Ann. Sci. Kanazawa Univ.
Vol. 28, pp. 9—18, 1991

Some singular measures in $M_p(G)$

Yuichi KANJIN and Enji SATO

(Received April 26, 1991)

1. Introduction

Let G be a locally compact abelian group with dual $\Gamma=\hat{G}$, $M(G)$ the convolution measure algebra of finite regular Borel measures on G. For $\mu \in M(G)$, let $\|\mu\|$ denote the total variation norm and $\mu^*(E)=\overline{\mu(-E)}$ for any Borel set E. We call μ a Hermitian measure if $\mu^*=\mu$. For $1\leq p\leq\infty$, let $L_p(G)$ be the L_p space on G with respect to the Haar measure of G, $\|\cdot\|_p$ the norm of $L_p(G)$, and $B(L_p)$ the set of all bounded linear operators on $L_p(G)$. For a function $f(x)$ on G and $y \in G$, we define $\tau_y f(x) = f(x-y)$. An operator $T \in B(L_p)$ is called an L_p-multiplier if $T\tau_y=\tau_y T$ for all $y \in G$. The set of all L_p-multipliers will be denoted by $M_p(G)$ and the norm of $T \in M_p(G)$ is defined by the operator norm on $L_p(G)$. Then $M_p(G)$ is a commutative Banach algebra with the identity I, where I is the identity mapping of $L_p(G)$. For $T \in M_p(G)$, let $\mathrm{sp}(T,M_p(G))=\{\lambda \in C;\ T-\lambda I$ is not invertible in $M_p(G)\}$, where C is the complex numbers. The set $\mathrm{sp}(T,M_p(G))$ is called the spectrum of T in $M_p(G)$. For $\mu \in M(G)$, we define $T_\mu f(x)=\mu * f(x)=\int_G f(x-y)\, d\mu(y)$ $(f \in L_p(G))$. Then $T_\mu \in M_p(G)$ for all $1\leq p\leq\infty$ and $I=T_{\delta(0)}$, where 0 is the unit of G, and $\delta(0)$ is the Dirac measure with unit mass at 0. We denote $\mathrm{sp}(\mu,M_p(G))=\mathrm{sp}(T_\mu, M_p(G))$. For $\mu \in M(G)$, let $\hat{\mu}$ be the Fourier-Stieltjes transform of μ, and $\|\hat{\mu}\|_\infty=\sup\{|\hat{\mu}(\gamma)|;\ \gamma \in \Gamma\}$. For later convenience, we here summarize some known properties of $M_p(G)$ on a nondiscrete locally compact abelian group G.

(1) The mapping $\mu\to T_\mu$ gives that $M(G)$ is isomorphic and isometric to $M_1(G)$.

(2) For $T \in M_p(G)$, there uniquely exists $\hat{T} \in L_\infty(\Gamma)$ such that $T(\hat{f})=\hat{T}\hat{f}$ for every $f \in L_p(G)\cap L_2(G)$.

(3) $M_2(G)$ is isomorphic to $L_\infty(\Gamma)$.

(4) $M_p(G)$ is isomorphic to $M_q(G)$ if $1/p+1/q=1$ $(1<p<\infty)$.

(5) $M_1(G) \subset M_p(G) \subset M_2(G)$ $(1\leq p\leq 2)$ (cf. [8]).

(6) For $\mu \in M(G)$, $\mathrm{sp}(\mu,M_1(G)) \supset \mathrm{sp}(\mu,M_p(G)) \supset \mathrm{sp}(\mu,M_2(G))=\mathrm{closure}(\hat{\mu}(\Gamma))$ $(1\leq p\leq 2)$, where closure $(\hat{\mu}(\Gamma))$ is the closure of $\hat{\mu}(\Gamma)$ in the complex plane.

(7) $\mathrm{sp}(T_f,M_p(G))=\hat{f}(\Gamma)\cup\{0\}$ for $1\leq p\leq\infty$ and $f \in L_1(G)$.

(8) There exists $\mu \in M(G)$ so that $\mathrm{sp}(\mu,M_1(G)) \supsetneqq \hat{\mu}(\Gamma)\cup\{0\}$ and $\hat{\mu}(\gamma)\to 0$ as $\gamma\to\infty$ $(\gamma \in \Gamma)$ (cf. [13]).

(9) If $1<p<2$, then $\mathrm{sp}(\mu,M_p(G))=\hat{\mu}(\Gamma)\cup\{0\}$, whenever $\mu \in M(G)$ with $\hat{\mu}(\gamma)\to 0$ as γ

$\to\infty (\gamma \in \Gamma)$ ([6]).

One of deep results on spectra of measures as L_p-multipliers, $1<p<2$ is the following theorem of Igari [7] .

THEOREM A ([7]). *Let G be a nondiscrete locally compact abelian group. For $1<p<2$, there exists $\mu \in M(G)$ such that $sp(\mu, M_p(G)) \supsetneqq$ closure $(\hat{\mu}(\Gamma))$.*

We remark that μ may depend on p.

