

# A Note on a Homeomorphism Type of an Aspherical Homogeneous Space $G/H$

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journal or publication title	金沢大学教育学部紀要.自然科学編
volume	52
page range	47-55
year	2003-02-28
URL	<a href="http://hdl.handle.net/2297/573">http://hdl.handle.net/2297/573</a>

## A Note on a Homeomorphism Type of an Aspherical Homogeneous Space $G/H$

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### Introduction

Let  $G$  be a connected simply connected Lie group and  $H$  a closed subgroup of  $G$ . In [3] V.V.Gorbacevič studied the structures of  $H$  of the compact aspherical homogeneous space  $G/H$  on which  $G$  acts irreducibly, i.e. no proper subgroup of  $G$  acts transitively on  $G/H$ . When  $G$  is a solvable group, the homeomorphism type of  $G/H$  (solvmanifold) is uniquely determined by its fundamental group ([4] [6]). In [8] we studied the homeomorphism type of 4 dimensional compact aspherical homogeneous space.

In this note we shall consider the homeomorphism type of  $G/H$  where  $G$  is a semisimple Lie group and prove the following

**Theorem.** *let  $G$  be a connected simply connected semisimple Lie group without compact factors and  $H$  a closed subgroup of  $G$  such that  $G/H$  is compact and aspherical on which  $G$  acts irreducibly. Let  $H$  be satisfying  $z(G) \subset H \subset z(G)T$ , where  $z(G)$  and  $T$  denote the center of  $G$  and the maximal triangular subgroup of  $G$  respectively. Then the fundamental group of  $G/H$  determines  $G/H$  uniquely up to a homeomorphism.*

In section 1, we shall list some basic facts about the structure of  $G$  and  $H$ , and in section 2, we shall prove that  $G/H$  is a principal  $T^k$  bundle over  $K \backslash S / \Gamma$ , where  $S$  is a product  $A_1 \times A_1 \times \cdots \times A_1$  ( $A_1$  is the universal covering of  $SL(2, R)$ ),  $K$  is a subgroup of  $S$  whose Lie algebra is maximal compact, and  $\Gamma$  is a torsion free cocompact discrete subgroup of  $S$ . In section 3, we shall prove that  $K \backslash S / \Gamma$  is uniquely determined up to a homeomorphism by its fundamental group. We shall prove the Theorem in a final section 4.

In this note we shall use the following notations.

1. For a Lie group  $G$ ,  $G^\circ$  denotes the identity component of  $G$ .
2. The short exact sequence of groups

$$1 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 1$$

means an exact sequence, unless the contrary stated explicitly.

3. Let  $G$  be a group and  $H$  its subgroup. Then  $N_G(H)$  denotes the normalizer of  $H$ .

4.  $T^k$  denotes a  $k$ -dimensional toral group.
5.  $[G, G]$  denotes the derived subgroup of  $G$ .
6. For a finitely generated abelian group  $A$ ,  $rkA$  denotes the rank of  $A$ .

## 1. Preliminaries

Let  $G$  and  $H$  be as in Theorem. We have the following several facts.

Fact 1 ([3]).  $G$  is a product of  $A$ 's, where  $A$  is the universal covering of  $SL(2, R)$ , i.e.  $G = A_1 \times \cdots \times A_r$  ( $A_i = A$ ).

Facts 2 ([3]).  $H^\circ$  is solvable and contained in  $[T, T]$ , where  $T$  is a connected maximal triangular subgroup of  $G$ . Note  $T = T_1 \times \cdots \times T_r$  ( $T_i$  is a maximal connected triangular subgroup of  $A_i$ ).

Fact 3 ([3]). Let  $\lambda_i : G \rightarrow A_i$  be the projection. Then  $\dim \lambda_i(H) \leq 1$ .

Fact 4 ([3]).  $N_G(H^\circ)$  contains some maximal connected triangular subgroup  $T = T_1 \times \cdots \times T_r$ .

The normal subgroup of  $T$  is of the form

$$(\times_{i \in I_1} T_i) \times (\times_{j \in I_2} [T_j, T_j]),$$

where  $I_1 \cup I_2 \subset \{1, 2, \dots, r\}$  and  $I_1 \cap I_2 = \emptyset$ . As  $H^\circ$  being a normal subgroup of  $T$ , we have

$$H^\circ = \times_{j \in I_2} [T_j, T_j]$$

By renumbering the indicis, if necessary, we may assume

$$H^\circ = \times_{j \in \{s+1, \dots, r\}} [T_j, T_j].$$

Then we have

$$N_G(H^\circ)^\circ = A_1 \times \cdots \times A_s \times T_{s+1} \times \cdots \times T_r.$$

Fact 5. Let  $T$  be a connected maximal triangular subgroup of  $A$ . Then  $N_A([T, T]) = ZT$ ,

where  $Z$  is the center of  $A$ . This fact follows from the structure of  $A$ .

