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WEIGHTED WEAK TYPE $(1, 1)$ ESTIMATES FOR OSCILLATORY SINGULAR INTEGRALS WITH DINI KERNELS

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ABSTRACT. We consider A_1 -weights and prove weighted weak type $(1, 1)$ estimates for oscillatory singular integrals with kernels satisfying a Dini condition.

1. INTRODUCTION

We consider an oscillatory singular integral operator of the form:

$$T(f)(x) = \text{p. v.} \int_{\mathbf{R}^n} e^{iP(x,y)} K(x-y) f(y) dy = \lim_{\epsilon \rightarrow 0} \int_{|x-y| > \epsilon} e^{iP(x,y)} K(x-y) f(y) dy,$$

where P is a real-valued polynomial:

$$(1.1) \quad P(x, y) = \sum_{|\alpha| \leq M, |\beta| \leq N} a_{\alpha\beta} x^\alpha y^\beta,$$

and $f \in \mathfrak{S}(\mathbf{R}^n)$ (the Schwartz space).

Let $K \in C^1(\mathbf{R}^n \setminus \{0\})$ satisfy

$$(1.2) \quad |K(x)| \leq c|x|^{-n}, \quad |\nabla K(x)| \leq c|x|^{-n-1};$$

$$(1.3) \quad \int_{a < |x| < b} K(x) dx = 0 \quad \text{for all } a, b \text{ with } 0 < a < b.$$

The smallest constant for which (1.2) holds will be denoted by $C(K)$. The following results are known.

Theorem A. (Ricci-Stein [5]) *Let $1 < p < \infty$. Then, T is bounded on $L^p(\mathbf{R}^n)$ with the operator norm bounded by a constant depending only on the total degree of P , $C(K)$, p and the dimension n .*

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Theorem B. (Chanillo-Christ [1]) *The operator T is bounded from $L^1(\mathbf{R}^n)$ to the weak $L^1(\mathbf{R}^n)$ space with the operator norm bounded by a constant depending only on the total degree of P , $C(K)$ and the dimension n .*

Let w be a locally integrable positive function on \mathbf{R}^n . We say that $w \in A_1$ if there is a constant c such that

$$(1.4) \quad M(w)(x) \leq cw(x) \quad \text{a.e.}$$

where M denotes the Hardy-Littlewood maximal operator. The smallest constant for which (1.4) holds will be denoted by $C_1(w)$.

It is known that T is bounded from L_w^1 to $L_w^{1,\infty}$ (the weak L_w^1 space).

Theorem C. ([8]) *There exists a constant c depending only on the total degree of P , $C(K)$, $C_1(w)$ and the dimension n such that*

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbf{R}^n : |T(f)(x)| > \lambda\}) \leq c \|f\|_{L_w^1},$$

where $w(E) = \int_E w(x) dx$ and $\|f\|_{L_w^1} = \int |f(x)|w(x) dx$.

Let K be locally integrable away from the origin. Put, for $r \geq 1$, $0 < t \leq 1$ and $R > 0$,

$$\omega_{r,R}(t) = \sup_{|y| \leq Rt/2} \left(R^{-n} \int_{R \leq |x| \leq 2R} |R^n (K(x-y) - K(x))|^r dx \right)^{1/r}.$$

We say that the kernel K satisfies the D_r -condition if

$$B_r = \int_0^1 \omega_r(t) \frac{dt}{t} < \infty \quad \text{where} \quad \omega_r(t) = \sup_{R>0} \omega_{r,R}(t);$$

$$C_r = \sup_{R>0} \left(R^{-n} \int_{R \leq |x| \leq 2R} |R^n K(x)|^r dx \right)^{1/r} < \infty.$$

By the usual modifications we can also define the D_∞ -condition. In this note we shall prove the following results, which will improve Theorems B and C.

Theorem 1. *Let $r > 1$ and $1/r + 1/u = 1$. Suppose the kernel K satisfy the D_r -condition and (1.3), and suppose $w^u \in A_1$. Then, there exists a constant c depending only on the total degree of P , B_r , C_r , $C_1(w^u)$, r and the dimension n such that*

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbf{R}^n : |T(f)(x)| > \lambda\}) \leq c \|f\|_{L_w^1}.$$

Theorem 2. *Suppose that K satisfies the D_1 -condition and (1.3). Then, there exists a constant c depending only on the total degree of P , B_1 , C_1 and the dimension n such that*

$$\sup_{\lambda > 0} \lambda |\{x \in \mathbf{R}^n : |T(f)(x)| > \lambda\}| \leq c \|f\|_{L^1}.$$

Every kernel satisfying (1.2) satisfies the D_∞ -condition. If $K(x) = |x|^{-n} \Omega(x')$, $x' = x/|x|$, and if Ω satisfies the L^r -Dini condition on S^{n-1} , then K satisfies the D_r -condition.

These theorems will be proved by a double induction as in [5], [1] and [8]. In this note we shall prove only Theorem 1. Theorem 2 can be proved similarly. Let P be a polynomial of the form in (1.1). We assume that there exists α such that $|\alpha| = M$ and $a_{\alpha\beta} \neq 0$ for some β . We write

$$(1.5) \quad P(x, y) = \sum_{|\alpha| \leq M} x^\alpha Q_\alpha(y)$$

and define $L = \max\{\deg(Q_\alpha) : Q_\alpha \neq 0, |\alpha| = M\}$. Then $0 \leq L \leq N$. We assume that $L \geq 1$ and $\max_{|\alpha|=M, |\beta|=L} |a_{\alpha\beta}| = 1$. Under this assumption on a polynomial P , we define

$$T_\infty(f)(x) = \int_{|x-y| > 1} e^{iP(x,y)} K(x-y) f(y) dy.$$

To prove Theorem 1, we shall use the following result in the induction.

Proposition 1. *Let $\eta, \rho > 0$ and let the kernel K , the weight w and the exponents r, u be as in Theorem 1. Then, there exists a constant c depending only on η, ρ , the total degree of P , r and the dimension n such that if $C_1(w^u) \leq \eta$, $B_r, C_r \leq \rho$,*

$$\sup_{\lambda > 0} \lambda w(\{x \in \mathbf{R}^n : |T_\infty(f)(x)| > \lambda\}) \leq c \|f\|_{L^1_w}.$$

Let $A(f)(x) = \text{p. v. } K * f(x)$. We need the following result for the first step of induction for the proof of Theorem 1.

Proposition 2. *Let the kernel K , the weight w and the exponents r, u be as in Theorem 1. Let $\eta, \rho > 0$. There exists a constant c depending only on η, ρ, r and the dimension n such that if $C_1(w^u) \leq \eta$, $B_r, C_r \leq \rho$, then*

$$\sup_{\lambda > 0} \lambda w(\{x \in \mathbf{R}^n : |A(f)(x)| > \lambda\}) \leq c \|f\|_{L^1_w}.$$

Since A is bounded on L^2 (see [6, pp. 25–26]), if A is as in Proposition 2, we see that A is a singular integral operator considered in [6, p. 13]. Hence the conclusion of Proposition 2 will follow from [6, p. 15, Theorem 1.6].

We shall give the outlines of the proofs of Theorem 1 and Proposition 1 in Sections 2 and 4, respectively. Our proof of Proposition 1 is based on the techniques in Christ [3] for the proofs of the weak (1, 1) estimates for rough operators (see also Christ-Rubio [4] and Sato [7]). We also use the geometrical argument of Chanillo-Christ [1]. We have to prove a key estimate (Lemma 8 in §5) in the unweighted case in order to apply the method of Vargas [9] involving an interpolation with change of measure. To prove Lemma 8, we need a geometrical result for polynomials (Lemma 6 in §5). We shall prove Lemma 6 in §6 by using the results appearing in the proof of Chanillo-Christ [1, LEMMA 4.1]. Lemmas 6 and 8 have been proved in [8]. We include the proofs and some other parts of [8] almost verbatim for the sake of completeness.

2. OUTLINE OF PROOF OF THEOREM 1

To apply the induction argument of [5] we need some preparation. We may assume that $M \geq 1$ and $N \geq 1$; otherwise Theorem 1 reduces to Proposition 2.