On the other hand, Sarnak [15] gave a bound of the region of the spectrum of a measure in $M_p(G)$.

THEOREM B ([15]). *Let G be a locally compact abelian group, $\mu \in M(G)$ a Hermitian measure with $\| \mu \| =1$ and $1 \leq p \leq 2$. Then $sp(\mu, M_p(G))$ is contained in the region bounded by $\gamma_p = \{z \in \boldsymbol{C}; \theta(z) = \pi/2 + \pi(p-1)/p\}$, where $\theta(z)$ is the angle substanded at z by the line $[-1,1]$.*

It is natural to ask the existence of a measure $\mu \in M(G)$ satisfying that the region $\mathrm{sp}(\mu, M_p(G))$ shrinks to closure $(\hat{\mu}(\Gamma))$ as $p \to 2$. The following theorem is an answer.

THEOREM C ([16]). *Let G be a nondiscrete locally compact abelian group. Then there exists $\mu \in M(G)$ a Hermitian probability measure such that $sp(\mu, M_p(G)) \supsetneqq$ closure $(\hat{\mu}(\Gamma))$ for all $1 \leq p < 2$.*

In this paper, we shall investigate properties of a measure in THEOREM C in detail. Let $C(\mu) = \{c \in \boldsymbol{C}; \gamma_n \to c$ in the weak $*$ topology of $L_\infty(\mu)$ for some $\{\gamma_n\} \subset \Gamma\}$ for a probability measure $\mu \in M(G)$. Then one of our results in §2 is as follows: There exists $\mu \in M(\boldsymbol{T})$ a Hermitian probability measure such that $C(\mu) = [-1, 1]$ and $\mathrm{sp}(\mu, M_p(\boldsymbol{T})) \supsetneqq$ closure $(\hat{\mu}(\Gamma))$ for all $1 \leq p < 2$ (THEOREM 2.1). In particular, μ has strongly independent powers (cf. [4]). Here, $\boldsymbol{T}$ is the circle group. We treat spectra of measures in $M_p(G)$ in the case that G is a dyadic group in §3, and in the case that G is a group of bounded order in §4. In the last section §5, we discuss spectra of measures in $M_p(G)$ when G is a locally compact abelian group.

Remark. By [2] , [3] and [11] , we obtain that for an infinite compact abelian group G, there exists $\mu \in M(G)$ a Hermitian probability measure such that $\mathrm{sp}(\mu, M(G)) = \{ |z| \leq 1\}$, and $\mathrm{sp}(\mu, M_p(G)) =$ closure $(\hat{\mu}(\Gamma)) = \{1\} \cup [-1/2, 1/2]$ for all $1<p\leq 2$. This measure is one of L_p-improving measures (cf. [3], [5], [11]).

We can say that the spectrum of this measure is *thin*. On the other hand, the spectrum of our measure is *thick*.

2. Spectra of measures in $M_p(T)$

We first prove several lemmas.

LEMMA 2.1 ([12]). *Suppose $1 \leq p < 2$, and r a natural number. Then there exist a constant $K_p > 1$ and a real-valued trigonometric polynomial $Q(x)$ on $\boldsymbol{T}$ such that $\Sigma \mid \hat{Q}(n) \mid \leq r$, and*

$$\{\Sigma_{n \in \mathbf{Z}} \mid \exp(iQ(x))\hat{}(n) \mid^p\}^{1/p} > K_p{}^r.$$

DEFINITION 2.2. Let K be a compact subset of $\boldsymbol{T}$. K is called a *Kronecker set*, if for each $\varepsilon > 0$ and $f \in C(K)$ with $\mid f \mid = 1$ on K, there exists $\gamma \in \hat{\boldsymbol{T}}$ such that $\| f - \gamma \|_K < \varepsilon$.

LEMMA 2.3. *Suppose $1 \leq p < 2$ and j a natural number. Then there exists a Hermitian probability measure $\mu = \mu_{p,j}$ on $\boldsymbol{T}$ such that $\mu = (\mu_0 + \mu_0{}^*)/2$, where μ_0 is a continuous probability measure on some Kronecker set, and $\| \exp(ij\mu) \|_{M_p(T)} > K_p{}^j$.*