Fact 6.  $N_G(H^\circ) = A_1 \times \cdots \times A_s \times Z_{s+1} T_{s+1} \times \cdots \times Z_r T_r$ , where  $Z_i$  is the center of  $A_i$ . This follows from facts 4 and 5.

Fact 7 ([10]). Let  $G$  be a connected semisimple Lie group without compact factors and  $\Gamma$  a subgroup of  $G$  which is discrete and  $G/\Gamma$  is compact. Then  $N_G(\Gamma)$  is also discrete,  $G/N_G(\Gamma)$  is compact and  $N_G(\Gamma)/\Gamma$  is finite.

Fact 8 ([11]). Let  $S$  be a connected semisimple Lie group without compact factors. If  $\Gamma$  is a discrete and cocompact subgroup of  $S \times \mathbb{R}^n$ , then the image of  $\Gamma$  by the projection  $S \times \mathbb{R}^n \rightarrow S$  is also discrete and cocompact.

## 2. A principal torus fibration

Let  $G$  and  $H$  be as in Theorem. Then we may assume

$$G = A_1 \times \cdots \times A_r, \quad H^\circ = [T_{s+1}, T_{s+1}] \times \cdots \times [T_r, T_r].$$

Let  $S_1 = A_1 \times \cdots \times A_s$ ,  $S_2 = A_{s+1} \times \cdots \times A_r$  and  $p_1 : G \rightarrow S_1$  the projection. Then we have the following

**Proposition 1.**  $p_1(H)$  is a discrete and cocompact subgroup  $S_1$ .

PROOF. Define  $H_1 = HN_G(H^\circ)^\circ$  and  $\Gamma = H \cap H_1^\circ / (H \cap H_1^\circ)^\circ$ . Then  $\Gamma$  is a discrete and cocompact subgroup of  $G_1 = H_1^\circ / (H \cap H_1^\circ)^\circ$ . Clearly  $G_1$  is  $S_1 \times \mathbb{R}^{-s}$ . It follows from fact 8 that  $q(\Gamma)$  is a discrete and cocompact subgroup of  $S_1$ , where  $q$  is the projection  $G_1 \rightarrow S_1$ . Note that  $q(\Gamma) = p_1(H \cap H_1^\circ)$ . It is easy to see that  $p_1(H) \subset N_{S_1}(p_1(H \cap H_1^\circ))$ . Fact 7 implies that  $p_1(H)$  is discrete and cocompact in  $S_1$ . This completes the proof of Proposition 1.

Q.E.D.

We have the following

**Proposition 2.** We have homeomorphisms

$$\begin{aligned} G/H &\cong (G/H^\circ)/(H/H^\circ) \\ &\cong ((S_2/H^\circ) \times S_1)/(H/H^\circ) \\ &\cong ((S_2/H \cap S_2) \times S_1)/p_1(H). \end{aligned}$$

PROOF. Define a map  $h_1 : G/H \rightarrow (G/H^\circ)/(H/H^\circ)$  by  $h_1(gH) = [gH^\circ]$ , where  $[gH^\circ]$  is the orbit of  $gH^\circ$  under the action of  $H/H^\circ$  on  $G/H^\circ$  defined by  $(hH^\circ)(gH^\circ) = gh^{-1}H^\circ$ . This is well defined. In fact, we have

$$\begin{aligned} gH = g'H &\Rightarrow g^{-1}g' \in H \\ &\Rightarrow g' = gh^{-1} \quad (h \in H) \\ &\Rightarrow g'H^\circ = gh^{-1}H^\circ = (hH^\circ)(gH^\circ). \end{aligned}$$

Clearly  $h_1$  is a homeomorphism.

Define an action of  $H/H^\circ$  on  $S_2/H^\circ \times S_1$  by

$$(hH^\circ)(g_2H^\circ, g_1) = (g_2h_2^{-1}H^\circ, g_1h_1^{-1}),$$

where  $h = (h_1, h_2) \in H \subset G = S_1 \times S_2$ .

Define a map  $h'_2 : G/H^\circ \rightarrow S_2/H^\circ \times S_1$  by  $h'_2(gH^\circ) = (g_2H^\circ, g_1)$ , where  $g = (g_1, g_2)$ . Clearly  $h'_2$  is a homeomorphism and  $H/H^\circ$ -equivariant.