We write a polynomial in (1.1) as follows:

$$P(x, y) = \sum_{j=0}^M \sum_{|\alpha|=j} x^\alpha Q_\alpha(y) =: \sum_{j=0}^M P_j(x, y).$$

We further decompose P_j as follows:

$$P_j(x, y) = \sum_{t=0}^N \sum_{\substack{|\alpha|=j \\ |\beta|=t}} a_{\alpha\beta} x^\alpha y^\beta =: \sum_{t=0}^N P_{jt}(x, y).$$

For $j = 1, 2, \dots, M$ and $k = 0, 1, \dots, N$, define

$$(2.1) \quad R_{jk}(x, y) = \sum_{s=0}^{j-1} P_s(x, y) + \sum_{t=0}^k P_{jt}(x, y).$$

Note that $R_{jN} = \sum_{s=0}^j P_s$ ($j = 1, 2, \dots, M$).

For $j = 1, 2, \dots, M$ and $k = 0, 1, \dots, N$, we consider the following propositions.

Proposition $A(j, k)$. *Let $\eta, \rho > 0$. There exists a constant c depending only on η, ρ, j, N, r and the dimension n such that if $C_1(w^u) \leq \eta$, $B_r, C_r \leq \rho$ and if R_{jk} is a polynomial of the form in (2.1), then*

$$\sup_{\lambda > 0} \lambda w(\{x \in \mathbf{R}^n : |T_{jk}(f)(x)| > \lambda\}) \leq c \|f\|_{L_w^1},$$

where

$$T_{jk}(f)(x) = \text{p. v.} \int_{\mathbf{R}^n} e^{iR_{jk}(x, y)} K(x - y) f(y) dy.$$

Then, Theorem 1 follows from Proposition $A(M, N)$. We shall prove it by double induction. We first note that $A(1, 0)$ follows from the boundedness of the operator A .

Next, we observe that if $M \geq 2$ and if $A(j, N)$ ($1 \leq j \leq M - 1$) is true, so is $A(j + 1, 0)$ since

$$R_{j+1,0}(x, y) = R_{jN}(x, y) + \sum_{|\alpha|=j+1} a_{\alpha 0} x^\alpha$$

and hence $|T_{j+1,0}(f)(x)| = |T_{jN}(f)(x)|$. Thus, to complete the induction starting from $A(1, 0)$ and arriving at $A(M, N)$, it is sufficient to prove $A(j, k + 1)$ assuming $A(j, k)$ ($0 \leq k < N, 1 \leq j \leq M$). To achieve this, put $R = R_{j, k+1}$, $R_0 = R_{jk}$, $T_{j, k+1} = S$. We note that

$$R(x, y) = R_0(x, y) + \sum_{\substack{|\alpha|=j \\ |\beta|=k+1}} a_{\alpha\beta} x^\alpha y^\beta.$$

We have only to deal with the case $C_{jk} = \max_{|\alpha|=j, |\beta|=k+1} |a_{\alpha\beta}| \neq 0$. Then, by a suitable dilation we may assume $C_{jk} = 1$. This can be seen as follows. We first note that, for $a > 0$,

$$S(f)(ax) = \text{p. v.} \int e^{iR(ax, ay)} K_a(x-y) f(ay) dy,$$

where $K_a(x) = a^n K(ax)$. Assume the boundedness of S for the case $C_{jk} = 1$. Then, choosing a to satisfy $a^{j+k+1} C_{jk} = 1$, and using the dilation invariance of both the class A_1 and the class of the kernels considered in Theorem 1, we get

$$\begin{aligned} w(\{x \in \mathbf{R}^n : |S(f)(x)| > \lambda\}) &= w_a(\{x \in \mathbf{R}^n : |S(f)(ax)| > \lambda\}) \\ &\leq c\lambda^{-1} \int |f(ax)| a^n w(ax) dx \\ &= c\lambda^{-1} \|f\|_{L_w^1}. \end{aligned}$$

We split the kernel K as $K = K_0 + K_\infty$, where $K_0(x) = K(x)$ if $|x| \leq 1$ and $K_\infty(x) = K(x)$ if $|x| > 1$. Assuming $C_{jk} = 1$, we consider the corresponding splitting $S = S_0 + S_\infty$:

$$S_0(f)(x) = \text{p. v.} \int e^{iR(x, y)} K_0(x-y) f(y) dy,$$

$$S_\infty(f)(x) = \int e^{iR(x, y)} K_\infty(x-y) f(y) dy.$$

In the next section, we shall prove

$$(2.2) \quad \sup_{\lambda > 0} \lambda w(\{x \in \mathbf{R}^n : |S_0(f)(x)| > \lambda\}) \leq c \|f\|_{L_w^1},$$

while by Proposition 1 we have

$$(2.3) \quad \sup_{\lambda > 0} \lambda w(\{x \in \mathbf{R}^n : |S_\infty(f)(x)| > \lambda\}) \leq c \|f\|_{L_w^1}.$$

Combining (2.2) and (2.3), we shall complete the proof of $A(j, k+1)$, which will finish the proof of Theorem 1.

3. ESTIMATE FOR S_0

In this section, we shall prove, under the assumption made in §2, that if $C_1(w) \leq \eta$, B_r , $C_r \leq \rho$ ($\eta, \rho > 0$), then S_0 is bounded from L_w^1 to $L_w^{1, \infty}$ with the operator norm bounded by a constant depending only on j, N, η, ρ, r and n ((2.2)).

First, we shall prove

$$(3.1) \quad w(\{x \in B(0, 1) : |S_0(f)(x)| > \lambda\}) \leq c\lambda^{-1} \int_{|y| < 2} |f(y)| w(y) dy,$$

where $B(x, r)$ denotes the closed ball with center x and radius $r > 0$.

Lemma 1. Let $w, w^u (1 \leq u < \infty) \in A_1$. Let T be an operator of the form:

$$T(f)(x) = \text{p. v.} \int_{\mathbf{R}^n} K(x, y) f(y) dy = \lim_{\epsilon \rightarrow 0} \int_{|x-y| > \epsilon} K(x, y) f(y) dy$$

for $f \in \mathfrak{S}(\mathbf{R}^n)$. Let $1/r + 1/u = 1$ and consider a non-negative function L on $\mathbf{R}^n \setminus \{0\}$ satisfying $J_r < \infty$, where

$$J_r = \sup_{R > 0} \left(R^{-n} \int_{R \leq |x| \leq 2R} (R^n L(x))^r dx \right)^{1/r}$$

for $r < \infty$ and J_∞ can be defined by the usual modification. Suppose the kernel K satisfies $|K(x, y)| \leq L(x - y)$. For $\epsilon > 0$, put

$$T_\epsilon(f)(x) = \text{p. v.} \int_{|x-y| < \epsilon} K(x, y) f(y) dy.$$

Suppose

$$\sup_{\lambda > 0} \lambda w(\{x \in \mathbf{R}^n : |T(f)(x)| > \lambda\}) \leq c_w \|f\|_{L_w^1}.$$

Then

$$\sup_{\lambda > 0} \lambda w(\{x \in \mathbf{R}^n : |T_\epsilon(f)(x)| > \lambda\}) \leq c(c_w + J_r C_1 (w^u)^{1/u}) \|f\|_{L_w^1}.$$

Proof. The proof is similar to that of LEMMA in [5, p. 187]. We shall prove

$$(3.2) \quad w(\{x \in B(h, \epsilon/4) : |T_\epsilon(f)(x)| > \lambda\}) \leq c(c_w + J_r C_1 (w^u)^{1/u}) \lambda^{-1} \int_{|y-h| < 6\epsilon/4} |f(y)| w(y) dy$$

uniformly in $h \in \mathbf{R}^n$. Integrating both sides of the inequality in (3.2) with respect to h , we get the conclusion of Lemma 1.

Split f into 3 pieces: $f = f_1 + f_2 + f_3$, where $f_i \in \mathfrak{S}(\mathbf{R}^n)$, $|f_i| \leq c|f|$ ($i = 1, 2, 3$); $\text{supp}(f_1) \subset B(h, \epsilon/2)$, $\text{supp}(f_2) \subset B(h, 11\epsilon/8) \setminus B(h, 3\epsilon/8)$, $\text{supp}(f_3) \subset \{x : |x-h| \geq 5\epsilon/4\}$. Note that if $|x-h| \leq \epsilon/4$, then $T_\epsilon(f_1)(x) = T(f_1)(x)$; since $|y-h| \leq \epsilon/2$ and $|x-h| \leq \epsilon/4$ imply $|x-y| < \epsilon$. So by the assumption on T , we have

$$w(\{x \in B(h, \epsilon/4) : |T_\epsilon(f_1)(x)| > \lambda\}) \leq c_w \lambda^{-1} \int_{|y-h| < 6\epsilon/4} |f(y)| w(y) dy.$$

Next, by Chebyshev's inequality, Hölder's inequality and the fact $w^u \in A_1$ we easily see that

$$w(\{x \in B(h, \epsilon/4) : |T_\epsilon(f_2)(x)| > \lambda\}) \leq c J_r C_1 (w^u)^{1/u} \lambda^{-1} \int_{|y-h| < 6\epsilon/4} |f(y)| w(y) dy.$$

Finally, if $|x - h| \leq \epsilon/4$ and $|y - h| \geq 5\epsilon/4$, then $|x - y| \geq \epsilon$, and so $T_\epsilon(f_3)(x) = 0$. Combining these results, we get (3.2). This completes the proof of Lemma 1.

