{-1})$ for graph $G_{(p,q,r)}$

$$\begin{aligned}
\chi(G_{(p,q,r)}; -t, -t^{-1}) &= t^{-2} + 2t^{-1} + 1 + (-t^{-1} - 1)\sum_{i=0}^{p}(-t)^i \\
&+ \sum_{j=0}^{q}(-t)^j \Big(-t^{-1} - 1 + \sum_{i=0}^{p}(-t)^i \Big) \\
&+ \sum_{k=0}^{r} \Big((-t)^{k-1} + \sum_{l=0}^{q+p}(-t)^{k+l+1} \Big) \\
&= \underbrace{t^{-2} + 1}_{A} + \sum_{i=0}^{p-1}(-t)^i + \sum_{i=0}^{q-1}(-t)^i \\
&+ \sum_{j=1}^{q}(-t)^j \sum_{i=1}^{p}(-t)^i + \underbrace{\sum_{k=0}^{r}(-t)^{k-1}}_{B_r} \\
&+ \underbrace{\sum_{k=0}^{r}\sum_{l=0}^{q+p}(-t)^{k+l+1}}_{C_{p,q,r}}
\end{aligned}$$

$$\begin{aligned}
A &= t^{-2} \quad + 1 \\
B_r &= \quad - t^{-1} + 1 - t + t^2 - t^3 + \cdots + (-t)^{r-1} \\
C_{p,q,r} &= \quad - t + t^2 - t^3 + \cdots + (-t)^{q+p+1} \\
&\quad + t^2 - t^3 + t^4 + \cdots + (-t)^{q+p+2} \\
&\quad - t^3 + t^4 - t^5 + \cdots + (-t)^{q+p+3} \\
&\quad + (-t)^{r+1} + (-t)^{r+2} + (-t)^{r+3} + \cdots + (-t)^{r+q+p+1}
\end{aligned}$$

All coefficients of polynomial terms $A + B_r + C_{p,q,r}$ are non-zero. Define $\deg^-$ be the lowest degree of its terms. Then, $\deg^-\left(\chi(G_{(p,q,r)}; -t, -t^{-1})\right) = \deg^-(A + B_r + C_{p,q,r}) = -2$ and $\deg\left(\chi(G_{(p,q,r)}; -t, -t^{-1})\right) = \deg(A + B_r + C_{p,q,r}) = r + q + p + 1$. Therefore, all coefficients of polynomial $\chi(G_{(p,q,r)}; -t, -t^{-1})$ are non-zero.□

Corollary 1. *All coefficients in the Jones polynomial of 3-tuple of pretzel link are non-zero.*
Proof. Let L be a 3-tuple of pretzel link.
Consider $\chi(G_{(p,q,r)}; -t, -t^{-1}) = \sum_{i=-2}^{r+q+p+1} (-1)^i a_i t^i$ be the Tutte polynomial with alternating sign and non-vanishing term ($a_i \neq 0$).
Let $\phi(t) = (-1)^h t^k$ be a sign and multiplication by power of t.
Since $V_L(t) = \phi(t)\chi(G_{(p,q,r)}; -t, -t^{-1}) = (-1)^h t^k \sum_{i=-2}^{r+q+p+1} (-1)^i a_i t^i = \sum_{i=-2}^{r+q+p+1} (-1)^{h+i} a_i t^{k+i}$.
Therefore, all coefficients of $V_L(t)$ are non-zero.□
Corollary 2. *All coefficients in the Jones polynomial of n-tuple pretzel link are non-zero.*
Proof. Let $P(c_1, c_2, \dots, c_n)$ be a pretzel link determined an n-tuple, $G(c_1, c_2, \dots, c_n)$ is a graph associated with $P(c_1, c_2, \dots, c_n)$. Choose p, q, r as the first three largest number of $c_1, c_2, \dots, c_n$ where $p \leq q \leq r$. Then we say that, all coefficients in $\chi(G_{(c_1,c_2,\dots,c_n)}; -t, -t^{-1})$ are non-zero with $\deg\{\chi(G_{(c_1,c_2,\dots,c_n)}; -t, -t^{-1})\} = m = r + q + p + 1$.
Let $\phi(t) = (-1)^h t^k$ be a sign and multiplication by power of t.
Since $V_{P(c_1,c_2,\dots,c_n)}(t) = \phi(t)\chi(G_{(c_1,c_2,\dots,c_n)}; -t, -t^{-1}) = (-1)^h t^k \sum_{i=-2}^{m} (-1)^i a_i t^i = \sum_{i=-2}^{m} (-1)^{h+i} a_i t^{k+i}$.
Therefore, all coefficients of $V_{P(c_1,c_2,\dots,c_n)}(t)$ are non-zero.□

Acknowledgment

We would like to thank Japanese Goverment (Monbukagakusho: MEXT) for the financial support in our research.
A special thank to Prof. Hiroshi Iwasaki, Andine Astiany, DDP KU-ITB 2013, our family, Computational Science of ITB and Kanazawa University's lecturers, and Kanazawa University for all the support.

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