Let  $h'_3 : S_2/H^\circ \times S_1 \rightarrow S_2/H \cap S_2 \times S_1$  be the map projection  $\times$  identity. Note that  $p_1(H) = H/S_2 \cap H$ . Define an action of  $p_1(H)$  on  $S_2/H^\circ \times S_1$  by

$$h(S_2 \cap H)(g_2(H \cap S_2), g_1) = (g_2h_2^{-1}(H \cap S_2), g_1h_1^{-1}),$$

where  $h = (h_1, h_2) \in H$ .

The map  $h'_3$  is an equivariant map between  $H/H^\circ$ -manifold  $S_2/H^\circ \times S_1$  and  $p_1(H)$ -manifold  $S_2/(H \cap S_2) \times S_1$ . In fact, we have

$$\begin{aligned} h'_3((hH^\circ)(g_2H^\circ, g_1)) &= h'_3(g_2h_2^{-1}(H \cap S_2), g_1h_1^{-1}) \\ &= h(S_2 \cap H)(h'_3(g_2H^\circ, g_1)), \end{aligned}$$

where  $h = (h_1, h_2) \in H$ .

Hence  $h'_3$  induces a homeomorphism

$$h_3 : (S_2/H^\circ \times S_1)/(H/H^\circ) \cong (S_2/(H \cap S_2) \times S_1)/p_1(H).$$

This completes the proof of Proposition 2.

Q.E.D.

In the following sections we shall assume  $z(G) \subset H \subset z(G)T$  and prove the following

**Theorem 1.**  $G/H$  admits a free  $T^{s+2(r-s)}$ -action.

PROOF. We have gotten the homeomorphism

$$G/H \cong (S_2/S_2 \cap H \times S_1)/p_1(H).$$

Put  $H_2 = H \cap S_2$ .

Let  $K_i$  be the subgroup of  $S_i$  whose Lie algebra is maximal compact for  $i = 1, 2$ . We show that  $G/H$  admits a free action of  $K_1/z(p_1(H))$ . In fact, define an action of  $K_1$  on  $G/H$  by

$$k[g_2H_2, g_1] = [g_2H_2, kg_1],$$

where  $[\cdot]$  denotes an equivalence class of  $(S_2/S_2 \cap H \times S_1)$  under the action of  $p_1(H)$ . We first prove this action induces an effective action of  $K_1/z(p_1(H))$ . Let  $[g_2H_2, g_1] = k[g_2H_2, g_1]$  for any  $g_1 \in S_1$  and any  $g_2 \in S_2$ . Then there exists  $h_1 \in p_1(H)$ ,  $h = (h_1, h_2) \in H$  satisfying  $(g_2h_2^{-1}H_2, g_1h_1^{-1}) = (g_2H_2, kg_1)$ . So we obtain  $h_2 \in H_2$  and  $g_1h_1^{-1}g_1^{-1} = k$ . This induces  $h_1 \in z(S_1)$  and  $k \in z(p_1(H))$ . This proves an effectiveness of the action by  $K_1/z(p_1(H))$ .

This action induces a free action of  $K_1/z(p_1(H))$ . In fact,

$$\begin{aligned} [g_2H_2, kg_1] &= [g_2H_2, g_1] \\ \Rightarrow (g_2H_2, kg_1) &= (hH_2)(g_2H_2, g_1) \\ &= (g_2h_2^{-1}H_2, g_1h_1^{-1}) \\ \Rightarrow h_2 \in H_2, k &= g_1h_1^{-1}g_1^{-1} \in K_1 \Rightarrow h_1 \in p_1(H) \cap K_1 = z(S_1) \\ &\Rightarrow k \in z(p_1(H)), \end{aligned}$$

where  $h = (h_1, h_2)$ .

Next define an action of  $K_2$  on  $S_2/H_2$  by

$$k(g_2H_2) = (kg_2)H_2.$$

The restriction of this action to  $H \cap K_2$  is trivial, because  $H \cap K_2 \subset z(S_2)$ . Moreover we can define an action of  $T_2$  (a maximal connected triangular subgroup of  $S_2$  such that  $K_2T_2 = S_2$ ) on  $S_2/H_2$  by

$$n(g_2H_2) = (g_2n^{-1})H_2.$$

This is well defined. In fact,

$$g_2^{-1}g_2' \in H \cap S_2 \Rightarrow n(g_2^{-1}g_2')n^{-1} \in H \cap S_2$$

because  $H \cap S_2 \subset z(S_2)T_2$ . Naturally this action induces an action of  $T_2/T_2 \cap H$ .