PROOF. Let Q be a real-valued trigonometric polynomial on $\boldsymbol{T}$. Then $\{\exp(iQ)\hat{}(n)\}_{n \in \boldsymbol{Z}} \in L_p(\boldsymbol{Z})$ for all $1 \leq p \leq \infty$, and $\| \exp(iQ)\hat{} \|_{L_p(Z)} \leq \| \exp(iQ)\hat{} \|_{M_p(Z)}$. By LEMMA 2.1, there exist a constant $K_p > 1$ and a real-valued trigonometric polynomial Q_j such that $Q_j = \Sigma_{k=1}^{j} \hat{Q}_j(n_k) \cos n_k x$ for some natural numbers $n_1, \cdots, n_j$, and $\| \exp(iQ_j)\hat{} \|_{L_p(Z)} > K_p{}^j$. So $\| \exp(iQ)\hat{} \|_{M_p(Z)} > K_p{}^j$. Also by [7; THEOREM A], $\exp(iQ_j(x)) \in M_p(\boldsymbol{R})$ and $\| \exp(iQ)\hat{} \|_{M_p(R)} = \| \exp(iQ)\hat{} \|_{M_p(Z)}$. On the other hand, for $\lambda > 0$ putting $\nu_{j,\lambda} = (1/j)\Sigma_{k=1}^{j} (\delta(\lambda n_k) + \delta(-\lambda n_k))/2$, we get $\hat{\nu}_{j,\lambda}(n) = Q_j(\lambda n)(n \in \boldsymbol{Z})$. Moreover, by [7; THEOREM B] $\lim_{\lambda \to 0} \| \exp(ij\nu_{j,\lambda}) \|_{M_p(T)} = \| \exp(iQ_j)\hat{} \|_{M_p(R)} > K_p{}^j$. Then if λ is sufficiently small, $\| \exp(ij\nu_{j,\lambda}) \|_{M_p(T)} > K_p{}^j$ and $\| j\nu_{j,\lambda} \|_{M_1(T)} = \| \{\hat{Q}_j\} \|_{L_1(T)} = j$. So there exist trigonometric polynomials f_j and h_j such that $\| f_j \|_p = 1$, $\| h_j \|_q = 1 (1/p + 1/q = 1)$, and

(1) $\quad \mid \int (\exp(ij\nu_{j,\lambda}) * f_j)(x) h_j(x)\, dx \mid > K_p{}^j.$

Now putting $g_j(y) = \int f_j(x-y) h_j(x)\, dx, g_j$ is a trigonometric polynomial, and

(2) $\quad \int (\exp(ji\mu_{j,\lambda}) * f_j)(x) h_j(x)\, dx = \Sigma_{n=0}^{\infty} (i^n j^n / n!) \int g_j\, d(\nu_{j,\lambda})^n.$

Then for any $\eta > 0$, there exists a natural number N_0 such that

$$\Sigma_{n=N_0+1}^{\infty} (j^n/n!) \| g_j \|_\infty < \eta/3.$$

So we obtain

(3) $\quad \mid (2) - \Sigma_{n=0}^{N_0} (i^n j^n / n!) \Sigma_k \hat{g}_j(k) \hat{\nu}_{j,\lambda}^n(-k) \mid < \eta/3.$

Also for $\eta_1 > 0$, there exist U_t the compact neighborhood of λn_t such that $\{U_t\}_{t=1}^{j}$ is pairwise disjoint, and $\mid \exp(i\lambda n_t k) - \exp(ixk) \mid < \eta_1 / j$ for all $x \in U_t$, and $k \in \operatorname{supp} \hat{g}_j$. Then there exist perfect sets $K_t \subset U_t$ such that $K_1 \cup \cdots \cup K_j$ is a Kronecker set.

Now let μ_t be a continuous probability measure on K_t. Putting $\tau=(1/j)\ \Sigma_t\mu_t$, τ is a continuous probability measure on $K_1\cup\cdots\cup K_j$. Then for $\mu=(\tau+\tau^*)/2$, we get

(4) $\quad |\hat{\mu}(-k)-\hat{\nu}_{j,\lambda}(-k)|<\eta_1$ for all $k\in$ supp $\hat{g}_j$.

Hence by (3) and (4), for sufficiently small $\eta_1>0$, we obtain

(5) $\quad |\Sigma_{n=0}^{N_0}(ij^n/n!)\ \Sigma_k\hat{g}_j(k)\hat{\nu}_{j,\lambda}(-k)^n-\Sigma_{n=0}^{N_0}(ij^n/n!)\ \Sigma_k\hat{g}_j(k)\hat{\mu}(-k)^n|<\eta/3.$

Thus by (1) and (5), for sufficiently small $\eta>0$, we get $|\int(\exp(ij\mu)*f_j)(x)h_j(x)dx|>K_p{}^j$. Then $\|\exp(ij\mu)\|_{M_p(T)}>K_p{}^j$. Q. E. D.

By the properties of Fejér kernel, it is easy to prove the next lemma.

LEMMA 2.4. *Let $F_n(x)$ be the Fejér kernel of degree n, and $\mu_{p,j}$ in* LEMMA 2.3. *We put $\phi_{p,j,n}=\mu_{p,j}*F_n$. Then there exists a natural number n_0 such that $\|\exp(ij\phi_{p,j,n})\|_{M_p(T)}>K_p{}^j$ for any $n\geq n_0$.*

LEMMA 2.5 ([9]). *Let K be a perfect Kronecker set, and ν a continuous probability measure on K. Then $\{|z|\leq 1\}=\{z\in\boldsymbol{C};\hat{\nu}(\gamma_n)\to z$ as $\gamma_n\to\infty$ for some $\{\gamma_n\}\subset\hat{\boldsymbol{T}}\}$ $(=C(\nu))$.*

THEOREM 2.6. *There exist a Hermitian probability measure $\mu\in M(\boldsymbol{T})$ such that*

(i) $C(\mu)=[-1,1]$,

(ii) *$\delta(x)*\mu^n\perp\mu^m$ if $0\leq m<n$ and $x\in\boldsymbol{T}$, and*

(iii) *$sp(\mu, M_p(\boldsymbol{T}))\supsetneqq$ closure $(\hat{\mu}(\boldsymbol{Z}))$ for all $1\leq p<2$.*

In particular, μ is a continuous singular measure.