Now we return to the proof of (3.1). If $|x| \leq 1$ and $|y| \leq 2$, then

$$\left| \exp(iR(x, y)) - \exp\left(i\left(R_0(x, y) + \sum_{\substack{|\alpha|=j \\ |\beta|=k+1}} a_{\alpha\beta} y^{\alpha+\beta}\right)\right)\right| \leq c|x - y|,$$

where c depends only on k, j and n .

Hence, if $|x| \leq 1$,

$$|S_0(f)(x)| \leq \left| U\left(\exp\left(i\sum_{\substack{|\alpha|=j \\ |\beta|=k+1}} a_{\alpha\beta} y^{\alpha+\beta}\right) f(y)\right)(x) \right| + cI(f)(x),$$

where

$$U(f)(x) = \text{p. v.} \int e^{iR_0(x, y)} K_0(x - y) f(y) dy, \quad I(f)(x) = \int_{|x-y|<1} |x - y| L(x - y) |f(y)| dy.$$

Note that $U(f)(x) = U(f\chi_{B(0,2)})(x)$, $I(f)(x) = I(f\chi_{B(0,2)})(x)$ if $|x| < 1$. By the induction hypothesis $A(j, k)$ and Lemma 1, we see that U is bounded from L_w^1 to $L_w^{1,\infty}$. On the other hand, it is easy to see that

$$\int_{|x-y|<1} |x - y| L(x - y) w(x) dx \leq \sum_{j \leq 0} 2^j \int_{2^{j-1} \leq |x-y| \leq 2^j} L(x - y) w(x) dx \leq cJ_r M_u(w)(y),$$

where $M_u(w) = M(w^u)^{1/u}$. Thus, by Chebyshev's inequality and the fact $w^u \in A_1$ we have

$$w(\{x \in B(0, 1) : I(f)(x) > \lambda\}) \leq cJ_r C_1 (w^u)^{1/u} \lambda^{-1} \int_{|y|<2} |f(y)| w(y) dy.$$

Combining these results, we get (3.1).

Similarly we can prove

$$(3.3) \quad w(\{x \in B(h, 1) : |S_0(f)(x)| > \lambda\}) \leq c\lambda^{-1} \int_{|y-h|<2} |f(y)| w(y) dy,$$

where c is independent of $h \in \mathbf{R}^n$. To see this, we first note that

$$S_0(f)(x + h) = \text{p. v.} \int e^{iR(x+h, y+h)} K_0(x - y) f(y + h) dy$$

and

$$R(x + h, y + h) = R_1(x, y, h) + \sum_{\substack{|\alpha|=j \\ |\beta|=k+1}} a_{\alpha\beta} x^\alpha y^\beta.$$

We can apply the induction hypothesis $A(j, k)$ to the operator

$$\text{p. v.} \int e^{iR_1(x, y, h)} K(x - y) f(y) dy$$

to get its boundedness from L_w^1 to $L_w^{1, \infty}$. Thus, by the same argument that leads to (3.1) we get

$$\begin{aligned} w(\{x \in B(h, 1) : |S_0(f)(x)| > \lambda\}) &= \tau_h w(\{x \in B(0, 1) : |S_0(f)(x + h)| > \lambda\}) \\ &\leq c\lambda^{-1} \int_{|y| < 2} |f(y + h)| w(y + h) dy \\ &\leq c\lambda^{-1} \int_{|y-h| < 2} |f(y)| w(y) dy, \end{aligned}$$

where $\tau_h w(x) = w(x + h)$, and we have used the translation invariance of the class A_1 . Integrating both sides of the inequality (3.3) with respect to h , we get (2.2).

4. OUTLINE OF PROOF OF PROPOSITION 1

Let $f \in \mathfrak{S}(\mathbb{R}^n)$. By Calderón-Zygmund decomposition at height $\lambda > 0$ we have a collection $\{Q\}$ of non-overlapping closed dyadic cubes and functions g, b such that

$$(4.1) \quad f = g + b;$$

$$(4.2) \quad \lambda \leq |Q|^{-1} \int_Q |f| \leq c\lambda;$$

$$(4.3) \quad v(\cup Q) \leq c_v \|f\|_{L_v^1} / \lambda \quad \text{for all } v \in A_1;$$

$$(4.4) \quad \|g\|_\infty \leq c\lambda, \quad \|g\|_{L_v^1} \leq c_v \|f\|_{L_v^1} \quad \text{for all } v \in A_1;$$

$$(4.5) \quad b = \sum_Q b_Q, \quad \text{supp}(b_Q) \subset Q, \quad \|b_Q\|_{L^1} \leq c\lambda|Q|.$$

Let a polynomial P be as in Proposition 1. We assume as we may that $M \geq 1$ as in the outline of the proof of Theorem 1 in §2. We write P as in (1.5). Then, let $q(y) = \sum_{|\beta| \leq L} c_\beta y^\beta$ be the coefficient of x_1^M . By a rotation of coordinates and a normalization, to prove Proposition 1 we may assume $\max_{|\beta|=L} |c_\beta| = 1$ (see [1, p. 151] and Sublemma 2 in §6).

We take a non-negative $\varphi \in C_0^\infty(\mathbf{R}^n)$ such that

$$\text{supp}(\varphi) \subset \{1/2 \leq |x| \leq 2\}, \quad \sum_{j=0}^{\infty} \varphi(2^{-j}x) = 1 \quad \text{if } |x| \geq 1.$$

Put $K_j(x, y) = \varphi(2^{-j}(x - y))K_\infty(x, y)$, where $K_\infty(x, y) = e^{iP(x, y)}K_\infty(x - y)$ ($K_\infty(x)$ is as in §2) and decompose $K_\infty(x, y)$ as $K_\infty(x, y) = \sum_{j=0}^{\infty} K_j(x, y)$.

Define

$$V_j(f)(x) = \int K_j(x, y) f(y) dy \quad \text{for } j \geq 0$$

and put

$$V(f)(x) = \sum_{j=1}^{\infty} V_j(f)(x).$$

Then $T_{\infty} = V_0 + V$. We have only to deal with V since we easily see that V_0 is bounded on L_w^1 ($w^u \in A_1$).

We set (see [3, 4])

$$B_i = \sum_{|Q|=2^{in}} b_Q \quad (i \geq 1), \quad B_0 = \sum_{|Q| \leq 1} b_Q.$$

Put $\mathcal{U} = \cup \tilde{Q}$, where \tilde{Q} denotes the cube with the same center as Q and with sidelength 100 times that of Q . (Throughout this note we consider the cubes with sides parallel to the coordinate axes.)

When $x \in \mathbf{R}^n \setminus \mathcal{U}$, we observe that

$$\begin{aligned} (4.6) \quad V(b)(x) &= V \left(\sum_{i \geq 0} B_i \right) (x) \\ &= \sum_{i \geq 0} \sum_{j \geq 1} \int K_j(x, y) B_i(y) dy = \sum_{i \geq 0} \sum_{j \geq i+1} \int K_j(x, y) B_i(y) dy \\ &= \sum_{s \geq 1} \sum_{j \geq s} \int K_j(x, y) B_{j-s}(y) dy = \sum_{s \geq 1} \sum_{j \geq s} V_j(B_{j-s})(x). \end{aligned}$$

To prove Proposition 1 we need the following results (Lemmas 2, 3 and 4).