Now we define the action of  $T^{2(r-s)}$  on  $S_2/H_2 \times S_1$  where  $T^{2(r-s)} = K_2/H \cap K_2 \times T_2/T_2 \cap H$ , by

$$(\bar{n}_1, \bar{n}_2)(g_2H_2, g_1) = (n_1g_2n_2^{-1}H_2, g_1),$$

where  $\bar{n}_1 \in K_2/H \cap K_2$  and  $\bar{n}_2 \in T_2/T_2 \cap H$ . This is compatible with the action of  $p_1(H)$ . In fact, note that  $H^\circ \subset [T, T]$ ,  $H \subset z(G)T = z(S_1)T_1 \times z(S_2)T_2$ . Any element  $h = (h_1, h_2) \in H$  is written as a product  $h_2 = z_2t_2$ , where  $z_2 \in z(S_2)$  and  $t_2 \in T_2$ .

Then we have

$$\begin{aligned} &(\bar{n}_1, \bar{n}_2)\{hH_2(g_2H_2, g_1)\} \\ &= (\bar{n}_1, \bar{n}_2)\{(g_2h_2^{-1}H_2, g_1h_1^{-1})\} \\ &= (n_1g_2h_2^{-1}n_2^{-1}H_2, g_1h_1^{-1}) \\ &= (n_1g_2z_2^{-1}t_2^{-1}n_2^{-1}H_2, g_1h_1^{-1}) \\ &= (n_1g_2t_2^{-1}n_2^{-1}z_2^{-1}H_2, g_1h_1^{-1}) \end{aligned}$$

Since  $[T_2, T_2] \subset H_2$ , we have

$$ht_2n_2 = n_2t_2, h \in H_2 \quad \text{for every } t_2, n_2 \in T_2.$$

We have then

$$\begin{aligned} & (n_1g_2t_2^{-1}n_2^{-1}z_2^{-1}H_2, g_1h_1^{-1}) \\ &= (n_1g_2n_2^{-1}t_2^{-1}z_2^{-1}H_2, g_1h_1^{-1}) \\ &= hH_2\{(\bar{n}_1, \bar{n}_2)(g_2H_2, g_1)\}. \end{aligned}$$

This implies the compatibility.

Next we shall prove the freeness of the action. Assume  $(\bar{n}_1, \bar{n}_2)(g_2H_2, g_1) = (g_2H_2, g_1)$ . Then we have

$$n_1g_2n_2^{-1} = g_2h, (h \in H_2)$$

Since  $g_2 = k_2t_2, h = z_2s_2$  ( $k_2 \in K_2, t_2 \in T_2, s_2 \in T_2, z_2 \in z(S_2)$ ), we have

$$\begin{aligned} n_1k_2t_2 &= k_2t_2z_2s_2n_2 \\ \Rightarrow z_2^{-1}n_1k_2 &= k_2t_2s_2n_2t_2^{-1} \\ \Rightarrow k_2^{-1}z_2^{-1}n_1k_2 &= t_2s_2n_2t_2^{-1}. \end{aligned}$$

All elements of left hand side being in  $K_2$ , we have

$$\begin{aligned} z_2^{-1}n_1 &= t_2s_2t_2^{-1}n_2h', (h' \in H_2^\circ) \\ &= s_2n_2h'', (h'' \in H_2^\circ) \end{aligned}$$

Hence we have  $z_2^{-1}n_1 = s_2n_2h''$  in  $K_2 \cap T_2$ , which implies  $z_2^{-1}n_1 \in z(S_2)$ . Thus we have shown that  $\bar{n}_1 = 1$ , which induces  $\bar{n}_2 = 1$ . This completes the proof of Theorem 1.

Q.E.D.

It follows from the Theorem 1 that we have a principal bundle;

$$T^{s+2(r-s)} \rightarrow G/H \rightarrow M.$$

We have the following

**Theorem 2.**  $M \cong K_1 \backslash S_1 / p_1(H)$ .

PROOF. We have proved that

$$G/H \cong (S_2/H_2 \times S_1) / p_1(H),$$

and the action of  $T^{2(r-s)}$  on  $G/H$  is induced by the action of  $T^{2(r-s)}$  on  $S_2/H_2$ . Since  $\dim S_2/H_2 = 2(r-s)$ , this action is transitive. Thus we have

$$\begin{aligned} (G/H) / (T^s \times T^{2(r-s)}) &\cong ((G/H) / T^{2(r-s)}) / T^s \\ &\cong (S_1 / p_1(H)) / T^s. \end{aligned}$$

Moreover from the definition of the action of  $T^s$  on  $S_1/p_1(H)$ , it follows that

$$(S_1/p_1(H))/T^s \cong K_1 \backslash S_1/p_1(H).$$

This proves the Theorem 2.

Q.E.D.

