素因子についての2、3の注意（英文）

著者	長沢 良一
雑誌名	金沢大学教育学部紀要.自然科学編
巻	44
ページ	45-48
発行年	1995-02-28
URL	http://hdl.handle.net/2297/493

Some Remarks on Prime Divisors

Ryoichi NAGASAWA

In this note we discuss some aspects of the theory of prime divisors in commutative algebra and also explain some simple proofs of known results. We know many different kinds of prime divisors attached to an ideal in a Noetherian ring. Recently, certain stable characters of large powers of ideals were deeply studied and many interesting results were obtained by various reserchers of this area.

These topics were systematically treated in [2] and [3]. Among various kinds of prime divisors we are much interested in so-called essential prime divisors which are closely related to certain ring extensions, and the prototype of these divisors may be found in [1].

Let X be a locally Noetherian prescheme and let Z be a subset of X stable under specialization. Then we defined an O_X-Module $H^o_{X/Z}(F)$ for an O_X-Module F such that $H^o_{X/Z}(F)$ is quasi-coherent for any F quasi-coherent. For any point x in $Ass(F) \cap (X-Z)$, let Y_x be the reduced subprescheme of X defined on $\bar{x}$, the closure of $\{x\}$ and let $Z_x = Z \cap \bar{x}$. If $H^o_{X/Z}(F)$ is a coherent O_X-Module, then codim $(Z_x, Y_x) \geq 2$ for all x as above, and the converse holds if every point of X has an open neighbourhood isomorphic to a subscheme of a regular scheme ([1], (5. 11. 1)). In particular, let R be a local ring, R^* its completion and f the canonical morphism of $X' = Spec(R^*)$ to $X = Spec(R)$. For any $Z \subset X$ as above, $Z' = f^{-1}(Z)$ is also stable under specialization, and $H^o_{X/Z}(F)$ is coherent if and only if $H^o_{X'/Z'}(f^*(F))$ is coherent. Here we remark that for a Noetherian domain R, $H^o_{X/Z}(O_X)$ is the O_X-*Algebra* $\tilde{S}$, where S is the intersection of all R_P with P in $X-Z$. Thus, with the notation as above we have the following:

For a local domain R, the overdomain S is finite integral over R if and only if Z contains no prime ideal P such that the completion of R_P contains a depth 1 prime divisor of zero. In particular, $R^{(1)}$, the intersection of all R_P with height $P=1$, is finite integral over R if and only if *height* $(P/Q) \geq 2$ for any prime ideals $Q \subset P$ in R^* such that Q is a prime divisor of zero and height $(P \cap R) \geq 2$ (cf. [1], (7.2.3), (7.2.4)).

As in the above statement, a concept of essential prime divisors is naturally obtained and its importance can be understood as well.

In the above statement, a certain gap may be found in the expressions of prime divisors, but it would be resolved in the following:

Let P be a regular prime ideal in a local ring R. Then the completion $(R_P)^*$ has a depth 1 prime divisor of zero if and only if there exist prime ideals $Q^* \subset P^*$ of R^* with $P^* \cap R = P$, $Q^* \in Ass(R^*)$ and *height* $(P^*/Q^*) = 1$.

This is a known result ([2], Corollary 10.16). We can prove the assertion by an indirect use of ideal transform, but here we give a somewhat different proof. The theorem of transition(cf.

Received September 16th. 1994.

〔4]) will be be used tacitly in the following. For a Noetherian ring R, $E(R)$ denotes the set of prime ideals P of R such that $(R_P)^*$ has a depth 1 prime divisor of zero. Now let P be a regular prime ideal in a local ring R. Let P^* be a prime ideal which is minimal over PR^* and let $T=(R^*)_{P^*}$. Suppose that $(R_P)^*$ has a depth 1 prime divisor of zero, P_1. Let P_2 be a prime ideal minimal over P_1T^*. Then we see that depth$(P_2)=1$ and height$(P_2)=$height(P_1). Let $P_3=P_2\cap T$ and $Q^*=R_3\cap R^*\in Ass(R^*)$. Then $S=T/P_3$ is naturally isomorphic to the localization of R^*/Q^* at P^*/Q^*. Clearly $P_3\in$Ass(T), $S^*=T^*/P_3T^*$ and $P_2/P_3T^*\in$Ass(S^*). Since R^*/Q^* is a complete local domain, depth$(P_2)=1$ implies dim $S=1$, hence height $(P^*/Q^*)=1$. Conversely let P^* and Q^* be as in the statement (2). Put $T=(R^*)_{P^*}$. Since $Q^*\in$Ass(R^*) and dim$(T/Q^*T)=1$, there is a prime ideal $Q'\in$Ass(T^*) with depth$(Q')=1$ and $Q'\cap T=Q^*T$. Let c be an element of T^* with $(0):c=Q'$. Then there would be an integer $k>0$ such that $c\notin p^kT^*$ and $JT^*:c\supset Q'+JT^*\supsetneqq Q'$ for any regular ideal $J\subset P^kR_P$. Since depth$(Q')=1$, we see that $JT^*:c$ is P^*T^*-primary and$P^*T^*\in Ass(T^*/JT^*)$, that is, $PR_P=P^*T^*\cap R_P\in Ass(R_P/J)$. This implies that for any regular element a in P, $A=(R_P)^*$ has a prime ideal $P'\in Ass(A)$ such that PA is minimal over $aA+P$ (cf. [3], (1.3)). Clearly depth$(P')=1$ and so $P\in E(R)$.

