

4次元非輪状等質空間についての注意(英文)

著者	齊藤 和夫
雑誌名	金沢大学教育学部紀要.自然科学編
巻	45
ページ	39-44
発行年	1996-02-28
URL	http://hdl.handle.net/2297/505

A remark on 4 dimensional compact aspherical homogeneous space

Kazuo SAITO

Introduction

In this note we shall consider the homeomorphism type of the compact aspherical homogeneous space of dimension 4. Let G be a connected, simply connected Lie group and H a closed subgroup of G such that G/H is a compact aspherical homogeneous space of dimension 4 and G acts on G/H irreducibly, i. e. no proper subgroup of G does not act transitively. When G is a solvable group, the homeomorphism type of the solvmanifold is uniquely determined by its fundamental group ([5], [6]). It is proved that if M and N are both compact connected negatively curved Riemannian manifolds with isomorphic fundamental groups, then M and N are homeomorphic provided $\dim M \neq 3$ and 4 ([2]). Recently it is shown that the homeomorphism type of the compact homogeneous space is determined up to a finite covering by its fundamental group ([7]). We shall prove the following

THEOREM. *Let G/H be a 4 dimensional compact homogeneous space, where G is a connected, simply connected Lie group and H is a closed subgroup of G . If G acts on G/H irreducibly, the fundamental group of G/H is solvable.*

From the result that compact aspherical homogeneous spaces with isomorphic solvable fundamental groups are homeomorphic ([4]), we have the following

COROLLARY. *Let G_i be a connected, simply connected Lie group and H_i its closed subgroup such that G_i/H_i is a closed 4 dimensional aspherical manifold on which G_i acts irreducibly for $i=1, 2$. If the fundamental groups of G_1/H_1 and G_2/H_2 are isomorphic, then G_1/H_1 and G_2/H_2 are homeomorphic.*

1. Preliminaries

In this note we use the following notations.

- (1) \mathbf{Z} , \mathbf{R} and \mathbf{N} denote the ring of integers, the field of real numbers and the set of natural numbers, respectively.
- (2) For a Lie group G , G° denotes its identity component.
- (3) $[G, G]$ denotes the derived subgroup of group G .
- (4) Let G be a group and H its subgroup. $N_G(H)$ denotes the normalizer of H in G .
- (5) For a Lie group G , $L(G)$ denotes its Lie algebra.
- (6) For a group G , $z(G)$ denotes the center of G .

Then we have the following definitions.

Definition. A manifold M is said aspherical or of type $K(\pi, 1)$ if all its homotopy groups, except possibly its fundamental group, are equal to zero.

Definition. A transitive action of a Lie group G on manifold M is said irreducible if the following conditions are satisfied ;

- (1) G acts on M effectively
- (2) G does not contain proper subgroups which are transitive on M .

Definition. Let G be a Lie group. A subgroup T of G is said to be triangular if its image $ad(T)$ by the adjoint representation of G is a triangular group of transformations of $L(G)$.

Let G be a connected Lie group and $G = S \cdot R$ be a Levi decomposition of G , where S is a semisimple subgroup of G and R is the radical of G . There are well known following facts.

THEOREM 1 ([3]). *Let $M = G/H$ be compact and aspherical, and let G be simply connected and locally effective on M . Then the following assertions are true ;*

- (1) G is diffeomorphic to \mathbf{R}^n
- (2) $H^\circ \subset T \cdot R$, where T is some maximal connected triangular subgroup of S
- (3) if G is irreducible on M , then $H^\circ \subset [T, T] \cdot R$.

It follows, from this Theorem 1, that if $G = S \cdot R$ is a Levi decomposition of G which is diffeomorphic to \mathbf{R}^n , then S is isomorphic to $A_1 \times \cdots \times A_p$, where A_i is a simple Lie group which is isomorphic to the universal covering group A of $SL(2, \mathbf{R})$ and diffeomorphic to \mathbf{R}^3 .

THEOREM 2 ([4]). *Let M be a compact homogeneous space of type $K(\pi, 1)$ with π solvable. Then the following assertions are true;*

- (1) M is homeomorphic to solvmanifold
- (2) if M_1 is another compact homogeneous space of type $K(\pi, 1)$, then M_1 is homeomorphic to M .

This Theorem 2 means that the homeomorphism type of the compact aspherical homogeneous space is determined by its fundamental group, if it is solvable.