Thus we have the fiber bundle;

$$T^{s+2(r-s)} \rightarrow G/H \rightarrow K_1 \backslash S_1/p_1(H).$$

Hence we have the following exact sequence of the fundamental groups;

$$1 \rightarrow Z^{s+2(r-s)} \xrightarrow{i_*} \pi_1(G/H) \xrightarrow{p_*} \pi_1(M) \rightarrow 1.$$

It follows from a result in[1] that the image of  $i_*$  is contained in the center  $z(\pi_1(G/H))$ .

We have the following

**Proposition 3.**  $\text{rank } z(\pi_1(G/H)) = s + 2(r - s)$ .

PROOF. We know the following facts (see [3]);

- (1)  $H^\circ = 1 \times [T_2, T_2]$
- (2)  $N_G(H^\circ)^\circ = S_1 \times T_2$
- (3)  $N_G(H^\circ) = S_1 \times Z_2 T_2$
- (4)  $H \subset N_G(H^\circ)$ .

Consider the following exact sequence;

$$\begin{array}{ccccccc} 1 & \longrightarrow & H_2/H_2 \cap H^\circ & \longrightarrow & H/H^\circ & \xrightarrow{p_1} & p_1(H/H^\circ) \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & (H_2/H_2 \cap H^\circ) \cap z(H/H^\circ) & \longrightarrow & z(H/H^\circ) & \longrightarrow & p_1(z(H/H^\circ)) \longrightarrow 1 \end{array}$$

Since  $(H_2/H_2 \cap H^\circ) \cap z(H/H^\circ) \subset z(H_2/H_2 \cap H^\circ)$  and  $p_1(z(H/H^\circ)) \subset z(p_1(H/H^\circ))$ , we have

$$\begin{aligned} \text{rank } z(H/H^\circ) &= \text{rank } p_1(z(H/H^\circ)) + \text{rank } ((H_2/H_2 \cap H^\circ) \cap z(H/H^\circ)) \\ &\leq s + 2(r - s). \end{aligned}$$

This completes the proof.



Q.E.D.

Since  $K_1 \backslash S_1 / p_1(H)$  is a closed aspherical manifold, its fundamental group is torsion free. Hence we have  $i_*(Z^{s+2(r-s)}) = z(\pi_1(G/H))$ .

Let  $H_1, H_2$  be the subgroups of  $G$  as  $H$  in Theorem. Assume  $\pi_1(G/H_1) \cong \pi_1(G/H_2)$ . We have the following

**Proposition 4.**  $\dim H_1 = \dim H_2$ .

PROOF. By the assumption, we have  $\text{cd}(H_1/H_1^\circ) = \text{cd}(H_2/H_2^\circ)$ , where  $\text{cd}(\ )$  is the cohomological dimension of a group. This means that  $\dim G/H_1^\circ = \dim G/H_2^\circ$ . Then  $\dim G/H_1 = \dim G/H_2$ . This completes the proof.

Q.E.D.

It follows from the arguments in Fact 4 that

$$H_i^\circ = \times_{j \in I_i} [T_j, T_j] \quad (i = 1, 2) \text{ and } \#I_1 = \#I_2.$$

Then we may assume that  $H_1^\circ = H_2^\circ$ , and that the decomposition of  $G$  into the product  $S_1 \times S_2$  in the above argument for  $H_2$  is the same one for  $H_1$ . Thus we have the following commutative diagram;

$$\begin{array}{ccccccc}
 1 & \rightarrow & Z^{s+2(r-s)} & \rightarrow & \pi_1(G/H_1) & \rightarrow & \pi_1(K_1 \backslash S_1 / p_1(H_1)) \rightarrow 1 \\
 (*) & & \gamma \downarrow & & \alpha \downarrow & & \beta \downarrow \\
 1 & \rightarrow & Z^{s+2(r-s)} & \rightarrow & \pi_1(G/H_2) & \rightarrow & \pi_1(K_1 \backslash S_1 / p_1(H_2)) \rightarrow 1,
 \end{array}$$

where  $\alpha$  is the given isomorphism,  $\gamma$  is the restriction (note that  $Z^{s+2(r-s)} = z(\pi_1(G/H_i))$ ) and  $\beta$  is the isomorphism induced by  $\alpha$ . We have the following

**Proposition 5.**  $\pi_1(K_1 \backslash S_1 / p_1(H_1)) \cong \pi_1(K_1 \backslash S_1 / p_1(H_2))$

### 3. The structure of $K \backslash S / \Gamma$

In this section, we shall prove the following

**Theorem 3.** *Let  $G$  be a product  $A_1 \times \cdots \times A_1$ ,  $K$  a subgroup of  $G$  whose Lie algebra is maximal compact and  $\Gamma$  a torsion free cocompact discrete subgroup of  $G$ . Then  $K \backslash S / \Gamma$  is uniquely determined, up to a homeomorphism, by its fundamental group.*

This Theorem is true when  $\dim K \backslash S / \Gamma \neq 4$ . In fact, if  $\dim K \backslash S / \Gamma = 2$ , this is a classical result. Assume  $\dim K \backslash S / \Gamma \geq 5$ . Then  $K \backslash S / \Gamma$  supports the structure of

nonpositively curved Riemannian manifold. By a result in [2] the homeomorphism type of such a manifold is uniquely determined by its fundamental group.