PROOF. Let $\boldsymbol{Q}$ be the set of all rational numbers. For each natural number n, let p_n be a rational number satisfying $1\leq p_n<2$ such that $\{p_n;n\geq 1\}=\boldsymbol{Q}\cap\{p;1\leq p<2\}$ and that each $p\in\boldsymbol{Q}$ with $1\leq p<2$ appears infinitely often in the p_n's. For each a positive integer $p=p_j$, there exists a positive integer n_j such that $\{(\mu_{p,j}*F_{n_j})\hat{}(k);k\in\boldsymbol{Z}\}$ has an intersection with any interval of j equal parts of $[-1,1]$. Putting $\phi_j=\phi_{p,j,n_j}$, there exists a trigonometric polynomial f_j such that

(1) $\quad \|f_j\|_p=1$ and $\|f_j*\exp(ij\phi_j)\|_p>K_p{}^j$.

So there is a positive integer m_j such that

(2) $\quad$ (supp $\hat{\phi}_j$) $\cup$ (supp $\hat{f}_j$) $\subset\{0,\pm 1,\cdots,\pm m_j\}$.

Now let $r_1=1, r_2, r_3,\cdots$ be an increasing sequence of natural numbers of $\boldsymbol{Z}$. Then by the proof of [10; LEMMA 5], we get the measures defined by

(3) $\quad d\mu_n(t)=[\phi_1(r_1t)\cdots\phi_n(r_nt)]\ d\lambda_T$.

They are probability measures, and converges to a probability measure $\mu\in M(\boldsymbol{T})$ in the weak* topology such that

(4) $\quad \hat{\mu}(k_1r_1+\cdots+k_nr_n)=\Pi_{j=1}^n\hat{\phi}_j(k_j)$

whenever the k_j are integers such that

(5) $|k_j| \leq m_j$ $(j=1,\cdots,n)$, and

(6) $\hat{\mu}(m)=0$ for all other integers m.

Let c be a real number in $[-1, 1]$. By the construction of μ, there exist $s_k \in r_k\boldsymbol{Z}$ with $\hat{\mu}(s_k) \to c$. Then $\exp(-is_k x)$ converges to c in the weak* topology of $L_\infty(\mu)$. So we get $C(\mu)=[-1, 1]$.

For a natural number j, we define a trigonometric polynomial g_j by setting $g_j(t)=f_j(r_j t)$ for $t \in \boldsymbol{T}$. Then,

(7) $\| g_j \|_p=1$ and supp $\hat{g}_j \subset \{kr_j;\ k=0, \pm 1, \cdots, \pm m_j\}$, by (1) and (2). Moreover, $\hat{g}_j(kr_j)=\hat{f}_j(k)$ for all $k \in \boldsymbol{Z}$. It follows from (1), (4), (5), and (6) that

(8) $\| \exp(ij\mu) * g_j \|_p = \| \exp(ij\phi_j) * f_j \|_p > K_p{}^j$ for $p=p_j$.

Thus, we obtain

$$\lim_{j\to\infty} \| \exp(ij\mu) \|_{M_p}^{1/j} > 1$$

for all $1 \leq p < 2$ (cf. [16]). Therefore, we get the desired result.

Q. E. D.

COROLLARY 2.7. *There exists a Hermitian probability measure* $\mu \in M(\boldsymbol{T})$ *such that*

(i) $sp(\mu, M_2(\boldsymbol{T}))=closure\ (\hat{\mu}(\boldsymbol{Z}))=[-1, 1]$, *and*

(ii) $sp(\mu, M_1(\boldsymbol{T}))=\{|z| \leq 1\} \supsetneqq sp(\mu, M_p(\boldsymbol{T})) \supsetneqq sp(\mu, M_2(\boldsymbol{T}))$ *for all* $1<p<2$.

PROOF. Let c be in $0<c<1$. By THEOREM 2.6, there exists a generalized character $\xi=\{\xi_\nu\}_{\nu \in M(T)}$ such that $\xi_n = c$ a. e. ν(cf. [4]). Then for any complex number z with Re $z>0$, ξ^z is a generalized character of $M(\boldsymbol{T})$ (cf. [4]). Hence we get sp$(\mu, M_1(\boldsymbol{T})) \supset \{\int \xi_\mu^z\, d\mu;\ Re\ z>0\}$ $(0<|z|<1)$, and sp$(\mu, M_1(\boldsymbol{T}))=\{|z| \leq 1\}$. Also by THEOREM B, we get $\{|z| \leq 1\} \supsetneqq$ sp$(\mu, M_p(\boldsymbol{T}))$ for all $p>1$. Q. E. D.