Lemma 2. *Suppose $w \in A_1$. Let $\{L_j\}_{j \geq 1}$ be a family of kernels satisfying*

$$\text{supp}(L_j) \subset \{2^{j-6} \leq |x| \leq 2^{j+6}\}, \quad |L_j(x)| \leq c_1 |x|^{-n}, \quad |\nabla L_j(x)| \leq c_2 |x|^{-n-1}.$$

Let

$$G_j(f)(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} L_j(x-y) f(y) dy.$$

Put

$$E_{\lambda}^s = \left\{ x \in \mathbf{R}^n : \left| \sum_{j \geq s} G_j(B_{j-s})(x) \right| > \lambda \right\}.$$

Then there exists $\epsilon, \eta > 0$ such that, for any positive integer s ,

$$w \left(E_{\lambda c_{\eta} 2^{-\eta s}}^s \right) \leq c 2^{-\epsilon s} \lambda^{-1} \|f\|_{L_w^1},$$

where c_{η} is a positive constant satisfying $\sum_{s=1}^{\infty} c_{\eta} 2^{-\eta s/2} = 1$.

We shall prove this in §5.

Lemma 3. *Let L_j and G_j be as in Lemma 2. Then, for $j \geq 1$,*

$$\|G_j\|_2 \leq c2^{-j\epsilon} \quad \text{for some } \epsilon > 0,$$

where $\|G_j\|_2$ denotes the operator norm on L^2 .

This follows from Ricci-Stein [5]. See also [8] for an alternative proof.

Lemma 4. *If $w^u \in A_1$, then the operator V is bounded on L_w^2 .*

Proof. Let

$$N_j(x) = \varphi(2^{-j}x)K(x), \quad L_j(x) = N_j * \psi_{2^{-j+\delta j}}(x) \quad (\delta > 0),$$

where $\psi \in C^\infty(\mathbf{R}^n)$ which is supported in $\{|x| < 2^{-10}\}$ and satisfying $\int \psi = 1$. Then L_j satisfies all the conditions of Lemma 2 with $c_1 = c2^{n\delta j}$, $c_2 = c2^{(n+1)\delta j}$, and we find

$$(4.7) \quad \|L_j\|_{L^1} \leq cC_1,$$

$$(4.8) \quad \|L_j\|_{L^r} \leq cC_r 2^{-jn/u}.$$

Put

$$R_j(x) = N_j(x) - L_j(x) = \int (N_j(x) - N_j(x-y)) \psi_{2^{-j+\delta j}}(y) dy.$$

Then, it is easy to see that

$$(4.9) \quad \|R_j\|_{L^1} \leq c\omega_1(2^{-\delta j}) + c2^{-\delta j} \leq c\omega_r(2^{-\delta j}) + c2^{-\delta j},$$

$$(4.10) \quad \|R_j\|_{L^r} \leq c(\omega_r(2^{-\delta j}) + c2^{-\delta j})2^{-jn/u}.$$

Put

$$U_j(f)(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} L_j(x-y)f(y) dy, \quad W_j(f)(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} R_j(x-y)f(y) dy.$$

First we estimate U_j . By Hölder's inequality and (4.7), (4.8) we have

$$(4.11) \quad \|U_j(f)\|_{L_w^2}^2 \leq c \int \left(\int |L_j(x-y)|w(x) dx \right) |f(y)|^2 dy \leq c \int |f(y)|^2 M_u(w)(y) dy.$$

On the other hand, if δ is small enough, by Lemma 3

$$(4.12) \quad \|U_j(f)\|_{L^2}^2 \leq c2^{-\epsilon j} \|f\|_2^2 \quad \text{for some } \epsilon > 0.$$

Interpolating between the estimates (4.11) and (4.12), we get

$$\|U_j(f)\|_{L_{w^\theta}^2}^2 \leq c2^{-\epsilon(1-\theta)j} \int |f(y)|^2 M_u(w)(y)^\theta dy,$$

for $\theta \in (0, 1)$. Substituting $w^{1/\theta}$ for w , we have

$$(4.13) \quad \|U_j(f)\|_{L_w^2}^2 \leq c2^{-\epsilon(s-u)j/s} \int |f(y)|^2 M_s(w)(y) dy \quad \text{for all } s > u.$$

Next we estimate W_j . By Hölder's inequality and (4.9), (4.10)

$$(4.14) \quad \begin{aligned} \|W_j(f)\|_{L_w^2}^2 &\leq c(\omega_r(2^{-\delta j}) + c2^{-\delta j}) \int \left(\int |R_j(x-y)|w(x) dx \right) |f(y)|^2 dy \\ &\leq c(\omega_r(2^{-\delta j}) + c2^{-\delta j})^2 \int |f(y)|^2 M_u(w)(y) dy. \end{aligned}$$

By (4.13) and (4.14), for all $s > u$,

$$\|V(f)\|_{L_w^2} \leq c \sum_{j \geq 1} (\omega_r(2^{-\delta j}) + 2^{-\delta j} + 2^{-\epsilon(s-u)j/(2s)}) \|f\|_{L_{M_s(w)}^2} \leq c_s \|f\|_{L_{M_s(w)}^2}.$$

From this we get the conclusion of Lemma 4, since $w^s \in A_1$ for some $s > u$.

Using these results, we can prove Proposition 1. Let N_j and ψ be as in the proof of Lemma 4. For a positive integer s let

$$L_j^{(s)}(x) = N_j * \psi_{2^{-j+\delta s}}(x) \quad (\delta > 0).$$

Put

$$R_j^{(s)}(x) = N_j(x) - L_j^{(s)}(x) = \int (N_j(x) - N_j(x-y)) \psi_{2^{-j+\delta s}}(y) dy.$$

Then $L_j^{(s)}$ is supported in $\{2^{j-6} \leq |x| \leq 2^{j+6}\}$ and satisfies

$$|L_j^{(s)}(x)| \leq c2^{n\delta s}|x|^{-n}, \quad |\nabla L_j^{(s)}(x)| \leq c2^{(n+1)\delta s}|x|^{-n-1}.$$

Set

$$U_j^{(s)}(f)(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} L_j^{(s)}(x-y) f(y) dy, \quad W_j^{(s)}(f)(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} R_j^{(s)}(x-y) f(y) dy.$$

Put

$$F_\lambda^s = \left\{ x \in \mathbf{R}^n : \left| \sum_{j \geq s} U_j^{(s)}(B_{j-s})(x) \right| > \lambda \right\}.$$

Then, if $(n+1)\delta < \eta/2$ by Lemma 2

$$(4.15) \quad w \left(F_{c_\eta 2^{-\eta s/2} \lambda}^s \right) \leq c2^{-\epsilon s} \lambda^{-1} \|f\|_{L_w^1},$$

where ϵ , η and c_η are as in Lemma 2. Since $\sum_{s=1}^{\infty} c_\eta 2^{-\eta s/2} = 1$, we have

$$\left\{ x \in \mathbf{R}^n : \left| \sum_{s \geq 1} \sum_{j \geq s} U_j^{(s)}(B_{j-s})(x) \right| > \lambda \right\} \subset \bigcup_{s \geq 1} F_{c_\eta 2^{-\eta s/2} \lambda}^s.$$

Thus by (4.15)

$$(4.16) \quad w \left(\left\{ x \in \mathbf{R}^n : \left| \sum_{s \geq 1} \sum_{j \geq s} U_j^{(s)}(B_{j-s})(x) \right| > \lambda \right\} \right) \leq \sum_{s \geq 1} w \left(F_{c_\eta 2^{-\eta s/2} \lambda}^s \right) \leq c \lambda^{-1} \|f\|_{L_w^1}.$$

Since

$$\|R_j^{(s)}\|_{L^r} \leq c(\omega_r(2^{-\delta s}) + 2^{-\delta s})2^{-jn/u},$$

by Hölder's inequality and the condition that $w^u \in A_1$ we find

$$\left\| \sum_{j \geq s} W_j^{(s)}(B_{j-s}) \right\|_{L_w^1} \leq c(\omega_r(2^{-\delta s}) + 2^{-\delta s}) \|f\|_{L_w^1}.$$

Thus, by Chebyshev's inequality we have

$$(4.17) \quad w \left(\left\{ x \in \mathbf{R}^n : \left| \sum_{s \geq 1} \sum_{j \geq s} W_j^{(s)}(B_{j-s})(x) \right| > \lambda \right\} \right) \leq c \left(\sum_{s \geq 1} (\omega_r(2^{-\delta s}) + 2^{-\delta s}) \right) \lambda^{-1} \|f\|_{L_w^1}.$$

By (4.6), (4.16) and (4.17) we have

$$(4.18) \quad w(\{x \in \mathbf{R}^n \setminus \mathcal{U} : |V(b)(x)| > 2\lambda\}) \leq c \lambda^{-1} \|f\|_{L_w^1}.$$

By (4.3) we see that

$$(4.19) \quad w(\mathcal{U}) \leq c_w \lambda^{-1} \|f\|_{L_w^1}.$$

By Lemma 4 and (4.4)

$$(4.20) \quad w(\{x \in \mathbf{R}^n : |V(g)(x)| > \lambda\}) \leq c \lambda^{-1} \|f\|_{L_w^1}.$$

Combining (4.18), (4.19) and (4.20), we conclude the proof of Proposition 1.