Now we suppose that  $G = A_1 \times A_1$ . If  $\Gamma$  is irreducible, it follows from results in [7] that the homeomorphism type of  $K \backslash S / \Gamma$  is uniquely determined by its fundamental group. Then we may assume that  $\Gamma$  is not irreducible, i.e. the image of  $\Gamma$  by any projection  $G \rightarrow A_1$  is cocompact discrete, and hence the intersection  $\Gamma \cap A_1$  is also cocompact discrete.

We denote  $G = A_1^{(1)} \times A_1^{(2)}$  where  $A_1^{(i)} = A_1$ ,  $\Gamma_i = \Gamma \cap A_1^{(i)}$  and  $\Gamma^i = p_i(\Gamma)$  where  $p_i : G \rightarrow A_1^{(i)}$  is the projection. We put  $\overline{U} = U/z(U)$  for any group  $U$ . Then we have

- 1.  $\overline{\Gamma}_i = \overline{\Gamma} \cap A_1^{(i)}$ .
- 2.  $\overline{\Gamma}^i = \overline{p_i(\Gamma)}$ , where  $\overline{p_i}$  is the natural map induced by  $p_i$ .
- 3.  $\overline{\Gamma}_1 \times \overline{\Gamma}_2 \subset \overline{\Gamma}^1 \times \overline{\Gamma}^2$
- 4. We have the fiber bundles;

$$K_1 \backslash A_1^{(i)} / \Gamma_i \rightarrow K \backslash G / \Gamma \rightarrow K_1 \backslash A_1^{(3-i)} / \Gamma^{3-i} \quad i = 1, 2,$$

where  $K_1$  is a subgroup of  $A_1$  whose Lie algebra is maximal compact.

We note that  $K_1 \backslash A_1^{(i)} / \Gamma_i \cong \overline{K_1} \backslash \overline{A_1^{(i)}} / \overline{\Gamma}_i$  and  $K_1 \backslash A_1^{(i)} / \Gamma^i \cong \overline{K_1} \backslash \overline{A_1^{(i)}} / \overline{\Gamma}^i$ .

PROOF. We omit the proof since it is not difficult to prove this Proposition.

Q.E.D.

Since  $K \backslash G / \Gamma$  is aspherical, the fiber and the base are also aspherical and hence  $\overline{\Gamma}_i, \overline{\Gamma}^i$  are torsion free. Let  $\Gamma'$  be an another cocompact discrete subgroup of  $G$  which is isomorphic to  $\Gamma$  and  $\theta$  the isomorphism. In the following the natural isomorphism  $\overline{\Gamma'} \rightarrow \overline{\Gamma}$  induced by  $\theta$  is also denoted by  $\theta$ . It follows from a result in [7] (Theorem 4.1 in [7]) that  $\Gamma'$  is also not irreducible.  $\overline{\Gamma'}$  and  $\overline{\Gamma'^i}$  are also torsion free.

We have the following exact sequence;

$$1 \longrightarrow \overline{\Gamma}_i \longrightarrow \overline{\Gamma} \longrightarrow \overline{\Gamma^{(3-i)}} \longrightarrow 1$$

and

$$1 \longrightarrow \overline{\Gamma'_i} \longrightarrow \overline{\Gamma'} \longrightarrow \overline{\Gamma'^{(3-i)}} \longrightarrow 1.$$

Put  $\overline{\Gamma''_i} = \theta(\overline{\Gamma'_i})$ . We have the following commutative diagram;

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \rightarrow & \overline{\Gamma_i} \cap \overline{\Gamma_j''} & \rightarrow & \overline{\Gamma_j''} & \rightarrow & \overline{p_{3-i}(\Gamma_j'')} \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 (*) & 1 & \rightarrow & \overline{\Gamma_i} & \rightarrow & \overline{\Gamma} & \rightarrow \overline{\Gamma^{(3-i)}} \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \rightarrow & \overline{\Gamma_i} / \overline{\Gamma_i} \cap \overline{\Gamma_j''} & \rightarrow & \overline{\Gamma} / \overline{\Gamma_j''} & \rightarrow & \overline{\Gamma^{(3-i)}} / \overline{p_{3-i}(\Gamma_j'')} \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