By the theorem and [9], we have the following corollary.

COROLLARY 2.8. *Let F be a complex-valued function on* $[-1, 1]$, *and continuous on* (0, 1). *Suppose that* $F(\hat{\mu})$ *is a Fourier-Stieltjes transform of some element in* $M(\boldsymbol{T})$ *for the measure* μ *obtained in the theorem. Then F extends to a bounded analytic function on* Σ_1, *where* Σ_1 *is the set of all complex numbers of modulus less than* 1 *which are not nonpositive real numbers.*

3. Spectra of measures in $M_p(D_q)$

$D_q(q \geq 2)$ is defined by the complete direct sum of countably many copies of the cyclic group $\boldsymbol{Z}_q$ of order q. Let Δ_q be the dual group of D_q.

DEFINITION 3.1. For a compact subset K of D_q, K is called a K_q-*set* if for any f

$\in C(K)$ with $f^q=1$, there exists $\gamma \in \Delta_q$ such that $f=\gamma$ on K.

LEMMA 3.2 ([16]). *Let $1 \leq p<2$ and s be a positive integer. Then there exist a constant $K_p>1$, independent of s, and $\{\gamma_k\} \subset \Delta_q$ satisfying the following two conditions.*
(i) *For any $1<j\leq s$ and Λ_{j-1} the cyclic group generated by $\{\gamma_1,\cdots,\gamma_{j-1}\}$, every coordinate component in Λ_{j-1}, which is in the coordinate components over s_{j-1}-th, is $0(\in Z_q)$, and nonzero coordinate component of the cyclic group generated by γ_j is in the coordinate components over s_{j-1}-th coordinate component.*

(ii) $\{\sum_{\gamma\in\Delta_q} |(\exp(iQ))\hat{}(\gamma)|^p\}^{1/p}>K_p{}^j$ *for* $Q=\Sigma_{k=1}^s(\gamma_k(x)+\bar{\gamma}_k(x))/2$.

LEMMA 3.3 ([7; THEOREM 3]). *Let φ be a continuous function on D_q, and $\hat{\nu}=\varphi$ for some $\nu \in M(\Delta_q)$. Then we have*

$$\|\nu\|_{M_p(\Delta_q)}=\|\nu\|_{M_p(D_q)}.$$

LEMMA 3.4. *Let p be a real number in $1\leq p<2$, and j a positive integer. Then there exist a constant $K_p>1$ depending only on p and q, and $\mu_{p,j}$ a Hermitian probability measure on D_q such that $\mu_{p,j}=(\tau+\tau^*)/2$, where τ is a continuous probability measure on some K_q-set, and $\|\exp(ij\mu_{p,j})\|_{M_p(D_q)}>K_p{}^j$.*

PROOF. For a real-valued trigonometric polynomial Q on D_q, we have $\exp(iQ)\hat{} \in M_p(\Delta_q)$ and $\|\exp(iQ)\hat{}\|_{L_p(\Delta_q)}\leq \|\exp(iQ)\|_{M_p(\Delta_q)}$. By LEMMA 3.2, there exist $K_p>1$ and $Q_j=\Sigma_{k=1}^j(\gamma_k+\bar{\gamma}_k)/2$ for some $\gamma_k \in \Delta_q(1\leq k\leq j)$ such that $\|\exp(iQ_j)\hat{}\|_{L_p(\Delta_q)}>K_p{}^j$. Then we have $\|\exp(iQ_j)\|_{M_p(\Delta_q)}>K_p{}^j$. Also by LEMMA 3.3, there exists $\nu_j \in M(D_q)$ with $\nu_j=Q_j$, and $\|\exp(i\nu_j)\|_{M_p(D_q)}\dot{}=\|\exp(iQ_j)\hat{}\|_{M_p(\Delta_q)}$. Then $\|\exp(i\nu_j)\|_{M_p(D_q)}>K_p{}^j$.

Now we define $\tau_i=\nu_j/j$. So there exist trigonometric polynomials f_j and h_j such that $\|f_j\|_p=1$, $\|h_j\|_{p_1}=1$ $(1/p+1/p_1=1)$ and $|\int (\exp(ij\tau_j) * f_j)(x)h_j(x)\,dx|>K_p{}^j$. Since τ_j has finite support, by the above inequality, there exist $\tau \in M(D_q)$ and a perfect K_q-set K such that τ is a continuous probability measure on K and for $\mu_{p,j}=(\tau+\tau^*)/2$,

$$\left|\int (\exp(ij\mu_{p,j})*f_j)(x)h_j(x)\,dx-\int (\exp(ij\tau_j)*f_j)(x)h_j(x)\,dx\right|$$

is sufficiently small. Then we have the result. We omit the details.