THEOREM 3 ([6]). *Let G be a connected Lie group and H a closed subgroup of G such that G/H is compact. Then $N_G(H^\circ)$ contains a maximal connected triangular subgroup of G .*

2. Example

This section is mainly concerned with the manifold which is constructed in [4]. Let G be a Lie group and F, H closed subgroups of G such that $F \supset H$. We consider a fibration

$$F/H \rightarrow G/H \rightarrow G/F,$$

where G/H is the base space and F/H is the fiber space. If $G_1 \subset G$ is a subgroup which is transitive on G/F , then the fibration is homogeneous with respect to G_1 . The stationary group of the fiber is $F_1 = F \cap G_1$. We have the following

PROPOSITION 1. *If $F_1 \subset H$, then the action of F_1 on F/H is induced by the action of F_1 by inner automorphisms of F .*

PROOF. Let gH be an element of F/H , Then since $g_1 \in H$, we have

$$g_1 \cdot (gH) = g_1 g g_1^{-1} g_1 H = g_1 g g_1^{-1} H, g_1 \in F_1.$$

Thus the assertion is proved.

Let G be $SL(2, \mathbf{R}) \times_{ad} \mathbf{R}^3$, where ad is the adjoint representation of $SL(2, \mathbf{R})$ and let T be the group of the triangular matrices with positive elements on the diagonal. Then $T \times_{ad|T} \mathbf{R}^3$ is a solvable group and there exists a subspace $V \subset \mathbf{R}^3$ of codimension 1 which is invariant with respect to T . Hence $T \times_{ad|T} \mathbf{R}^3/V$ is 3 dimensional solvable group and so $[[T, T], \mathbf{R}^3] \subset V$. Thus we obtain an abelian Lie group $[T, T] \cdot \mathbf{R}^3/V$. The subgroup

$$D_\alpha = \left\{ \left[\begin{array}{cc} e^{k\alpha} & 0 \\ 0 & e^{-k\alpha} \end{array} \right] \mid k \in \mathbf{Z} \right\}$$

acts on $[T, T] = \mathbf{R}^1$ by the multiplication by $e^{2\alpha}$. If $\alpha \in \mathbf{R}$ is such that $e^{2\alpha} + e^{-2\alpha} \in \mathbf{N}$, then D_α is conjugate with the subgroups of integral valued matrices in the group of automorphisms of $[T, T] \cdot \mathbf{R}^3/V = \mathbf{R}^2$. Hence D_α preserves some lattice Γ in $[T, T] \cdot \mathbf{R}^3/V$. We set $H_1 = \phi^{-1}(\Gamma)$, where $\phi : [T, T] \cdot \mathbf{R}^3 \rightarrow [T, T] \cdot \mathbf{R}^3/V$ is a natural projection. Let H be the subgroup of G generated by D_α and H_1 . Then it follows that H is a closed in G and $G/H = M_\alpha$ is a compact 4 dimensional homogeneous space. If we set $H' = z(SL(2, \mathbf{R})) \cdot H$ in place of H , then we get another analogous manifold $G/H' = M'_\alpha$.

PROPOSITION 2. M_α and M'_α are diffeomorphic to solvmanifolds.

PROOF. Since $H \subset T \cdot \mathbf{R}^3$, we have the fibration $T \cdot \mathbf{R}^3/H \rightarrow G/H \rightarrow G/T \cdot \mathbf{R}^3$. Here the base space $G/T \cdot \mathbf{R}^3$ is diffeomorphic to S^1 . Then a maximal compact subgroup K of $SL(2, \mathbf{R})$ is transitive on $G/T \cdot \mathbf{R}^3$ and so, by Proposition 1, the diffeomorphism of the fiber space is trivial. Hence G/H is diffeomorphic to $T \cdot \mathbf{R}^3/H \times S^1$, because a bundle over S^1 is determined by a diffeomorphism of the fiber space. This manifold is a solvmanifold. For M'_α this argument is also applicable, because $z(SL(2, \mathbf{R}))$ acts trivially on \mathbf{R}^3 .

3. Proof of Theorem

In this section we shall prove Theorem in Introduction. let G be a connected, simply connected Lie group which is irreducible on $M = G/H$ and $G = S \cdot R$ be a Levi decomposition with the natural projection $q : G \rightarrow S$. Then, from the remark following Theorem 1, we have $S = S_1 \times \cdots \times S_p$. Let $p_i : S \rightarrow A_i$ be the natural projection for $i = 1, \dots, p$. From Theorem 1 we have that $\dim p_i \cdot q(H) \leq 1$. Since $\dim M = 4$, we get that $p \leq 2$.