We have the following

**Lemma 1.**  $\overline{\Gamma} / \overline{\Gamma_j''}$  is torsion free.

PROOF. This follows immediately from the following commutative diagram;

$$\begin{array}{ccccccc}
 1 & \rightarrow & \overline{\Gamma_j''} & \rightarrow & \overline{\Gamma} & \rightarrow & \overline{\Gamma^{(3-i)}} \rightarrow 1 \\
 & & \tilde{\theta} \downarrow & & \theta \downarrow & & \bar{\theta} \downarrow \\
 1 & \rightarrow & \overline{\Gamma_j''} & \rightarrow & \overline{\Gamma} & \rightarrow & \overline{\Gamma} / \overline{\Gamma_j''} \rightarrow 1
 \end{array}$$

where  $\tilde{\theta}$  is the restriction of  $\theta$ .

Q.E.D.

Since the surface group  $\overline{\Gamma^{(3-i)}}$  has a trivial center, it contains no non-trivial normal finitely generated subgroup of infinite index, and hence  $\overline{p_{3-i}(\Gamma_j'')}$  is trivial or a finite index in  $\overline{\Gamma^{(3-i)}}$ .

Suppose the  $\overline{p_2(\Gamma_1'')}$  is trivial. Then we have  $\overline{\Gamma_1} \cap \overline{\Gamma_1''} = \overline{\Gamma_1''}$ . The group  $\overline{\Gamma_1} / \overline{\Gamma_1} \cap \overline{\Gamma_1''}$  is also finite. It follows from Lemma 1 that this group is trivial. Thus we have the following commutative diagram;

$$\begin{array}{ccccccc}
 1 & \rightarrow & \overline{\Gamma_1} & \rightarrow & \overline{\Gamma} & \rightarrow & \overline{\Gamma^{v2}} \rightarrow 1 \\
 (\#) & & \tilde{\theta} \downarrow & & \theta \downarrow & & \bar{\theta} \downarrow \\
 1 & \rightarrow & \overline{\Gamma_1} & \rightarrow & \overline{\Gamma} & \rightarrow & \overline{\Gamma} / \overline{\Gamma_1} \rightarrow 1,
 \end{array}$$

where  $\tilde{\theta}$  is the restriction of  $\theta$  and the vertical maps are isomorphisms.

Note that manifolds  $K \backslash G / \Gamma$  and  $K \backslash G / \Gamma'$  are considered as 4 dimensional codimension 2 foliated manifolds with all leaves compact. The diagram (#) shows that the isomorphism  $\theta : \pi_1(K \backslash G / \Gamma') \rightarrow \pi_1(K \backslash G / \Gamma)$  restricts to an isomorphism  $\tilde{\theta} : \pi_1(F') \rightarrow \pi_1(F)$ , where  $F, F'$  are typical fibers. Then the argument of the proof of Theorem(5.3) and Theorem (4.1) in [9] shows that  $K \backslash G / \Gamma$  and  $K \backslash G / \Gamma'$  are homeomorphic.

Next we suppose that  $\bar{p}_2(\bar{\Gamma}_1'') \neq 1$ . Since  $\bar{\Gamma}^2$  is a surface group of genus  $> 1$ ,  $\bar{p}_2(\bar{\Gamma}_1'')$  is a subgroup of  $\bar{\Gamma}^2$  of finite index and  $\bar{\Gamma}_1 \cap \bar{\Gamma}_1''$  is trivial. Then the diagram (\*) is now as follows;

$$\begin{array}{ccccccc}
 & & & 1 & & 1 & \\
 & & & \downarrow & & \downarrow & \\
 & & & \bar{\Gamma}_2'' & \xrightarrow{\bar{p}_2} & \bar{p}_2(\bar{\Gamma}_2'') & \rightarrow 1 \\
 & & 1 & \rightarrow & \bar{\Gamma}_1 & \xrightarrow{i} & \bar{\Gamma} & \xrightarrow{\bar{p}_2} & \bar{\Gamma}^2 & \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & \bar{q} \downarrow & & q \downarrow & & \bar{q} \downarrow & & & \\
 1 & \rightarrow & \bar{\Gamma}_1 & \xrightarrow{\bar{i}} & \bar{\Gamma} / \bar{\Gamma}_2'' & \xrightarrow{\bar{p}_2} & \bar{\Gamma}^2 / \bar{p}_2(\bar{\Gamma}_2'') & \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & & \\
 & & 1 & & 1 & & 1 & & & 
 \end{array}$$