Q. E. D.

LEMMA 3.5 ([1]). *Let G be a metrizable compact abelian group. Then there exist trigonometric polynomials $\{F_n\}$ on G such that $\|F_n\|_1=1$, $F_n\geq 0$, $\hat{F}_n\geq 0$, $\lim_{n\to\infty}\|f-f*F_n\|_1=0$ for all $f \in L_1(G)$, and for any finite set $E \subset \Gamma$, $\{\hat{F}_n(\gamma)\}$ uniformly converges to 1 on E.*

By LEMMA 3.5, it is easy to get the following lemma.

LEMMA 3.6. *Let $\{F_n\}$ be in* LEMMA 3.15, *and $\{\mu_{p,j}\}$ in* LEMMA 3.4. *Then for*

$\phi_{p,j,n}=\mu_{p,j}*F_n$, *there exists a positive integer* n_0 *with*

$$\| \exp(ij\phi_{p,j,n}) \|_{M_p(D_q)} > K_p{}^j \text{ for all } n \geq n_0.$$

LEMMA 3.7 ([9]). *Let* K *be a perfect* K_q*-set of* D_q, *and* ν *a continuous probability measure on* K. *Then closure* $(\hat{\nu}(\Delta_q))$ *is the closed convex hull of* q*-th roots of unity.*

THEOREM 3.8. *There exists* μ *a Hermitian probability measure such that*

(i) $C(\mu)=\begin{cases} [-1, 1] & (q\text{: even}) \\ [-\cos(\pi/q), 1] & (q\text{: odd}), \end{cases}$

(ii) $\delta(x)*\mu^n \perp \mu^m$ *if* $0 \leq m < n$ *and* $x \in D_q$, *and*

(iii) $sp(\mu, M_p(D_q)) \supsetneqq closure\ (\hat{\mu}(\Delta_q))$ *for all* $1 \leq p < 2$.

In particular, μ *is a continuous singular measure.*

PROOF. Let $\boldsymbol{Q}$ be the set of all rational numbers. For $1 \leq p < 2$ and $T \in M_p(D_q)$, we write $\| T \|_{M(p,\Delta_q)}$ for $\| T \|_{M_p(D_q)}$. We choose $\{p_n;\ n \geq 1\} = \boldsymbol{Q} \cap \{p;\ 1 \leq p < 2\}$ in the same way of the proof of THEOREM 2.6. For natural numbers $m < n$, we write $G(m,n)=\Pi_{k=m+1}^{n} \boldsymbol{Z}_q$, and $\Gamma(m,n)=\Pi_{k=m+1}^{n} \hat{\boldsymbol{Z}}_q$, where $\hat{\boldsymbol{Z}}_q$ is the dual group of $\boldsymbol{Z}_q$. We shall identify $G(m,n)$ and $\Gamma(m,n)$ with the naturally corresponding subgroups of D_q and Δ_q, respectively.

Now we choose natural numbers n_j $(j \geq 0)$ as follows. Put $n_0=1$ and suppose that $n_0 < n_1 < \cdots < n_{j-1}$ have been chosen for some $j \geq 1$. For j, by LEMMA 3.6 and 3.7 with $p=p_j$, there exist m_j and $\mu_{p,j} \in M(D_q)$ such that if q is even(resp. odd), $\{(\mu_{p,j}*F_{m_j})\hat{}(\gamma);\ \gamma \in \Delta_q\} \cap I_k \neq \phi$ for each interval I_k, where $\{I_k\}_{k=1}^{n}$ consists of n equal parts of $[-1, 1]$ (resp. $[-\cos(\pi/q), 1]$). Then there exists $n_j > n_{j-1}$ such that $\mu_j=\mu_{p,j}*F_{m_j} \in M(G(n_{j-1},n_j))$, and $\| \exp(ij\mu) \|_{M_p(G(n_{j-1},n_j))} > K_p{}^j$ for $p=p_j$. Identify D_q with the product group $\Pi_{j=1}^{\infty} G(n_{j-1}, n_j)$, and put $\mu=\mu_1 \times \mu_2 \times \cdots$, the product measure of all μ_j $(j \geq 1)$. Clearly, μ is a Hermitian probability measure on D_q. Writing $\Gamma_j=\Gamma(n_{j-1}, n_j) \subset \Delta_q$ for each $j \geq 1$, we also have $\| \exp(ij\mu) \|_{M_p(D_q)} \geq \| \exp(ij\mu) \|_{M(p,\Gamma_j)}$ for $p=p_j$, since $\hat{\mu}=\hat{\mu}_j$ on Γ_j(cf. [14]). Thus in the same way of THEOREM 2.6, it is routine to show that $\lim \| \exp(in\mu) \|_{M_p(D_q)}^{1/n} > 1$ for all $1 \leq p < 2$. Hence, $sp(\mu, M_p(D_q)) \supset$ closure $(\hat{\mu}(\Delta_q))$ for all $1 \leq p < 2$. Also since $\hat{\mu}=\hat{\mu}_j$ on Γ_j and the choice of m_j, we have that if q is even (resp. odd), $C(\mu)= [-1, 1]$ (resp. $[-\cos(\pi q), 1]$). We omit the details. Q. E. D.