(1) Case of $p=2$

In this case we have that $G = (A_1 \times A_2) \cdot R$ and $\dim q(H) \leq 2$. If $R \neq \{e\}$, then $R \subset H$. So by the reason of irreducibility we have $G = A_1 \times A_2$. Then $H^\circ = [T_1, T_1] \times [T_2, T_2]$, where T_i is a maximal triangular subgroup of A_i for $i = 1, 2$, and $N_G(H^\circ) = z(A_1)T_1 \times z(A_2)T_2$. Since $\pi_1(G/H) = H/H^\circ$ and $H \subset N_G(H^\circ)$, $H/H^\circ \subset N_G(H^\circ)/H^\circ = z(A_1)\mathbf{R} \times z(A_2)\mathbf{R}$ and so $\pi_1(G/H) = \mathbf{Z}^4$.

Let $H_1 = H \cdot N_G(H^\circ)^\circ$. Since $H \subset H_1 \subset N_G(H^\circ)$ and $H_1^\circ = N_G(H^\circ)^\circ = T_1 \times T_2$, it follows that H_1° is a closed subgroup of G , while by the construction H_1° is transitive on H_1/H . We

consider the fibration and the corresponding exact sequence of the fundamental group ;

$$\begin{aligned} H_1/H &\rightarrow G/H \rightarrow G/H_1 \\ 1 &\rightarrow \pi_1(H_1/H) \rightarrow \pi_1(G/H) \rightarrow \pi_1(G/H_1) \rightarrow 1. \end{aligned}$$

Let K_i be a subgroup of A_i whose Lie algebra is maximal and compact. Since $K = K_1 \times K_2$ is an abelian subgroup of G and is transitive on the compact base space G/H_1 , G/H_1 is a 2 dimensional torus. With respect to the fiber H_1/H ,

$$H_1/H = H_1^\circ/H_1^\circ \cap H = S^1 \times S^1.$$

However $K \subset G$ is transitive on G/H_1 and $K \cap H_1 \subset z(G) \cap H$, because $K \cap H_1 \subset N_G(H^\circ)$. So the diffeomorphism of the fiber, which is induced by an inner automorphism by the element of $K \cap H_1$, is trivial and hence, applying Proposition 1, this fibration is trivial over the loops S^1 of the base. Thus G/H is 4 dimensional torus([8]).

(2) Case of $p = 1$

In this case we have $G = A \cdot R$ and further the following two subcases (i) and (ii) because $\dim q(H) \leq 1$.

(i) $\dim q(H) = 0$ and $H^\circ \subset R$, $\text{codim}_R H^\circ = 1$.

Claim 1. R is abelian and if we write $G = A \times_\varphi \mathbf{R}^n$, then φ is an irreducible representation.

If R is nonabelian, then either $[R, R] \cdot H = R$ or $[R, R] \subset H$. If $[R, R] \cdot H = R$, then $G' = A \times_\varphi [R, R]$ is transitive on M , which contradicts an irreducibility of G on M . If $[R, R] \subset H$, then this contradicts a local effectiveness of the action of G on M . Thus R is abelian.

Assume φ is reducible. Let $R = V_1 + V_2$ be a direct sum decomposition of two φ invariant subspaces of R , $V_i \neq \{0\}$ for $i = 1, 2$. Then either $V_1 + H^\circ = R$ or $V_2 + H^\circ = R$ and so $A \times_\varphi V_1$ or $A \times_\varphi V_2$ is transitive on M . This contradicts an irreducibility of G on M . Thus φ is irreducible.

Claim 2. φ is unimodular.

We put $F = N_G(H^\circ)^\circ$. By Theorem 3 there exists a maximal connected triangular subgroup T of A such that $T \cdot R \subset F$. But this induces $T \cdot R = F$ and F/H° is a 3 dimensional solvable Lie group with a lattice $H \cap F/H^\circ$. This solvable Lie group has the following splitting

$$\begin{array}{ccccccc} 1 & \longrightarrow & [T, T] \cdot R/H^\circ & \longrightarrow & F/H^\circ & \longrightarrow & T/[T, T] \longrightarrow 1. \\ & & \parallel & & & & \parallel \\ & & \mathbf{R}^2 & & & & \mathbf{R}^1 \end{array}$$

We get that $F/H^\circ = \mathbf{R}^1 \times_\psi \mathbf{R}^2$, where $\psi(t) = (Ad(t), \varphi(t))$, Ad is an adjoint map of T on $[T, T]$, for $t \in T/[T, T]$. By the reason of the unimodularity of ψ and ad , φ is unimodular.

By the above claims, φ coincides with the adjoint representation used in Proposition 2. Hence $G = A \times_{ad} \mathbf{R}^3$ and $G/H = M_a$.

(ii) $\dim q(H) = 1$ and $\text{codim}_R R \cap H = 2$.

Claim 3. If $G = A \cdot R$, then $R = N$, where N is the nilradical of R .