Assume  $\bar{\Gamma}_1 \cap \bar{\Gamma}_2'' = 1$ . Then we have  $\bar{i}(\bar{\Gamma}_1) \cap q(\bar{\Gamma}_2'') = 1$ . In fact, let  $x = \bar{i}(\bar{q}(x_1)) = q(x_2)$ ,

i.e.  $q(x_2) = q(i(x_1))$ , where  $x_2 \in \bar{\Gamma}_2''$  and  $x_1 \in \bar{\Gamma}_1$ . Since  $\bar{\Gamma}_1 \cap \bar{\Gamma}_2'' = 1$  and  $\bar{\Gamma}_1'' \cap \bar{\Gamma}_2'' = 1$ ,  $q|_{\bar{\Gamma}_2''}$  and  $q|_{i(\bar{\Gamma}_1)}$  are injective. Hence we have  $x_2 = i(x_1)$ , which implies  $\bar{\Gamma}_2'' \cap \bar{\Gamma}_1 \neq 1$ , which contradicts the assumption. Thus  $\bar{p}_2|_{q(\bar{\Gamma}_2'')}$  is infinite. But  $\bar{\Gamma}^2 / \bar{p}_2(\bar{\Gamma}_2'')$  is finite. This is a contradiction. Thus we have proved that  $\bar{\Gamma}_2'' \cap \bar{\Gamma}_1 \neq 1$ . We have the diagram;

$$\begin{array}{ccccccc}
 & & & 1 & & 1 & \\
 & & & \downarrow & & \downarrow & \\
 1 & \rightarrow & \bar{\Gamma}_1 \cap \bar{\Gamma}_2'' & \rightarrow & \bar{\Gamma}_2'' & \rightarrow & \bar{p}_2(\bar{\Gamma}_2'') & \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 1 & \rightarrow & \bar{\Gamma}_1 & \rightarrow & \bar{\Gamma} & \rightarrow & \bar{\Gamma}^2 & \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 1 & \rightarrow & \bar{\Gamma}_1 / \bar{\Gamma}_1 \cap \bar{\Gamma}_2'' & \rightarrow & \bar{\Gamma} / \bar{\Gamma}_2'' & \rightarrow & \bar{\Gamma}^2 / \bar{p}_2(\bar{\Gamma}_2'') & \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 1 & & 1 & & 1 & 
 \end{array}$$

Since  $\bar{\Gamma}_1 \cap \bar{\Gamma}_2'' \neq 1$ , we have  $\bar{\Gamma}_1 / \bar{\Gamma}_1 \cap \bar{\Gamma}_2'' = 1$ . Thus we have the following diagram;

$$\begin{array}{ccccccc}
 1 & \rightarrow & \overline{\Gamma}_1 & \rightarrow & \overline{\Gamma} & \rightarrow & \overline{\Gamma}^2 \rightarrow 1 \\
 & & \tilde{\theta} \downarrow & & \theta \downarrow & & \bar{\theta} \downarrow \\
 1 & \rightarrow & \overline{\Gamma}_1 & \rightarrow & \overline{\Gamma} & \rightarrow & \overline{\Gamma}^2 \rightarrow 1.
 \end{array}$$

By the same argument as above, we have a homeomorphism  $K \backslash G / \Gamma' \cong K \backslash G / \Gamma$ . This completes the proof of Theorem 3.

#### 4. The Proof of Theorem

In this section, we shall prove the Theorem in Introduction. We use the same notations in section 2. It follows from Theorem 3 and Proposition 5 that we have the following

##### Proposition 7.

$$K_1 \backslash S_1 / p_1(H_1) \cong K_1 \backslash S_1 / p_1(H_2).$$

Since the principal fibration

$$T^{s+2(r-s)} \rightarrow G/H_i \rightarrow K_1 \backslash S_1 / p_1(H_i), i = 1, 2$$

is uniquely determined up to an equivalence by its characteristic class

$$c \in H^2(K_1 \backslash S_1 / p_1(H_i); Z^{s+2(r-s)}).$$

It follows from a result in [5] that this class is naturally identified with the characteristic class corresponding to the exact sequence;

$$1 \rightarrow Z^{s+2(r-s)} \rightarrow \pi_1(G/H_i) \rightarrow \pi_1(K_1 \backslash S_1 / p_1(H_i)) \rightarrow 1$$

The diagram ( $\star$ ) in section 2 shows that two fibrations over  $K_1 \backslash S_1 / p_1(H_i)$

$$T^{s+2(r-s)} \rightarrow G/H_i \rightarrow K_1 \backslash S_1 / p_1(H_i), i = 1, 2$$

are equivalent, which means  $G/H_1$  and  $G/H_2$  are homeomorphic. This completes the proof of the Theorem in Introduction.

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