COROLLARY 3.9. *There exists* μ *a Hermitian probability measure in* $M(D_q)$ *such that*

(i) $sp(\mu, M_2(D_q))= closure\ (\hat{\mu}(\Delta_q))$

$$= \begin{cases} [-1, 1] & (q;\ \text{even}) \\ [-\cos(\pi/q), 1] & (q\text{: odd}), \text{ and} \end{cases}$$

(ii) $sp(\mu, M_1(D_q))= \{ |z| \leq 1\} \supsetneqq sp(\mu, M_p(D_q)) \supsetneqq sp(\mu, M_2(D_q))$

for all $1 < p < 2$.

COROLLARY 3.10. *For even q (resp. odd q), let F be a complex-valued function on* $[-1,1]$ *(resp.* $[-\cos(\pi/q), 1]$ *), and be continuous in* (0, 1). *Let* μ *be the measure obtained in* THEOREM 3.8. Suppose $F(\hat{\mu})$ *be a Fourier-Stieljes transform of some element in* $M(D_q)$. *Then F extends to a bounded analytic function on* Σ_1, *where* Σ_1 *is in* COROLLARY 2.8.

The proof of COROLLARY 3.9 (resp. 3.10) is as similar to that of COROLLARY 2.7 (resp. 2.8).

4. Spectra of measures in $M_p(G)$ with a group of bounded order

THEOREM 4.1. *Let G be a group of unbounded order. Then, there exists a Hermitian probability measure* $\mu \in M(G)$ *such that*

(i) $C(\mu) = [-1, 1]$,

(ii) $\delta(x) * \mu^n \perp \mu^m$ *if* $0 \leq m < n$ *and* $x \in G$, *and*

(iii) $sp(\mu, M_p(G)) \supsetneqq$ *closure* $(\hat{\mu}(\Gamma))$ *for all* $1 \leq p < 2$.

In particular, μ *is a continuous singular measure.*

PROOF. Let $\{p_j\}_j$ be in the proof of THEOREM 2.6. Then by THEOREM 2.6, for any natural number j there exist trigonometric polynomials ϕ_j and f_j on $\boldsymbol{T}$ such that

(1) $\phi_j \geq 0$, $\|\phi_j\| = 1$, ϕ_j: real-valued, and

(2) $\|f_j\|_p \leq 1$, $\|g_j * \exp(ij\phi_j)\|_p > K_p{}^j$ for $p = p_j$.

Also there exists a natural number N_j such that (supp $\hat{\phi}_j$) $\cup$ (supp $\hat{f}_j$) $\subset \{-N_j, \cdots, -1, 0, 1, \cdots, N_j\}$. By [13], there exist $\{\gamma_s\}_{s=1}^{\infty} \subset \Gamma$ such that as $S_m = \{\gamma = k_1\gamma_2 + \cdots + k_m\gamma_m;\ |k_j| \leq N_j,\ j = 1, \cdots, m\}$, we have

(3) $0 \in S_1$, and

(4) $\{j\gamma_m;\ j = \pm 1, \cdots, \pm 2N_{m+1}\} \cap (S_m - S_m) = \phi$ for all $m \geq 1$.

Suppose that

$$\phi_j = \sum_{|k| \leq N_j} \hat{f}_j(k)\gamma_j{}^k(x)\ (j \geq 1),\quad d\mu_n = \phi_1 \cdots \phi_n\, d\lambda_G\ (n \geq 1), \text{ and}$$

μ the weak*-limit of $\{d\mu_n\} \subset M(G)$. Then we obtain

(5) $\hat{\mu}(k_1\gamma_1 + \cdots + k_n\gamma_n) = \Pi\hat{\phi}_j(k_j)$ ($|k_j| \leq N_j$), and

(6) $\hat{\mu}(\gamma) = 0$ on $\Gamma / \bigcup_{K=1}^{\infty} \{k_1\gamma_1 + \cdots + k_n\gamma_n;\ |k_j| \leq N_j,\ 1 \leq j \leq n\}$.

Now let c be fixed in $[-1, 1]$. Then there exists $\{k_{s_j}\} \subset \boldsymbol{Z}$ ($|k_{s_j}| \leq N_j$) such that $\hat{\phi}_j(k_{s_j}) \to c$ as $j \to \infty$, and $k_{s_j}\gamma_{s_j} \to c$ as $j \to \infty$ in the weak* topology of $L_\infty(\mu)$. Hence $C(\mu) = [-1, 1]$.

On the other hand, suppose that

$$\Psi_j(x) = \sum_{|k| \leq N_j} \hat{g}_j(k)\gamma_j{}^k(x),\quad \text{supp } \hat{\Psi}_j \subset \{k\gamma_j;\ |k| \leq N_j\},$$

and $\hat{\Psi}_j(k\gamma_j)=\hat{g}_j(k)$ for all $k \in \boldsymbol{Z}$. So by (1), (5), and (6) we have $\exp(ij\hat{\mu}(k\gamma_j))\hat{\Psi}_j(k\gamma_j)=\exp(ij\hat{\phi}_j(k))\hat{g}_j(k)$. Then in the same way of [16; p. 338], we obtain $\lim_{j\to\infty} \| \exp(ij\mu) \|_{M_p}^{1/j}>1$ for all $1\leq p<2$. So we get the desired result. Q. E. D.