Assume $R \neq N$. Let $H_2 = H \cap R$ and then $H_2 \cdot N \subset R$. Since both cases $N \subset H_2$ and $H_2 \cdot N = R$ are impossible by the reason of the irreducibility of G on M , we have $\text{codim}_R H_2 \cdot N = 1$. Let $p : R \rightarrow R/N$ be the natural projection. Since $p(H_2 \cdot N) = H_2/H_2 \cap H$ has

codimension 1 in the abelian Lie group R/N , there exists a 1 dimensional subgroup V such that $V \cdot p(H_2 \cdot N) = R/N$ and $V \cap p(H_2 \cdot N) = \{e\}$. Since A acts on R/N trivially, $p^{-1}(V) \subset R$ is invariant with respect to A and hence $G' = A \cdot p^{-1}(V)$ is a proper subgroup of G . Since $L(G') + L(H) = L(G)$, G' is transitive on M , which contradicts the irreducibility of G on M . Thus $R=N$.

Claim 4. N is abelian.

Since $T \cdot N$ is a maximal connected triangular subgroup of G , we have $N_G(H^\circ) = T \cdot N$ by Theorem 3. Let $C \subset T$ be a Cartan subgroup such that $C \cdot [T, T] = T$ and $C \cap [T, T] = \{e\}$. Since $\dim q(H) = 1$, $q(H^\circ) = [T, T]$. We have $H \subset N_G(H^\circ) = z(A)T \cdot N$ and so $H^\circ \subset T \cdot N$. Then 3 dimensional solvable Lie group $C \cdot N/H^\circ \cap C \cdot N$ is transitive on the compact solvmanifold $T \cdot N/H^\circ \cap C \cdot N$ of dimension 3. $C \cdot N/H^\circ \cap C \cdot N$ contains N/H_2 as a normal subgroup of dimension 2. From the description of 3 dimensional solvable Lie groups having lattices ([1]), N/H_2 is abelian and hence $[N, N] \subset H_2$. But assume $[N, N] \neq \{e\}$, this contradicts the local effectiveness of the action of G on M . Thus N is abelian.

Claim 5. If we write $G = A \times_\varphi \mathbf{R}^n$, φ is trivial and so $G = A \times \mathbf{R}^n$.

Since T normalizes H° , φ induces a representation $\tilde{\varphi} : T \rightarrow GL(\mathbf{R}^n/H_2^\circ)$, which is trivial. In fact \mathbf{R}^n is abelian and $T \cdot \mathbf{R}^n/H^\circ$ is 3 dimensional solvable group and so $[T, \mathbf{R}^n] \subset H_2$. Let $V_1 \subset \mathbf{R}^n$ be the space of the fixed elements of φ and V_2 its invariant complement. From the triviality of $\tilde{\varphi}$ it follows that $V_2 \subset H_2$. Assume $V_2 \neq \{0\}$. Then the subgroup $G' = A \times_\varphi V_1$ is a proper subgroup of G and transitive on M , which contradicts the irreducibility of G on M . Thus $V_1 = \mathbf{R}^n$ and so φ is trivial.

By the same argument as in case (1), $M = G/H$ is diffeomorphic to a 4 dimensional torus.

(3) Case of $p=0$

In this case we have $G=R$ and so G/H is a solvmanifold.

In all cases we proved that the fundamental group of G/H is solvable since the fundamental group of solvmanifold is solvable. This completed the proof of Theorem.

References

- [1] L. Auslander, L. Green and F. Haken, "Flows on homogeneous spaces," Princeton Univ. Press, Princeton N. J., 1963.
- [2] F. Farrell and L. Jones, *Compact negatively curved connected manifolds (of dim $\neq 3, 4$) are topologically rigid*, Proc. Nat. Acad. Sci. USA **86** (1989), 3461-3463.
- [3] V. Gorbacevič, *On aspherical homogeneous space*, Math. USSR Sb. **29** (1976), 223-238.
- [4] V. Gorbacevič, *On Lie groups, transitive on compact solvmanifolds*, Math. USSR Izv. **11** (1977), 271-292.
- [5] G. Mostow, *Factor spaces of solvable groups*, Ann. of Math. **60** (1954), 1-27.
- [6] A. Oniščik, *On Lie groups transitive on compact manifolds*, Math. USSR Sb. **3** (1967), 59-72.
- [7] S. Ogose, K. Kawabe and T. Watabe, *A note on aspherical homogeneous manifolds* (preprint).
- [8] K. Sakamoto and S. Fukuhara, *Classification of T^2 -bundles over T^2* , Tokyo J. Math. **6** (1983), 311-327.