5. PROOF OF LEMMA 2

In this section we shall prove Lemma 2 in §4. For $k, m \geq 1$, put

$$(5.1) \quad H_{km}(x, y) = \int e^{-iP(z, x) + iP(z, y)} \bar{L}_k(z - x) L_m(z - y) dz.$$

Then $G_k^* G_m(f)(x) = \int H_{km}(x, y) f(y) dy$, where G_k^* denotes the adjoint of G_k .

Lemma 5. *Let $k \geq m \geq 1$. Then, $H_{km}(x, y) = 0$ unless $|x - y| \leq 2^{k+7}$; and*

$$(1) \quad |H_{km}(x, y)| \leq c2^{-kn},$$

$$(2) \quad |H_{km}(x, y)| \leq c2^{-kn}2^{-m}|q(x) - q(y)|^{-1/M}.$$

Proof. We prove only the estimate of (2) since the other assertions immediately follow from the definition of H_{km} in (5.1). We first note that

$$(\partial/\partial z_1)^M (P(z, x) - P(z, y)) = M!(q(x) - q(y)).$$

Hence, from van der Corput's lemma it follows that

$$\left| \int_a^b e^{i(P(z, x) - P(z, y))} dz_1 \right| \leq c|q(x) - q(y)|^{-1/M},$$

for any a and b (see [1, p.152]).

Therefore by integration by parts in variable z_1 in the formula of (5.1) we get the conclusion.

For the rest of this note, we denote by $P(x)$ a real-valued polynomial on \mathbf{R}^n .

Definition 1. For a polynomial $P(x) = \sum_{|\alpha| \leq N} a_\alpha x^\alpha$ of degree N , define

$$\|P\| = \max_{|\alpha|=N} |a_\alpha|.$$

Definition 2. For a polynomial P and $\beta > 0$, let

$$\mathcal{R}(P, \beta) = \{x \in \mathbf{R}^n : |P(x)| \leq \beta\}.$$

Let $d(E, F)$ denote the distance between sets E and F . We now state a geometrical lemma for polynomials, which will be proved in §6.

Lemma 6. *Let k, m be integers such that $k \geq m$. Suppose $N \geq 1$. Then, for any polynomial P of degree N satisfying $\|P\| = 1$ and for any $\gamma > 0$, there exists a positive constant $C_{n, N, \gamma}$ depending only on n, N and γ such that*

$$|\{x \in B(a, 2^k) : d(x, \mathcal{R}(P, 2^{N^m})) \leq \gamma 2^m\}| \leq C_{n, N, \gamma} 2^{(n-1)k} 2^m$$

uniformly in $a \in \mathbf{R}^n$.

Let $\lambda > 0$ and let $\{\mathcal{B}_j\}_{j \geq 0}$ be a family of measurable functions such that

$$(5.2) \quad \int_Q |\mathcal{B}_j| \leq \lambda |Q|$$

for all cubes Q in \mathbf{R}^n with sidelength $\ell(Q) = 2^j$.

Then we have the following.

Lemma 7. *Let the kernels H_{ji} be as in Lemma 5. Then, we can find a constant c such that*

$$\sum_{i=s}^j \sup_{x \in \mathbf{R}^n} \left| \int \mathcal{B}_{i-s}(y) H_{ji}(x, y) dy \right| \leq c\lambda 2^{-s}$$

for all integers j and s such that $0 < s \leq j$.

Definition 3. For $m \in \mathbf{Z}$ (the set of all integers), let \mathcal{D}_m be the family of all closed dyadic cubes Q with sidelength $\ell(Q) = 2^m$.

Proof of Lemma 7. Fix $x \in \mathbf{R}^n$. Let

$$\mathcal{F} = \{Q \in \mathcal{D}_{i-s} : Q \cap B(x, 2^{j+2}) \neq \emptyset\} \quad (0 < s \leq i \leq j).$$

Then clearly $\sum_{Q \in \mathcal{F}} |Q| \leq c2^{jn}$.

Decompose $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$, where

$$\mathcal{F}_0 = \left\{ Q \in \mathcal{F} : Q \cap \mathcal{R}(q(\cdot) - q(x), 2^{L(i-s)}) \neq \emptyset \right\}$$

and $\mathcal{F}_1 = \mathcal{F} \setminus \mathcal{F}_0$. Then by Lemma 6 we have

$$(5.3) \quad \sum_{Q \in \mathcal{F}_0} |Q| \leq c2^{(n-1)j} 2^{i-s}.$$

By Lemma 5 (1), (5.2) and (5.3), we see that

$$(5.4) \quad \begin{aligned} \sum_{Q \in \mathcal{F}_0} \int_Q |\mathcal{B}_{i-s}(y) H_{ji}(x, y)| dy &\leq c2^{-jn} \sum_{Q \in \mathcal{F}_0} \int_Q |\mathcal{B}_{i-s}(y)| dy \\ &\leq c2^{-jn} \lambda \sum_{Q \in \mathcal{F}_0} |Q| \leq c2^{-jn} \lambda 2^{(n-1)j} 2^{i-s} = c\lambda 2^{i-j-s}. \end{aligned}$$

Next, by Lemma 5 (2), (5.2) and the estimate $\sum_{Q \in \mathcal{F}_1} |Q| \leq c2^{jn}$, we have

$$(5.5) \quad \begin{aligned} \sum_{Q \in \mathcal{F}_1} \int_Q |\mathcal{B}_{i-s}(y) H_{ji}(x, y)| dy &\leq c2^{-jn} 2^{-i} 2^{-L(i-s)/M} \sum_{Q \in \mathcal{F}_1} \int_Q |\mathcal{B}_{i-s}(y)| dy \\ &\leq c2^{-jn} 2^{-i} 2^{-L(i-s)/M} \lambda \sum_{Q \in \mathcal{F}_1} |Q| \leq c\lambda 2^{-i} 2^{-L(i-s)/M}. \end{aligned}$$

From (5.4) and (5.5) it follows that

$$\begin{aligned} \int |\mathcal{B}_{i-s}(y) H_{ji}(x, y)| dy &= \sum_{Q \in \mathcal{F}} \int_Q |\mathcal{B}_{i-s}(y) H_{ji}(x, y)| dy \\ &= \sum_{\nu=0}^1 \sum_{Q \in \mathcal{F}_\nu} \int_Q |\mathcal{B}_{i-s}(y) H_{ji}(x, y)| dy \leq c\lambda \left(2^{i-j-s} + 2^{-i} 2^{-L(i-s)/M} \right). \end{aligned}$$

Thus we see that

$$\sum_{i=s}^j \sup_{x \in \mathbf{R}^n} \int |\mathcal{B}_{i-s}(y) H_{ji}(x, y)| dy \leq c\lambda \sum_{i=s}^j \left(2^{i-j-s} + 2^{-i} 2^{-L(i-s)/M} \right) \leq c\lambda 2^{-s}.$$

This completes the proof of Lemma 7.

By Lemma 7 we readily get the following.

Lemma 8. *Let $\{\mathcal{B}_j\}_{j \geq 0}$ be as in Lemma 7. Suppose $\sum_{j \geq 0} \|\mathcal{B}_j\|_{L^1} < \infty$. Let G_j be as in Lemma 2. Then, for any positive integer s , we have*

$$\left\| \sum_{j \geq s} G_j(\mathcal{B}_{j-s}) \right\|_{L^2}^2 \leq c\lambda 2^{-s} \sum_{j \geq 0} \|\mathcal{B}_j\|_{L^1}.$$

Proof. Let $\langle \cdot, \cdot \rangle$ denote the inner product in L^2 . Using Lemma 7, we see that

$$\begin{aligned} \left\| \sum_{j \geq s} G_j(\mathcal{B}_{j-s}) \right\|_{L^2}^2 &\leq 2 \sum_{j \geq s} \sum_{i=s}^j |\langle G_j(\mathcal{B}_{j-s}), G_i(\mathcal{B}_{i-s}) \rangle| \leq 2 \sum_{j \geq s} \sum_{i=s}^j |\langle \mathcal{B}_{j-s}, G_j^* G_i(\mathcal{B}_{i-s}) \rangle| \\ &\leq 2 \sum_{j \geq s} \sum_{i=s}^j \|\mathcal{B}_{j-s}\|_{L^1} \|G_j^* G_i(\mathcal{B}_{i-s})\|_{L^\infty} \leq c\lambda 2^{-s} \sum_{j \geq s} \|\mathcal{B}_{j-s}\|_{L^1}. \end{aligned}$$

This completes the proof of Lemma 8.