COROLLARY 4.2. *Let G be a group of unbounded order. Then, there exists μ a Hermitian probability measure in M(G) such that*

(i) $sp(\mu, M_2(G)) = closure\ (\hat{\mu}(\Gamma))$, *and*

(ii) $sp(\mu, M_1(G))= \{ |z| \leq 1\} \supsetneqq sp(\mu, M_p(G)) \supsetneqq sp(\mu, M_2(G))$

for all $1<p<2$.

COROLLARY 4.3. *Let F be a complex-valued function on* $[-1, 1]$, *and continuous in* (0, 1). *Suppose* $F(\hat{\mu})$ *be a Fourier-Stieltjes transform of some element in M(G) for the measure μ obtained in* THEOREM 4.1. *Then F extends to a bounded analytic function on* Σ_1, *where* Σ_1 *is in* COROLLARY 2.8.

5. Measures as L_p multipliers on a locally compact abelian group

We first assume that G is an infinite compact abelian group. Then, Γ contains a subgroup $\Lambda=\boldsymbol{Z}$ or Δ_q $(q\geq 2)$ or a group of unbounded order. Thus, by routine argument we have that there exists a Hermitian probability measure $\mu \in M(G)$ such that $\mathrm{sp}(\mu, M_2(G))=\mathrm{closure}\ (\hat{\mu}(\Gamma))$, and $\mathrm{sp}(\mu, M_1(G))= \{ |z| \leq 1\} \supsetneqq \mathrm{sp}(\mu, M_p(G)) \supsetneqq \mathrm{sp}(\mu, M_2(G))$ for all $1<p<2$.

Let G be a nondiscrete locally compact abelian group. Suppose G be noncompact. Then, G contains an open subgroup $\boldsymbol{R}^n\times H$ with some $n\geq 0$ and a compact group H. We have the following theorem by applying the preceding results to $M_p(H)$ if H is infinite and to $M_p(\boldsymbol{R}^n)$ if H is finite. We omit the details.

THEOREM 5.1. *Let G be a nondiscrete locally compact abelian group. Then there exists a Hermitian probability measure* $\mu \in M(G)$ such that

(i) $\mathrm{sp}(\mu, M_2(G))=closure\ (\hat{\mu}(\Gamma))$, *and*

(ii) $sp(\mu, M_1(G))= \{ |z| \leq 1\} \supsetneqq sp(\mu, M_p(G)) \supsetneqq sp(\mu, M_2(G))$ *for all* $1<p<2$.

References

[1] J. J. Benedetto, Spectral synthesis, Academic Press, New York, London, San Francisco, 1975.
[2] G. Brown, Riesz products and generalized characters, Proc. of London Math. 30(1975), 209-238.
[3] C. C. Graham, K. E. Hare and D. L. Ritter, The size of L^p-improving measures, J. of Functional Analysis 84(1989), 472-495.
[4] C. C. Graham and O. C. McGehee, Essays in commutative harmonic analysis, Springer-Verlag, New-York 1979.
[5] K. E. Hare, A characterization of L^p-improving measures, Proc. Amer. Math. Soc.

102(1988), 295-299.
[6] L. Hörmander, Estimates for translation invariant operators in L^p spaces, Acta Math. 104(1960), 93-140.
[7] S. Igari, Functions of L^p-multipliers, Tôhoku Math. J. 21(1969), 304-320.
[8] R. Larsen, An introduction to the theory of multipliers, Springer-Verlag, 1970.
[9] W. Moran, The individual symbolic calculus for measures, Proc. London Math. Soc. 31(1975), 385-417.
[10] L. Pigno and S. Saeki, Constructions of singular measures with remarkable convolution properties, Studia Math. 69(1980), 133-141.
[11] D. Ritter, Most Riesz product measures are L^p-improving, Proc. Amer. Math. Soc. 97(1986), 291-295.
[12] W. Rudin, A strong converse of the Wiener-Levy theorem, Canad. J. Math. 14(1962), 694-701.
[13] W. Rudin, Fourier analysis on groups, 2nd. ed., Interscience Publishers, New York, 1967.
[14] S. Saeki, Translation invariant operators on groups, Tôhoku Math. J. 22(1970), 409-419.
[15] P. Sarnak, Spectra of singular measures as multipliers on L^p, J. of Functional Analysis 37(1980), 302-317.
[16] E. Sato, Spectra of measures as L^p multipliers, Tôhoku Math. J. 37(1985), 333-342.

Yuichi KANJIN
Department of Mathematics
College of Liberal Arts
Kanazawa University
Kanazawa 920
Japan

Enji SKTO
Department of Mathematics
Faculty of Science
Yamagata University
Yamagata 990
Japan