Thus the proof is complete.

There would be many interesting results concerning essential prime divisors. Particularly, we are interested in the behaviour of asymptotic prime divisors under finite ring extensions.

Let A be any finite ring extension of a Noetherian ring R in the total quotient ring $F(R)$ of R. Then, for a regular element $a\in R$, $Spec(A)\longrightarrow Spec(R)$ yields a surjective mapping $E(aA)\longrightarrow E(aR)$, where $E(aR)=E(R)\cap Ass(R/aR)$, etc. This is a characteristic property of essential prime divisors. In other words, a prime divisor P of aR is essential if and only if for any finite ring extension A in $F(R)$, there is a prime divisor Q of aA with $Q\cap R=P$.

Moreover there is a finite extension A in $R[1/a]$ for which $E(aA)=Ass(aA)$ holds(cf. [3], (8.4)). This result may be one of the most beautiful results of recent years in the theoy of prime divisors, and it may be clarified through an affine description of $H^{o}_{X/Z}(O_X)$, that is, ideal transforms. Suppose that R is a local ring with regular maximal ideal M. Let B be the product of all R^*/Q, where Q runs over the primary components of zero in R^*. Let Z be the subset of prime ideals P in $Spec(B)$ with $height(P)\geq 2$. Then $H^{o}_{Y/Z}(O_Y)$ with $Y=Spec(B)$ is made into $Spec(T)$, where T is the intersection of inverse images of all B_P, $P\in Y-Z$ in the total quotient ring $F(B)$ of B under the canonical homomorphism $F(B)\longrightarrow F(B_P)$. Since B is a complete semi-local ring, T is finite over B. Let A be the inverse image of T under the canonical homomorphism $F(R)\longrightarrow F(R^*)\longrightarrow F(B)$. From our construction of T, we can prove directly that A is a finite ring extension of R in the total quotient ring $F(R)$, and for any regular element a of A, every prime divisor of aA is essential, that is, $P(A)=E(A)$, where $P(A)$ denotes the set of P in $Spec(A)$ such that P is a prime divisor of some regular element. Thus the local case is directly proved in a strong form without making use of ideal transform. Similarly, we can treat the semi-local case. But the similar result would not be true for the global case, and it may be true for only finitely many regular principal ideals. In fact, let a_1 , $\cdots, a_n$ be regular non-units of a Noetherian ring R and let a be their product. With respect to this element a, construct a finite ring extension A in $A[1/a]$ as stated above. Then each prime divisor of a_iA is a prime divisor of aA, hence a essential prime divisor (Cf. [4], (12.6)). In particular, if R is a Noetherian domain such that the set of normal points in $Spec(R)$ is open, then R$[1/a]$ is normal for some a, and there is a finite ring extension A in $R[1/a]$ with $E(aA) = Ass(A/aA)$. Then we have $P(A) = E(R)$ since A_P is normal for any P in $P(A) - Ass (A/aA)$. Here we should note that the derived normal ring of R is not necessarily finite over R, and in fact, there is a Noetherian domain R such that $R[1/a]$ is normal as above and for some prime divisor P of aR, the completion of R_P has a depth 1 prime divisor Q with *height* $(Q) > 0$. For a bad Noetherian ring it may be impossible to find a finite ring extension A with $Ass(A) = E(A)$. The proof for the global case can be easily reduced to the (semi-) local case. The crucial point is to construct a finite ring extension B in $F(R)$ such that the essential prime divisors of aB form an isolated subset in $Ass(B/aB)$, that is, any prime divisor P which is contained in an essential prime divisor of aB is essential, and this can be settled by using the local case. An elegant and systematic treatment of the above result may be found in [3], and there the concept of essential prime divisors is extended for arbitrary ideals of Noetherian rings. For any ideal I of a Noetherian ring R, the so-called Rees ring $R(I)$ of R with respect to I is defined as $R(I) = R[u, It]$ with t an indeterminate and $u = t^{-1}$. Since $uR(I)$ is a regular principal ideal, we get the set $E(uR(I))$, and $E(I)$, the set of essential prime divisors of I, is defined as the set of prime ideals $Q \cap R$, $Q \in E(uR(I))$. Here we should note some difference between $R(I)$ and the graded ring $G(I) = R[IT]$. We consider a ring extension $R[u]$ with u an indeterminate and an extended ideal $J = (I, u)R[u]$. Then u is a regular superficial element of J such that $J^n : u = J^{n-1}$ for any $n > 0$.

Putting $t = u^{-1}$, we obtain the Rees ring $R(I) = R[u][Ju^{-1}] = R[u, It]$, and in fact, $JR(I) = uR(I)$ is the principalization of J.

Thus, $R(I)$ is a substitute for neighbourhood rings, and $Spec(R(I))$ is a nice affine part of the blowing up $Proj(R[u][Jz])$, where $J = (I, u)R[u]$ and z is an indeteminate.

References

[1] A. Grothendieck, EGA IV, Publ. Math. Inst. Hautes Etudes Sci. 24 (1965).

[2] S. McAdam, Asymptotic prime Divisors, Lecture Notes in Mathematics, No. 1023, SPringer-Verlag, New York, 1983.

[3] S. McAdam, Primes Associated to an Ideal, Contemporary Mathematics, 102, American Mathematical Society, 1989.

[4] M. Nagata, Local Rings, Interscience, New York, 1962.