Definition 4. For each $j \geq 0$, let \mathcal{G}_j be a family of non-overlapping closed dyadic cubes Q such that $\ell(Q) \leq 2^j$. We suppose that if $Q \in \mathcal{G}_j$, $R \in \mathcal{G}_k$ and $j \neq k$, then Q and R are non-overlapping and that $\sum_{j \geq 0} \sum_{Q \in \mathcal{G}_j} |Q| < \infty$. Put $\mathcal{G} = \cup_{j \geq 0} \mathcal{G}_j$.

Let $\lambda > 0$. To each $Q \in \mathcal{G}$ we associate $f_Q \in L^1$ such that

$$\int |f_Q| \leq \lambda |Q|, \quad \text{supp}(f_Q) \subset Q.$$

We define $\mathcal{A}_i = \sum_{Q \in \mathcal{G}_i} f_Q$.

Lemma 9. *Let G_j be as in Lemma 2 and let v be a locally integrable positive function. Then for a positive integer s we have*

$$\left\| \sum_{j \geq s} G_j(\mathcal{A}_{j-s}) \right\|_{L^1_+} \leq c\lambda \sum_{Q \in \mathcal{G}} |Q| \inf_Q M(v),$$

where $\inf_Q M(v) = \inf_{x \in Q} M(v)(x)$.

Proof. We easily see that

$$\begin{aligned} \left\| \sum_{j \geq s} G_j(\mathcal{A}_{j-s}) \right\|_{L^1_+} &\leq \sum_j \int |\mathcal{A}_{j-s}(y)| \left(\int |L_j(x-y)| v(x) dx \right) dy \\ &\leq \sum_j \sum_{Q \in \mathcal{G}_{j-s}} \int |f_Q(y)| \inf_{z \in Q} M(v)(z) dy \leq c \sum_{Q \in \mathcal{G}} \lambda |Q| \inf_Q M(v). \end{aligned}$$

We prove Lemma 2 by the estimates of Lemma 8 and Lemma 9. We slightly modify the interpolation argument of [9].

Lemma 10. *Let \mathcal{F} denote the family of dyadic cubes arising from the Calderón-Zygmund decomposition in §4. Define a set E_λ^s as in Lemma 2. Then, for all $t > 0$, we have*

$$(5.6) \quad \int_{E_\lambda^s} \min(v(x), t) dx \leq c \sum_{Q \in \mathcal{F}} |Q| \min \left(t2^{-s}, \inf_Q M(v) \right),$$

where s is a positive integer and v is a locally integrable positive function.

Proof. For $t > 0$, set $\mathcal{F}_t = \{Q \in \mathcal{F} : \inf_Q M(v) < t2^{-s}\}$ and $\mathcal{F}_t^* = \mathcal{F} \setminus \mathcal{F}_t$. Put

$$B_j' = \sum_{\substack{\ell(Q)=2^j \\ Q \in \mathcal{F}_t}} b_Q, \quad B_j'' = \sum_{\substack{\ell(Q)=2^j \\ Q \in \mathcal{F}_t^*}} b_Q \quad (j \geq 1); \quad B_0' = \sum_{\substack{|Q| \leq 1 \\ Q \in \mathcal{F}_t}} b_Q, \quad B_0'' = \sum_{\substack{|Q| \leq 1 \\ Q \in \mathcal{F}_t^*}} b_Q.$$

Define

$$E_\lambda' = \left\{ \left| \sum_{j \geq s} G_j(B_{j-s}') \right| > \lambda \right\}, \quad E_\lambda'' = \left\{ \left| \sum_{j \geq s} G_j(B_{j-s}'') \right| > \lambda \right\}.$$

Then we find $E_\lambda^s \subset E_{\lambda/2}' \cup E_{\lambda/2}''$, since $B_j = B_j' + B_j''$, and so

$$\begin{aligned} \int_{E_\lambda^s} \min(v(x), t) dx &\leq \int_{E_{\lambda/2}'} \min(v(x), t) dx + \int_{E_{\lambda/2}''} \min(v(x), t) dx \\ &\leq \int_{E_{\lambda/2}'} v(x) dx + \int_{E_{\lambda/2}''} t dx =: I + II. \end{aligned}$$

By Lemma 9 with $\mathcal{A}_j = cB_j'$, we get

$$I \leq c \sum_{Q \in \mathcal{F}_t} |Q| \inf_Q M(v) = c \sum_{Q \in \mathcal{F}_t} |Q| \min \left(t2^{-s}, \inf_Q M(v) \right).$$

By Lemma 8 with $\mathcal{B}_j = cB_j''$, we have

$$II \leq ct2^{-s} \sum_{Q \in \mathcal{F}_t^*} |Q| = c \sum_{Q \in \mathcal{F}_t^*} |Q| \min \left(t2^{-s}, \inf_Q M(v) \right).$$

Combining the estimates for I and II , we conclude the proof of Lemma 10.

Now we finish the proof of Lemma 2. Multiplying both sides of the inequality (5.6) by $t^{-\theta}$ ($\theta \in (0, 1)$), then integrating them on $(0, \infty)$ with respect to the measure dt/t , and using

$$\int_0^\infty \min(A, t) t^{-\theta} \frac{dt}{t} = c_\theta A^{1-\theta} \quad (A > 0),$$

we get

$$(5.7) \quad \begin{aligned} \int_{E_\lambda^s} v(x)^{1-\theta} dx &\leq c \sum_{Q \in \mathcal{F}} |Q| 2^{-\theta s} \inf_Q M(v)^{1-\theta} \\ &\leq c\lambda^{-1} 2^{-\theta s} \sum_{Q \in \mathcal{F}} \inf_Q M(v)^{1-\theta} \int_Q |f(x)| dx \leq c\lambda^{-1} 2^{-\theta s} \int |f(x)| M(v)(x)^{1-\theta} dx, \end{aligned}$$

where the second inequality follows from (4.2).

If $w \in A_1$, then $w^{1+\delta} \in A_1$ for some $\delta > 0$; so substituting $w^{1+\delta}$ for v and taking θ such that $1 - \theta = (1 + \delta)^{-1}$ in (5.7), we get

$$(5.8) \quad w(E_\lambda^s) \leq c\lambda^{-1}2^{-s\delta/(1+\delta)}\|f\|_{L_w^1}.$$

Checking the constants appearing in the proof of (5.8) and replacing L_j by $c2^{ns}L_j$, we get the desired estimate of Lemma 2.

6. PROOF OF LEMMA 6

Our proof is an application of the method for the proof of [1, LEMMA 4.1]. We use some tools and results given in [1].

Definition 5. Suppose $n \geq 2$. Let

$$S_m = \{Q_m + (0, 0, \dots, 0, j) : j \in \mathbf{Z}\},$$

where $m = (m_1, m_2, \dots, m_{n-1}) \in \mathbf{Z}^{n-1}$ and $Q_m = [0, 1]^n + (m_1, m_2, \dots, m_{n-1}, 0)$. We call S_m a strip.

Definition 6. Suppose $n \geq 2$. For $m \in \mathbf{Z}^{n-1}$, we define

$$I_m = \{Q_m + (0, 0, \dots, 0, j) : j_1 < j < j_2\},$$

where $j_1, j_2 \in \mathbf{Z} \cup \{-\infty, \infty\}$ and Q_m is as in Definition 5. We call I_m an interval.

Definition 7. For a set $E \subset \mathbf{R}^n$, we put

$$\mathcal{N}(E) = \{x \in \mathbf{R}^n : d(x, E) \leq 1\}.$$

Let P be a polynomial of degree N as in Lemma 6. We consider $\mathcal{R}(P, \beta)$ for $\beta > 0$ (see Definition 2).

Lemma 11. *Suppose that $n \geq 2$ and $N \geq 1$. There exists a positive integer $C_{n,N}$ depending only on n and N such that for $i = 1, 2, \dots, C_{n,N}$ we can find $U_i \in O(n)$ (the orthogonal group) and families of cubes $J_{m,i} \subset S_m$ ($m \in \mathbf{Z}^{n-1}$) so that*

$$(1) \quad \mathcal{N}(\mathcal{R}(P, \beta)) \subset \bigcup_{i=1}^{C_{n,N}} U_i(\mathcal{L}_i), \text{ where}$$

$$\mathcal{L}_i = \bigcup \left\{ Q : Q \in \bigcup_{m \in \mathbf{Z}^{n-1}} J_{m,i} \right\};$$

$$(2) \quad \text{card}(J_{m,i}) \leq c \text{ for some constant } c \text{ depending only on } n, N \text{ and } \beta.$$

Remark 1. If Lemma 11 holds, then we have, for any $\gamma > 0$,

$$\{x : d(x, \mathcal{R}(P, \beta)) \leq \gamma\} \subset \bigcup_{i=1}^{C_{n,N,\gamma}} U_i(\mathcal{L}_i)$$

for some positive integer $C_{n,N,\gamma}$ depending only on n, N and γ , where U_i and \mathcal{L}_i are as in Lemma 11. This can be proved by considering a finite number of polynomials which are defined by translating P and by applying Lemma 11 to each of them. (See [1, p. 149].)

To prove Lemma 11, we need the following results given in [1].

Sublemma 1. *Suppose $n \geq 2$. For any positive integer N , there exists a positive integer $C_{n,N}$ depending only on n and N such that for any strip S , any polynomial P of degree N and any $\gamma > 0$*

$$\{Q \in S : Q \cap \mathcal{R}(P, \gamma) \neq \emptyset\}$$

is a union of at most $C_{n,N}$ intervals. (See LEMMA 4.2 of [1].)

Sublemma 2. *Suppose $n \geq 2$. For any positive integer N , there exist positive constants $A_{n,N}$ and $B_{n,N}$ depending only on n and N such that*

$$A_{n,N}\|P\| \leq \|P \circ \Xi\| \leq B_{n,N}\|P\|$$

for all polynomial P of degree N and all $\Xi \in O(n)$, where $P \circ \Xi(x) = P(\Xi x)$.

Sublemma 3. *Suppose $n \geq 2$. For any positive integer N , there exists a positive constant $C_{n,N}$ depending only on n and N such that for any polynomial P of degree N we can find $\Theta \in O(n)$ so that*

$$\min_{1 \leq j \leq n} \|D_j(P \circ \Theta)\| \geq C_{n,N}\|P \circ \Theta\|,$$

where $D_j = \partial/\partial x_j$.

Now we prove Lemma 11. We use induction on the polynomial degree N . Let $A(N)$ be the assertion of Lemma 11 for polynomials of degree N .

Proof of $A(1)$. Let $P(x) = \sum_{i=1}^n a_i x_i + b$. First, we consider the case $|a_n| = 1$. Now we show that if I is an interval such that each cube of I intersects $\mathcal{R}(P, \beta)$, then $\text{card}(I) \leq c$ for some c depending only on n and β . Let $y \in Q \in I$ satisfy $|P(y)| \leq \beta$. We note that

$$P(y + de_n) - P(y) = da_n \quad \text{for } d \in \mathbf{R},$$

where e_j is the element of \mathbf{R}^n whose j th coordinate is 1 and whose other coordinates are all 0. Therefore, if $y + de_n \in Q' \in I$, we see that

$$\inf_{z \in Q'} |P(z)| \geq |P(y + de_n)| - \sum_{i=1}^n |a_i| \geq |da_n| - \beta - \sum_{i=1}^n |a_i| \geq |d| - \beta - n.$$

This easily implies that $\text{card}(I) \leq c$.

By this and Sublemma 1, there exists a constant c depending only on n and β such that

$$\text{card}(\{Q \in S : Q \cap \mathcal{R}(P, \beta) \neq \emptyset\}) \leq c$$

for all strips S .

Therefore, if we put

$$J_m = \{Q \in S_m : d(Q, \mathcal{R}(P, \beta)) \leq 1\},$$

then $\text{card}(J_m) \leq c$ for some c depending only on n and β ; and $\mathcal{N}(\mathcal{R}(P, \beta)) \subset \mathcal{L}$, where

$$\mathcal{L} = \cup \left\{ Q : Q \in \bigcup_{m \in \mathbf{Z}^{n-1}} J_m \right\}.$$

Next, we consider any polynomial P of degree 1 such that $\|P\| = 1$. Then if $P_1(x) = P(Ux)$ for a suitable $U \in O(n)$, we have $D_n P_1 = 1$. Hence, by what we have already proved we get $\mathcal{N}(\mathcal{R}(P_1, \beta)) \subset \mathcal{L}$. It follows that $\mathcal{N}(\mathcal{R}(P, \beta)) \subset U(\mathcal{L})$ since $\mathcal{N}(\mathcal{R}(P \circ U, \beta)) = U^{-1}\mathcal{N}(\mathcal{R}(P, \beta))$. This completes the proof of $A(1)$.

Now we assume $A(N-1)$ ($N \geq 2$) and prove $A(N)$. For a polynomial P of degree N such that $\|P\| = 1$, we take $\Theta \in O(n)$ as in Sublemma 3. Put

$$E_0 = \mathcal{R}(P \circ \Theta, \beta) \cap \left(\bigcup_{j=1}^n \mathcal{R}(D_j(P \circ \Theta), \beta) \right);$$

and for $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_n) \in \{-1, 1\}^n$ put

$$E_\kappa = \{x \in \mathcal{R}(P \circ \Theta, \beta) : \kappa_j D_j(P \circ \Theta)(x) > \beta \text{ for } j = 1, 2, \dots, n\}.$$

Then

$$\mathcal{R}(P \circ \Theta, \beta) = E_0 \cup \left(\bigcup_{\kappa \in \{-1, 1\}^n} E_\kappa \right)$$

and so

$$(6.1) \quad \mathcal{N}(\mathcal{R}(P \circ \Theta, \beta)) = \mathcal{N}(E_0) \cup \left(\bigcup_{\kappa \in \{-1, 1\}^n} \mathcal{N}(E_\kappa) \right).$$

We separately treat the $2^n + 1$ sets of the right hand side.

First, clearly

$$(6.2) \quad \mathcal{N}(E_0) \subset \bigcup_{j=1}^n \mathcal{N}(\mathcal{R}(D_j(P \circ \Theta), \beta)).$$

Since $C_j = \|D_j(P \circ \Theta)\| \sim 1$ (this means that $c^{-1} \leq \|D_j(P \circ \Theta)\| \leq c$ for some $c > 1$ depending only on n and N) and $\mathcal{R}(D_j(P \circ \Theta), \beta) = \mathcal{R}(C_j^{-1} D_j(P \circ \Theta), C_j^{-1} \beta)$, we can apply the induction hypothesis $A(N-1)$ to the right hand side of (6.2).

Next, we fix κ and consider $\mathcal{N}(E_\kappa)$. Take $O_\kappa \in O(n)$ such that $O_\kappa(e_n) = n^{-1/2}\kappa$. Define

$$\mathcal{D}_0^* = \mathcal{D}_0 \setminus \left\{ Q \in \mathcal{D}_0 : \left(\bigcup_{j=1}^n \mathcal{R}((D_j(P \circ \Theta)) \circ O_\kappa, \beta) \right) \cap Q \neq \emptyset \right\}.$$

Since $\|(D_j(P \circ \Theta)) \circ O_\kappa\| \sim 1$ by Sublemmas 2 and 3, we can apply the hypothesis $A(N-1)$ along with Remark 1 to

$$G = \cup \left\{ Q \in \mathcal{D}_0 : \left(\bigcup_{j=1}^n \mathcal{R}((D_j(P \circ \Theta)) \circ O_\kappa, \beta) \right) \cap Q \neq \emptyset \right\}$$

to get

$$(6.3) \quad \mathcal{N}(G) \subset \cup_i U_i'(\mathcal{L}'_i)$$

for some $U'_i \in O(n)$ and for some \mathcal{L}'_i such that

$$\mathcal{L}'_i = \cup \left\{ Q : Q \in \bigcup_{m \in \mathbf{Z}^{n-1}} J'_{m,i} \right\}$$

for some $J'_{m,i} (\subset S_m)$ satisfying $\text{card}(J'_{m,i}) \leq c$.

Therefore we have only to consider $O_{\kappa}^{-1}(E_{\kappa}) \cap (\cup \mathcal{D}_0^*)$. First, we note that if $O_{\kappa}^{-1}(E_{\kappa})$ intersects $Q, Q \in \mathcal{D}_0^*$, then

$$(6.4) \quad \min_{1 \leq j \leq n} \kappa_j D_j(P \circ \Theta)(O_{\kappa}y) > \beta \quad \text{for all } y \in Q.$$

This can be seen as follows. Suppose that there are j_0 and $y_0 \in Q$ such that $\kappa_{j_0} D_{j_0}(P \circ \Theta)(O_{\kappa}y_0) \leq \beta$. Then, since we have $\kappa_{j_0} D_{j_0}(P \circ \Theta)(O_{\kappa}x) > \beta$ for some $x \in Q$, by the intermediate value theorem we can find $z \in Q$ such that $|D_{j_0}(P \circ \Theta)(O_{\kappa}z)| \leq \beta$. This contradicts the fact that $Q \in \mathcal{D}_0^*$.

By (6.4) we have

$$(6.5) \quad O_{\kappa}^{-1}(E_{\kappa}) \cap (\cup \mathcal{D}_0^*) \subset \cup \left\{ Q \in \mathcal{D}_0 : \min_{1 \leq j \leq n} \kappa_j D_j(P \circ \Theta)(O_{\kappa}y) > \beta \quad \text{for all } y \in Q \right. \\ \left. \text{and } \mathcal{R}(P \circ \Theta \circ O_{\kappa}, \beta) \cap Q \neq \emptyset \right\}.$$

For a strip S , put

$$\mathcal{E} = \left\{ Q \in S : \min_{1 \leq j \leq n} \kappa_j D_j(P \circ \Theta)(O_{\kappa}y) > \beta \quad \text{for all } y \in Q \right. \\ \left. \text{and } \mathcal{R}(P \circ \Theta \circ O_{\kappa}, \beta) \cap Q \neq \emptyset \right\}.$$

We shall show $\text{card}(\mathcal{E}) \leq C_{n,N}$.

We first see that \mathcal{E} is a union of at most $C_{n,N}$ intervals. Put

$$\mathcal{E}' = \left\{ Q \in S : \min_{1 \leq j \leq n} |D_j(P \circ \Theta)(O_{\kappa}y)| > \beta \quad \text{for all } y \in Q \right. \\ \left. \text{and } \mathcal{R}(P \circ \Theta \circ O_{\kappa}, \beta) \cap Q \neq \emptyset \right\}.$$

Then

$$\mathcal{E}' = \left(\bigcap_{j=1}^n (S \setminus \{Q \in S : \mathcal{R}((D_j(P \circ \Theta)) \circ O_{\kappa}, \beta) \cap Q \neq \emptyset\}) \right) \\ \cap \{Q \in S : \mathcal{R}(P \circ \Theta \circ O_{\kappa}, \beta) \cap Q \neq \emptyset\}.$$

We observe that the complement of a finite union of intervals in a strip S is also a finite union of intervals, and the intersection of finite unions of intervals is also a finite union

of intervals. Hence, by Sublemma 1 we see that \mathcal{E}' is a union of at most $C_{n,N}$ intervals:
 $\mathcal{E}' = \cup_i J_i$.

Take any J_i . Then by the intermediate value theorem we have either

$$\min_{1 \leq j \leq n} \kappa_j D_j(P \circ \Theta)(O_\kappa y) > \beta \quad \text{for all } y \in \cup \{Q : Q \in J_i\}$$

or

$$\min_{1 \leq j \leq n} \kappa_j D_j(P \circ \Theta)(O_\kappa y) < -\beta \quad \text{for all } y \in \cup \{Q : Q \in J_i\}.$$

Thus \mathcal{E} is a union of a subfamily $\{I_i\}$ of $\{J_i\}$: $\mathcal{E} = \cup_i I_i$.

Let I be any interval in $\{I_i\}$. We need the following (see [1, p. 151]).

Sublemma 4. *There exists a constant c_n depending only on n such that if $x, y \in I$ and $y_n - x_n \geq c_n$, then*

$$y - x = \sum_{i=1}^n \lambda_i O_\kappa^{-1} e_i$$

for some $\lambda_i \in \mathbf{R}$ such that $\kappa_i \lambda_i \geq 3$.

Proof. We see that

$$\begin{aligned} O_\kappa(y - x) &= \sum_{i=1}^n (y_i - x_i) O_\kappa e_i = \sum_{i=1}^{n-1} (y_i - x_i) O_\kappa e_i + (y_n - x_n) n^{-1/2} \kappa e_n \\ &= \sum_{i=1}^n \left(n^{-1/2} (y_n - x_n) \kappa_i + b_i \right) e_i \end{aligned}$$

for some $b_i \in \mathbf{R}$ such that $|b_i| \leq c$, which is feasible since $|y_i - x_i| \leq 1$ for $i = 1, 2, \dots, n-1$. This readily implies the conclusion.

Put $Y = P \circ \Theta \circ O_\kappa$. Then $\nabla Y(x) = O_\kappa^{-1}(\nabla(P \circ \Theta)(O_\kappa x))$; so, if $x, y \in I$ and $y_n - x_n \geq c_n$, by Sublemma 4 we have

$$\begin{aligned} Y(y) - Y(x) &= \int_0^1 \langle y - x, (\nabla Y)(x + t(y - x)) \rangle dt \\ &= \int_0^1 \sum_{i=1}^n \lambda_i \langle O_\kappa^{-1} e_i, O_\kappa^{-1} (\nabla(P \circ \Theta)(O_\kappa(x + t(y - x)))) \rangle dt \\ &= \int_0^1 \sum_{i=1}^n \lambda_i D_i(P \circ \Theta)(O_\kappa(x + t(y - x))) dt \geq \sum_{i=1}^n \lambda_i \kappa_i \beta \geq 3n\beta > 3\beta, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbf{R}^n . Since $\mathcal{R}(Y, \beta) \cap Q \neq \emptyset$ for all $Q \in I$, we can conclude that $\text{card}(I) \leq c_n + 3$.

Combining the above results, we have $\text{card}(\mathcal{E}) \leq C_{n,N}$ as claimed. From this and (6.5) we easily see that

$$(6.6) \quad \mathcal{N}(O_\kappa^{-1}(E_\kappa) \cap (\cup \mathcal{D}_0^*)) \subset \mathcal{L},$$

where $\mathcal{L} = \cup \{Q : Q \in \cup_{m \in \mathbf{Z}^{n-1}} J_m\}$ for some $J_m \subset S_m$ with $\text{card}(J_m) \leq C_{n,N}$.

By (6.3) and (6.6) we have

$$\mathcal{N}(O_\kappa^{-1}(E_\kappa)) \subset \mathcal{N}(G) \cup \mathcal{N}(O_\kappa^{-1}(E_\kappa) \cap (\cup \mathcal{D}_0^*)) \subset (\cup_i U_i'(\mathcal{L}'_i)) \cup \mathcal{L};$$

and so, observing $\mathcal{N}(O_\kappa^{-1}(E_\kappa)) = O_\kappa^{-1}\mathcal{N}(E_\kappa)$,

$$(6.7) \quad \mathcal{N}(E_\kappa) \subset (\cup_i O_\kappa U_i'(\mathcal{L}'_i)) \cup O_\kappa(\mathcal{L}).$$

Since $\mathcal{N}(\mathcal{R}(P \circ \Theta, \beta)) = \Theta^{-1}\mathcal{N}(\mathcal{R}(P, \beta))$, by (6.1), (6.2) with $A(N-1)$ and (6.7) we get $A(N)$. This completes the proof of Lemma 11.

Proof of Lemma 6. We see that $\mathcal{R}(P, 2^{Nm}) = 2^m \mathcal{R}(\tilde{P}, 1)$, where

$$\tilde{P}(x) = 2^{-Nm} P(2^m x).$$

Note that $\|\tilde{P}\| = 1$. (See [1, p. 151].) This observation enables us to assume $m = 0$ to prove Lemma 6. Clearly, we may also assume $\gamma = 1$.

Thus it is sufficient to show, for $k \geq 0$,

$$(6.8) \quad \left| \{x \in B(a, 2^k) : d(x, \mathcal{R}(P, 1)) \leq 1\} \right| \leq C_{n,N} 2^{(n-1)k}$$

uniformly in $a \in \mathbf{R}^n$.

If $n = 1$, (6.8) easily follows from Chanillo-Christ [1, LEMMA 3.2] (see also [2]). Suppose $n \geq 2$. Then, (6.8) follows from Lemma 11 with $\beta = 1$ and the obvious estimate:

$$|B(a, 2^k) \cap U_i(\mathcal{L}_i)| \leq c 2^{(n-1)k},$$

where $U_i(\mathcal{L}_i)$ is as in Lemma 11. This completes the proof of Lemma 6.

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